Axiomatic Set Theory

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Lecture Notes

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Contents

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†Resources used in the preparation of this text include [Kun12, Kun13].

1 Motivation and Preliminaries

1.1 Cantor's Definition, Russell's Antinomy

In 1895 in [Can95], Georg Cantor defined a set as "any collection into a whole M of definite and separate objects m of our intuition or our thought". The objects m are called the *elements* of the set M and one writes $m \in M$ if, and only if, m is an element of M.

As it turns out, *naive set theory*, founded on Cantor's definition, is not suitable to be used in the foundation of mathematics. The problem lies in the possibility of obtaining contradictions such as *Russell's antinomy*, after Bertrand Russell, who described it in 1901, see [Rus80, Rus96].

Russell's antinomy is obtained when considering the set X of all sets that do not contain themselves as an element: When asking the question if $X \in X$, one obtains the contradiction that $X \in X \Leftrightarrow X \notin X$:

Suppose $X \in X$. Then X is a set that contains itself. But X was defined to contain only sets that do not contain themselves, i.e. $X \notin X$.

So suppose $X \notin X$. Then X is a set that does not contain itself. Thus, by the definition of X, $X \in X$.

Perhaps you think Russell's construction is rather academic, but it is easily translated into a practical situation. Consider a library. The catalog C of the library should contain all the library's books. Since the catalog itself is a book of the library, it should occur as an entry in the catalog. So there can be catalogs such as C that have themselves as an entry and there can be other catalogs that do not have themselves as an entry. Now one might want to have a catalog X of all catalogs that do not have themselves as an entry. As in Russell's antinomy, one is led to the contradiction that the catalog X must have itself as an entry if, and only if, it does not have itself as an entry.

One can construct arbitrarily many versions, which we will not do. Just one more: Consider a small town with a barber, who, each day, shaves all inhabitants, who do not shave themselves. The poor barber now faces a terrible dilemma: He will have to shave himself if, and only if, he does not shave himself.

To avoid contradictions such as Russell's antinomy, *axiomatic set theory* restricts the construction of sets via so-called axioms, as we will see below.

1.2 Mathematical Logic

The development and presentation of axiomatic set theory is based on mathematical logic. Indeed, mathematical logic is a large field in its own right and a thorough introduction is beyond the scope of this class – the interested reader may refer to [EFT07], [Kun12], and references therein. Still, it will be necessary to at least introduce some basic concepts. Occasionally, we will touch on some deeper logical issues and subtlebies, usually referring to the literature for further information.

One can view the central goal of mathematics as the rigorous proof of the truth or falsehood of statements. By a *statement* or *proposition*, we mean any sentence (any sequence of symbols) that can reasonably be assigned a *truth value*, i.e. a value of either *true*, abbreviated T, or *false*, abbreviated F. For example, "$2+3 = 5$" is a true statement, "$\sqrt{2} < 0$" is a false statement, whereas "$3 \cdot 5 + 7$" and "$x + 1 > 0$" are not statements at all.

Statements can be manipulated or combined into new statements using *logical operators*, where the truth value of the combined statements depends on the truth values of the original statements and on the type of logical operator facilitating the combination.

The simplest logical operator is *negation*, denoted $\neg$. It is a so-called *unary* operator, i.e. it does not combine statements, but is merely applied to one statement. For example, if A stands for the (true) statement "$2 + 3 = 5$", then $\neg A$ stands for the (false) statement "$2 + 3 \neq 5$"; if B stands for the (false) statement "$\sqrt{2} < 0$", then $\neg B$ stands for the (true) statement "$\sqrt{2}$ is not less than zero", which can also be expressed as "$\sqrt{2} \geq 0$".

To completely understand the action of a logical operator, one usually writes what is known as a *truth table*. For negation, the truth table is

A	$\neg A$
T	F
F	T

(1.1)

that means if the input statement A is true, then the output statement $\neg A$ is false; if the input statement A is false, then the output statement $\neg A$ is true.

We now proceed to discuss *binary* logical operators, i.e. logical operators combining precisely two statements. The following four operators are essential for mathematical reasoning:

Conjunction: A *and* B, usually denoted $A \wedge B$.

Disjunction: A *or* B, usually denoted $A \vee B$.

Implication: A *implies* B, usually denoted $A \Rightarrow B$.

Equivalence: A is *equivalent* to B, usually denoted $A \Leftrightarrow B$.

The corresponding truth table reads:

A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

(1.2)

Note that the disjunction $A \vee B$ is true if, and only if, at least one of the statements A, B is true. Here one already has to be a bit careful – $A \vee B$ defines the *inclusive* or, whereas "or" in common English is often understood to mean the exclusive or (which is false if both input statements are true). Instead of A implies B, one also says *if* A *then* B, B *is a consequence of* A, B *is concluded or inferred from* A, A *is sufficient for* B, or B *is necessary for* A.

The implication $A \Rightarrow B$ is always true, except if A is true and B is false. At first glance, it might be surprising that $A \Rightarrow B$ is defined to be true for A false and B true, however, this is precisely what distinguishes the implication from the equivalence. After a moment's contemplation, one will most likely notice that one is quite familiar with examples of incorrect statements implying correct statements: For instance, squaring the (false) equality of integers $-1 = 1$, implies the (true) equality of integers $1 = 1$. Of course, the implication $A \Rightarrow B$ is not really useful in situations, where the truth values of both A and B are already known. Rather, in a typical application, one tries to establish the truth of A to prove the truth of B (a strategy that will fail if A happens to be false).

The equivalence $A \Leftrightarrow B$ means A is true if, and only if, B is true. Analogous to the situation of implications, $A \Leftrightarrow B$ is not really useful if the truth values of both A and B are known a priori, but can be a powerful tool to prove B to be true or false by establishing the truth value of A.

Note that the expressions in the first row of the truth table (1.2) (e.g. $A \wedge B$) are *not*, actually, statements, as they contain the *statement variables* (also known as *propositional variables*) A or B. However, the expressions become statements if all statement variables are substituted with actual statements. We will call expressions of this form *propositional formulas*. Moreover, if a truth value is assigned to each statement variable of a propositional formula, then this uniquely determines the truth value of the formula. In other words, the truth value of the propositional formula can be *calculated* from the respective truth values of its statement variables – the presently discussed topic is, therefore, known as *propositional calculus*.

Example 1.1. **(a)** Consider the propositional formula $(A \wedge B) \vee (\neg B)$. Suppose A is true and B is false. The truth value of the formula is obtained according to the following truth table:

A	B	$A \wedge B$	$\neg B$	$(A \wedge B) \vee (\neg B)$
T	F	F	T	T

(1.3)

(b) The propositional formula $A \vee (\neg A)$, also known as the *law of the excluded middle*, has the remarkable property that its truth value is T for every possible choice of truth values for A:

A	$\neg A$	$A \vee (\neg A)$
T	F	T
F	T	T

(1.4)

Formulas with this property are of particular importance.

Definition 1.2. A propositional formula ϕ is called a *tautology* or *universally true* if, and only if, its truth value is T for all possible assignments of truth values to all the statement variables it contains. One writes $\vdash \phi$ if, and only if, ϕ is a tautology.

Definition 1.3. The propositional formulas ϕ and ψ are called *equivalent* if, and only if, $\phi \Leftrightarrow \psi$ is a tautology.

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For all logical purposes, two equivalent formulas are exactly the same – it does not matter if one uses one or the other. The following Th. 1.5 provides some important equivalences of propositional formulas. As too many parentheses tend to make formulas less readable, we first introduce some precedence conventions for logical operators:

Convention 1.4. $\neg$ takes precedence over $\wedge, \vee$, which take precedence over $\Rightarrow, \Leftrightarrow$. So, for example,

$$(A \vee \neg B \Rightarrow \neg B \wedge \neg A) \Leftrightarrow \neg C \wedge (A \vee \neg D)$$

is the same as

$$\Big(\big(A \vee (\neg B)\big) \Rightarrow \big((\neg B) \wedge (\neg A)\big)\Big) \Leftrightarrow \Big((\neg C) \wedge \big(A \vee (\neg D)\big)\Big).$$

Theorem 1.5. (a) $\vdash (A \Rightarrow B) \Leftrightarrow \neg A \vee B$. *This means one can actually define implication via negation and disjunction.*

(b) $\vdash (A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \wedge (B \Rightarrow A)\big)$, *i.e. A and B are equivalent if, and only if, A is both necessary and sufficient for B. One also calls the implication $B \Rightarrow A$ the* converse *of the implication $A \Rightarrow B$. Thus, A and B are equivalent if, and only if, both $A \Rightarrow B$ and its converse hold true.*

(c) *Commutativity of Conjunction:* $\vdash A \wedge B \Leftrightarrow B \wedge A$.

(d) *Commutativity of Disjunction:* $\vdash A \vee B \Leftrightarrow B \vee A$.

(e) *Associativity of Conjunction:* $\vdash (A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$.

(f) *Associativity of Disjunction:* $\vdash (A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$.

(g) *Distributivity I:* $\vdash A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$.

(h) *Distributivity II:* $\vdash A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$.

(i) *De Morgan's Law I:* $\vdash \neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$.

(j) *De Morgan's Law II:* $\vdash \neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$.

(k) *Double Negative:* $\vdash \neg\neg A \Leftrightarrow A$.

(l) *Contraposition:* $\vdash (A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$.

Proof. Each equivalence is proved by providing a suitable truth table, showing that the respective equivalence τ is a tautology: In each case, the final column of the truth table shows that, for all possible assignments of truth values to A, B, C (where applicable), τ has truth value T:

(a):

A	B	$\neg A$	$A \Rightarrow B$	$\neg A \vee B$	$(A \Rightarrow B) \Leftrightarrow \neg A \vee B$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(b) – (h): Exercise.

(i):

A	B	$\neg A$	$\neg B$	$A \wedge B$	$\neg(A \wedge B)$	$\neg A \vee \neg B$	$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

(j): Exercise.

(k):

A	$\neg A$	$\neg\neg A$	$\neg\neg A \Leftrightarrow A$
T	F	T	T
F	T	F	T

(l):

A	B	$\neg A$	$\neg B$	$A \Rightarrow B$	$\neg B \Rightarrow \neg A$	$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Having checked all the equivalences completes the proof of the theorem. ■

The importance of the rules provided by Th. 1.5 lies in their providing *proof techniques*, i.e. methods for establishing the truth of statements from statements known or assumed to be true. The rules of Th. 1.5 will be used frequently in proofs throughout this class.

Remark 1.6. Another important proof technique is the so-called *proof by contradiction*, also called *indirect proof*. It is based on the observation, called the *principle of contradiction*, that $A \wedge \neg A$ is always false:

A	$\neg A$	$A \wedge \neg A$
T	F	F
F	T	F

(1.5)

Thus, one possibility of proving a statement B to be true is to show $\neg B \Rightarrow A \wedge \neg A$ for some arbitrary statement A. Since the right-hand side of the implication is false, the left-hand side must also be false, proving B is true.

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Two more rules we will use regularly in subsequent proofs are the so-called transitivity of implication and the transitivity of equivalence:

Theorem 1.7. **(a)** *Transitivity of Implication:* $\vdash (A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$.

(b) *Transitivity of Equivalence:* $\vdash (A \Leftrightarrow B) \wedge (B \Leftrightarrow C) \Rightarrow (A \Leftrightarrow C)$.

Proof. Both implications are proved by providing a suitable truth table, showing that the respective implication $\tau(A, B, C)$ is a tautology: In each case, the final column of the truth table shows that, for all possible assignments of truth values to A, B, and C, $\tau(A, B, C)$ has truth value T. We carry out (a) and leave (b) as an exercise.

(a):

A	B	C	$A \Rightarrow B$	$B \Rightarrow C$	$(A \Rightarrow B) \wedge (B \Rightarrow C)$	$A \Rightarrow C$	$(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

(b): Exercise. ■

Definition and Remark 1.8. A *proof* of the statement B is a finite sequence of statements $A_1, A_2, \dots, A_n$ such that A_1 is true; for $1 \leq i < n$, A_i implies A_{i+1}, and A_n implies B. If there exists a proof for B, then Th. 1.7(a) guarantees that B is true[1].

1.3 Set-Theoretic Formulas

The contradiction of Russell's antinomy, as described in Sec. 1.1, is related to Cantor's sets not being hierarchical. Another source of contradictions in naive set theory is the imprecise nature of informal languages such as English. Suppose B is a set and $P(x)$ is a statement about an element x of B (a so-called *predicate* of x). Then one might define

$$A := \{x \in B : P(x)\}$$

to be the subset of B, consisting of all elements of B such that $P(x)$ is true. Now take $B := \mathbb{N} := \{1, 2, \dots\}$ to be the set of the natural numbers and let

$$P(x) := \text{“The number } x \text{ can be defined by fifty English words or less”}. \tag{1.6}$$

Then A is a finite subset of $\mathbb{N}$, since there are only finitely many English words (if you think there might be infinitely many English words, just restrict yourself to the words contained in some concrete dictionary). Then there is a smallest natural number n that is not in A. But then n is the smallest natural number that can not be defined by fifty English words or less, which, actually, defines n by less than fifty English words, in contradiction to $n \notin A$.

[1] Actually, more generally, a *proof* of the statement B is given by a finite sequence of statements $A_1, A_2, \dots, A_n$ such that A_1 is true; the logical disjunction $A_1 \vee \dots \vee A_i$ implies A_{i+1} for $1 \leq i < n$; and $A_1 \vee \dots \vee A_n$ implies B. It is then still correct that the existence of a proof of B guarantees B to be true.

To avoid contradictions of this type[2], we require $P(x)$ to be a so-called *set-theoretic formula*.

Definition 1.9. **(a)** The *language* of set theory consists precisely of the following symbols: $\wedge, \neg, \exists, (,), \in, =, v_j$, where $j = 1, 2, \dots$.

(b) A *set-theoretic formula* is a finite string of symbols from the above language of set theory that can be built using the following recursive rules:

(i) $v_i \in v_j$ is a set-theoretic formula for all $i, j = 1, 2, \dots$.

(ii) $v_i = v_j$ is a set-theoretic formula for all $i, j = 1, 2, \dots$.

(iii) If ϕ and ψ are set-theoretic formulas, then $(\phi) \wedge (\psi)$ is a set-theoretic formula.

(iv) If ϕ is a set-theoretic formula, then $\neg(\phi)$ is a set-theoretic formula.

(v) If ϕ is a set-theoretic formula, then $\exists v_j(\phi)$ is a set-theoretic formula for all $j = 1, 2, \dots$.

Example 1.10. Examples of set-theoretic formulas are $(v_3 \in v_5) \wedge (\neg(v_2 = v_3))$, $\exists v_1(\neg(v_1 = v_1))$; examples of symbol strings that are not set-theoretic formulas are $v_1 \in v_2 \in v_3$, $\exists\exists\neg$, and $\in v_3 \exists$.

Remark 1.11. It is noted that, for a given finite string of symbols, a computer can, in principle, check in finitely many steps, if the string constitutes a set-theoretic formula or not. The symbols that can occur in a set-theoretic formula are to be interpreted as follows[3]: The variables $v_1, v_2, \dots$ are variables for sets. The symbols $\wedge$ and $\neg$ are to be interpreted as the logical operators of conjunction and negation as described in Sec. 1.2. Moreover, $\exists$ stands for a so-called *existential quantifier*: The statement $\exists v_j(\phi)$ means "there exists a set v_j that has the property ϕ" (we will see many examples throughout this class). Parentheses (and) are used to make clear the scope of the logical symbols $\exists, \wedge, \neg$. Where the symbol $\in$ occurs, it is interpreted to mean that the set to the left of $\in$ is contained as an element in the set to the right of $\in$. Similarly, $=$ is interpreted to mean that the sets occurring to the left and to the right of $=$ are equal.

Remark 1.12. A disadvantage of set-theoretic formulas as defined in Def. 1.9 is that they quickly become lengthy and unreadable (at least to the human eye). To make formulas more readable and concise, one introduces additional symbols and notation.

[2]The described contradiction is a variant of the so-called *Berry paradox* (see, e.g., [Wik22a] for further information and references). While it is, clearly, not as easy to provide a variant of the Berry paradox when using set-theoretic formulas, it does not seem at all obvious if it is, actually, impossible.

[3]In the terminology of mathematical logic, Def. 1.9 provides the *syntax* of set-theoretic formulas, whereas the interpretations given by the present Rem. 1.11 provide the *semantics* of set-theoretic formulas

Formally, additional symbols and notation are always to be interpreted as abbreviations or transcriptions of actual set-theoretic formulas. For example, we use the rules of Th. 1.5 to *define* the additional logical symbols $\vee$, $\Rightarrow$, $\Leftrightarrow$ as abbreviations:

$$(\phi) \vee (\psi) \quad \text{is short for} \quad \neg((\neg(\phi)) \wedge (\neg(\psi))) \quad \text{(cf. Th. 1.5(j))}, \tag{1.7a}$$

$$(\phi) \Rightarrow (\psi) \quad \text{is short for} \quad (\neg(\phi)) \vee (\psi) \quad \text{(cf. Th. 1.5(a))}, \tag{1.7b}$$

$$(\phi) \Leftrightarrow (\psi) \quad \text{is short for} \quad ((\phi) \Rightarrow (\psi)) \wedge ((\psi) \Rightarrow (\phi)) \quad \text{(cf. Th. 1.5(b))}. \tag{1.7c}$$

We also define the *universal quantifier* $\forall$:

$$\forall v_j(\phi) \quad \text{is short for} \quad \neg(\exists v_j(\neg(\phi))), \tag{1.7d}$$

such that $\forall v_j(\phi)$ means "each set v_j has the property ϕ", which is equivalent to the statement "there does not exist a set v_j that does not have the property ϕ". Further abbreviations and transcriptions are obtained from omitting parentheses if it is clear from the context and/or from Convention 1.4 where to put them in, by writing variables bound by quantifiers under the respective quantifiers (to improve readability), and by using other symbols than v_j for set variables. For example,

$$\underset{x}{\forall} \; (\phi \Rightarrow \psi) \quad \text{transcribes} \quad \neg(\exists v_1(\neg((\neg(\phi)) \vee (\psi)))), \tag{1.7e}$$

$$\left(\underset{x}{\exists}\, \phi\right) \Leftrightarrow \left(\underset{y}{\exists}\, \psi\right) \quad \text{transcribes} \quad ((\exists v_1(\phi)) \Rightarrow (\exists v_2(\psi))) \wedge ((\exists v_2(\psi)) \Rightarrow (\exists v_1(\phi))). \tag{1.7f}$$

Moreover,

$$v_i \neq v_j \quad \text{is short for} \quad \neg(v_i = v_j); \tag{1.7g}$$

$$v_i \notin v_j \quad \text{is short for} \quad \neg(v_i \in v_j); \tag{1.7h}$$

$$v_i \subseteq v_j \quad \text{is short for} \quad \underset{x}{\forall} \Big(x \in v_i \;\Rightarrow\; x \in v_j \Big). \tag{1.7i}$$

Definition and Remark 1.13. We say that a variable v_j, occurring in a set-theoretic formula is *bound* by a quantifier or in the *scope* of a quantifier if, and only if, it occurs directly behind an existential quantifier (i.e. in the form $\exists v_j(\phi)$, cf. Def. 1.9(b)(v)) or directly behind a universal quantifier (i.e. in the form $\forall v_j(\phi)$, cf. (1.7d)); otherwise, we call the variable v_j *free*. Bound variables are sometimes also called *dummy* variables, since, if the bound variable v_j in, say, $\exists v_j(\phi)$ is replaced by v_k (and v_k is not free in ϕ, cf. Ex. 1.14(e) below), then $\exists v_j(\phi)$ and the formula with v_j replaced by v_k are equivalent. Thus, if one uses the transcriptions introduced in (1.7e) and (1.7f), then the bound variables are precisely those, occurring *under* a quantifier. In principle, it is not forbidden for the same variable (more precisely, the same variable *symbol*) to occur as both a free variable and a bound variable in the same formula, and it could also occur in the scope of several different quantifiers[4]. However, using the same variable symbol both free and bound and/or within several scopes tends to make formulas less readable and it can, actually, always be avoided, using additional variable symbols, see Ex. 1.14(c)-(e) below. One might already have encountered the analogous situation when writing integrals: For example, consider $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined by the formula

$$f(x) := x + \int_0^1 x \,\mathrm{d}x + \int_x^1 \left(x + \int_0^x \exp(x)\,\mathrm{d}x \right) \mathrm{d}x\,. \tag{1.8a}$$

In (1.8a), the variable x occurs as a bound variable with three different scopes (within the scope of each of the three integrals, x is used as the respective integrand's dummy variable) and also as a "free" variable (not bound by any integral), namely as the function argument of f. Successively replacing each bound version of x, starting with the innermost integral, one can write (1.8a) in the equivalent (and more readable) form

$$f(x) := x + \int_0^1 u \,\mathrm{d}u + \int_x^1 \left(z + \int_0^z \exp(y)\,\mathrm{d}y \right) \mathrm{d}z\,. \tag{1.8b}$$

Example 1.14. **(a)** $x \in y$ has x and y as free variables and no bound variables. It states that the set x is an element of the set y.

(b) $\exists_x (x \in y)$ has x bound and y free. It states that there exists a set x that is an element of the set y (i.e. that y is not the empty set).

(c) In the formula

$$\forall_y \Big(y \in x \;\Rightarrow\; \exists_x (x \in y) \Big),$$

y is bound, whereas x occurs both free and bound. If one replaces the bound version of x by z, then one obtains the equivalent formula

$$\forall_y \Big(y \in x \;\Rightarrow\; \exists_z (z \in y) \Big).$$

The formulas state that, if the set y is an element of the set x, then y contains an element z – in other words, the set x does not contain the empty set.

(d) The formula

$$\exists_x \forall_x (x \in x)$$

[4]Using the same variable symbol in such a way is similar to using the same variable name for different *local* variables when coding computer programs.

contains x as bound variables within two different scopes. Replacing the version of x in the scope of the all quantor by y yields the equivalent formula

$$\underset{x}{\exists} \underset{y}{\forall} (y \in y).$$

It is somewhat peculiar, as it has the form $\underset{x}{\exists} \phi$, where x does not occur as a free variable in ϕ. According to the interpretation given by Rem. 1.11, the formula is true if, and only if, there exists a set such that ϕ is true, i.e. if, and only if, the considered universe of sets in nonempty and every set in the universe contains itself.

(e) As stated in Def. and Rem. 1.13, dummy (i.e. bound) variables may be replaced by other symbols without changing the meaning of the formula, however, if replacing x in, say, $\underset{x}{\exists} (\phi)$, then one has to make sure that the replacement does not occur as a free variable in ϕ: For instance, in $\underset{x}{\exists} (x \in y)$ of (b), one can replace x with every variable symbol, except y: While $\underset{x}{\exists} (x \in y)$ states that the set y is not empty, the formula $\underset{y}{\exists} (y \in y)$ states the existence of a set that contains itself.

Remark 1.15. In Def. and Rem. 1.8, we defined a proof of statement B from statement A_1 as a finite sequence of statements $A_1, A_2, \dots, A_n$ such that, for $1 \leq i < n$, A_i implies A_{i+1}, and A_n implies B. In the field of *proof theory*, which, similar to mathematical logic, is a large field in its own right and a detailed treatment is beyond the scope of this class, proofs are formalized via a finite set of rules that can be applied to (set-theoretic) formulas (see, e.g., [EFT07, Sec. IV], [Kun12, Sec. II]). Once proofs have been formalized in this way, one can, in principle, *mechanically* check if a given sequence of symbols does, indeed, constitute a valid proof (without even having to understand the actual *meaning* of the statements). Indeed, several different computer programs have been devised that can be used for automatic proof checking, for example *Coq* [Wik22b], *HOL Light* [Wik21], *Isabelle* [Wik22c] and *Lean* [Wik22d] to name just a few.

1.4 Scope of ZFC: Zermelo-Fraenkel Set Theory Plus the Axiom of Choice

Axiomatic set theory seems to provide a solid and consistent foundation for conducting mathematics, and most mathematicians have accepted it as the basis of their everyday work. However, there do remain some deep, difficult, and subtle *philosophical issues* regarding the foundation of logic and mathematics (see, e.g., [Kun12, Sec. 0, Sec. III]).

Definition and Remark 1.16. An *axiom* is a statement that is assumed to be true without any formal logical justification. The most basic axioms (for example, the standard axioms of set theory) are taken to be justified by common sense or some underlying philosophy. However, on a less fundamental (and less philosophical) level, it is a common mathematical strategy to state a number of axioms (for example, the axioms defining the mathematical structure called a *group*), and then to study the logical consequences of these axioms (for example, *group theory* studies the statements that are true for all groups as a consequence of the group axioms). For a given system of axioms, the question if there exists an object satisfying all the axioms in the system (i.e. if the system of axioms is *consistent*, i.e. free of contradictions) can be extremely difficult (or even impossible) to answer.

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We are now in a position to formulate and discuss the axioms of axiomatic set theory. More precisely, we will present the axioms of *Zermelo-Fraenkel* set theory, usually abbreviated as ZF, which are Axiom 0 – Axiom 8 below. While there exist various set theories in the literature, each set theory defined by some collection of axioms, the axioms of ZFC, consisting of the axioms of ZF plus the axiom of choice (Axiom 9, see Sec. 7 below), are used as the foundation of mathematics currently accepted by most mathematicians.

2 The Most Basic Axioms

2.1 Existence, Extensionality

Axiom 0 *Existence:*

$$\underset{X}{\exists} \quad (X = X).$$

Recall that this is just meant to be a more readable transcription of the set-theoretic formula $\exists v_1(v_1 = v_1)$. The axiom of existence states that there exists (at least one) set X.

—

In naive set theory, based on Cantor's definition as described in Sec. 1.1, sets X and Y are defined to be equal if, and only if, they contain precisely the same elements. In axiomatic set theory, this is guaranteed by the axiom of extensionality:

Axiom 1 *Extensionality:*

$$\underset{X}{\forall} \; \underset{Y}{\forall} \quad \Big(\Big(\underset{z}{\forall} \, (z \in X \;\Leftrightarrow\; z \in Y) \Big) \;\Rightarrow\; X = Y \Big).$$

—

Following [Kun12], we assume that the substitution property of equality is part of the underlying logic, i.e. if $X = Y$, then X can be substituted for Y and vice versa without changing the truth value of a (set-theoretic) formula. In particular, this yields the converse to extensionality:

$$\underset{X}{\forall} \ \underset{Y}{\forall} \ \Big(X = Y \ \Rightarrow \ \Big(\underset{z}{\forall} \, (z \in X \ \Leftrightarrow \ z \in Y) \Big) \Big).$$

Before we discuss further consequences of extensionality, we would like to have the existence of the empty set. However, Axioms 0 and 1 do not suffice to prove the existence of an empty set as we will see in Ex. 2.2 below. We will take the opportunity to discuss, at an early stage, the idea of proving independence results via suitable models in the following section.

2.2 Models, Independence Results

One is often interested in proving the *independence* of an axiom A from a collection $\mathcal{C}$ of other axioms, which one does by providing one *model* of set theory in which all axioms in $\mathcal{C}$ hold as well as A, and a second model of set theory in which all axioms in $\mathcal{C}$ hold, but A fails. In Def. 2.1 below, we provide ten simple "toy models" (the first seven are the ones introduced in [Kun12, Sec. I.2]). Subsequently, we will check which of our axioms are satisfied by which model, providing a number of simple independence results.

Definition 2.1 (Toy Models)**.** Let a, b, c, d, e be distinct elements. For each index i in $\{1, 2, \dots, 10\}$, we define the model $M_i := (D_i, E_i)$, where M_i is the pair consisting of the "domain" D_i and a relation E_i on D_i (i.e. $E_i \subseteq D_i \times D_i$), where one thinks of D_i as modeling the universe of sets and of E_i as modeling the element relation $\in$ (one might be concerned that the construction of these models is not justified by the axioms that have, thus far, been introduced – this is a fair concern and we will address it further in

Rem. 2.3 below):

$$
\begin{aligned}
M_1 &:= (D_1, E_1), & D_1 &:= \{a\}, & E_1 &:= \emptyset,\\
M_2 &:= (D_2, E_2), & D_2 &:= \{a\}, & E_2 &:= \{(a,a)\},\\
M_3 &:= (D_3, E_3), & D_3 &:= \{a,b\}, & E_3 &:= \{(a,b),(b,a)\},\\
M_4 &:= (D_4, E_4), & D_4 &:= \{a,b,c\}, & E_4 &:= \{(a,b),(b,a),(a,c),(b,c)\},\\
M_5 &:= (D_5, E_5), & D_5 &:= \{a,b,c\}, & E_5 &:= \{(a,b),(a,c)\},\\
M_6 &:= (D_6, E_6), & D_6 &:= \{a,b,c,d\}, & E_6 &:= \{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\},\\
M_7 &:= (D_7, E_7), & D_7 &:= \{a,b,c\}, & E_7 &:= \{(a,b),(b,c)\},\\
M_8 &:= (D_8, E_8), & D_8 &:= \{a,b,c\}, & E_8 &:= \{(b,c)\},\\
M_9 &:= (D_9, E_9), & D_9 &:= \{a,b,c,d,e\}, & E_9 &:= \{(a,b),(b,c),(c,d),(b,e),(c,e)\},\\
M_{10} &:= (D_{10}, E_{10}), & D_{10} &:= \{a,b\}, & E_{10} &:= \{(a,b),(b,b)\}.
\end{aligned}
$$

Example 2.2. For each toy model M_i of Def. 2.1, we will check if it satisfies Axiom 0 (i.e. existence of a set), Axiom 1 (i.e. extensionality), and the (non-)existence of an empty set (c.f. (2.1) below). We will see that Axioms 0 and 1 are independent from each other and that Axioms 0 and 1 together neither imply nor refute the existence of an empty set.

(a) Axiom 0 holds in each of the above models M_i, $i \in \{1, \dots, 10\}$, since $D_i \neq \emptyset$ in each case.

(b) Axiom 1 holds in each M_i, $i \in \{1,2,3,4,6,7,9,10\}$, but is violated in M_5 and M_8:

Axiom 1 holds in M_i, $i \in \{1,2\}$, since D_i contains only 1 element.

Axiom 1 holds in M_3, since E_3 provides precisely the relations aE_3b and bE_3a, i.e., in this universe of sets, b has only a as an element and a has only b as an element – in particular, there are no distinct sets that contain precisely the same elements.

Axiom 1 is violated in M_5, since, according to E_5, both b and c contain precisely a as an element.

We leave M_4 and M_6 – M_{10} as an exercise.

(c) We check which of our toy models do not contain an "empty set", i.e. satisfy the "axiom"

$$\neg \left(\underset{X}{\exists} \ \underset{x}{\forall} \quad x \notin X \right) : \tag{2.1}$$

(2.1) holds in M_2, M_3, M_4, whereas $M_1, M_5, \dots, M_{10}$ do have an "empty set": In M_2, a contains a; in M_3, a contains b and b contains a; M_4 is an exercise; in each of the models $M_5, \dots, M_{10}$, a does not contain any elements (M_8 even has a second empty set, namely b).

From (a) – (c), we see, in particular, that M_2, M_3, M_4 satisfy Axioms 0, 1, plus (2.1); whereas M_1, M_6, M_7, M_9, M_{10} satisfy Axioms 0, 1, plus the existence of an "empty set".

Remark 2.3. Using models of set theory to prove independence results, as we have just done in Def. 2.1 and Ex. 2.2 is subject to some logical subtleties: The validity of such arguments relies on the admissibility of constructing the respective models: For example, one can obtain all the models of Def. 2.1, if one is allowed to form sets with up to 5 distinct elements, one is allowed to form ordered pairs from these elements, and one is also allowed to form sets, containing the obtained ordered pairs as elements (of course, each individual model can be obtained with weaker construction rules).

2.3 Comprehension

To obtain, among many other things, the existence of the empty set, we introduce the additional axiom of comprehension. More precisely, in the case of comprehension, we do not have a single axiom, but a scheme of infinitely many axioms, one for each set-theoretic formula that satisfies a certain condition. Its formulation makes use of the following definition:

Definition 2.4. One obtains the *universal closure* of a set-theoretic formula ϕ, by writing $\underset{v_j}{\forall}$ in front of ϕ for each variable v_j that occurs as a free variable in ϕ (recall from Def. and Rem. 1.13 that v_j is free in ϕ if, and only if, it is not bound by a quantifier in ϕ). While, if ϕ contains more than one free variable, the universal closure of ϕ is nonunique (as one can choose an arbitrary order of the $\underset{v_j}{\forall}$ in front of ϕ), this does not cause a problem, since all universal closures of ϕ are equivalent.

Axiom 2 *Comprehension Scheme:* For each set-theoretic formula ϕ, not containing Y as a free variable, the universal closure of

$$\underset{Y}{\exists} \ \underset{x}{\forall} \ \Big(x \in Y \ \Leftrightarrow \ (x \in X \wedge \phi)\Big) \tag{2.2}$$

is an axiom. Thus, the comprehension scheme states that, given the set X, there exists (at least one) set Y, containing precisely the elements of X that have the property ϕ (the importance of allowing ϕ in (2.2) to have free variables will be illustrated in Ex. 2.10 below, where Ex. 2.10(e) will also show, why Y must *not* be free in ϕ).

Lemma 2.5. *Axioms* 0 *and* 2 *(i.e. the existence of a set together with the comprehension scheme) imply the existence of (at least one) empty set, i.e. the validity of*

$$\underset{Y}{\exists} \ \underset{x}{\forall} \ x \notin Y. \tag{2.3}$$

Proof. According to Axiom 0, there exists a set X. Letting ϕ denote the set-theoretic formula $x \neq x$, Axiom 2 yields

$$\underset{Y}{\exists} \ \underset{x}{\forall} \ \Big(x \in Y \ \Leftrightarrow \ (x \in X \wedge x \neq x) \Big).$$

Since, for each x, the statement $x \in X \wedge x \neq x$ is false, $x \in Y$ must be false for each x as well, thereby proving (2.3). ■

Example 2.6. We check which of our toy models $M_1, \dots, M_{10}$ of Def. 2.1 satisfy Axiom 2 (i.e. the comprehension scheme):

We begin with some general considerations that will be useful for several of the models:

Claim 1: If X in (2.2) is an empty set, then (2.2) holds with $Y := X$: Indeed, both $x \in Y$ and $x \in X \wedge \phi$ are then false for each x and ϕ.

Claim 2: If the domain D_i contains elements A, B, C (not necessarily distinct), where A is empty and C contains precisely one element, namely B, then (2.2) holds for $X := C$: Indeed, there are four possible cases to check: (i) ϕ does not contain x as a free variable and ϕ is true (independently of x) – then (2.2) holds with $Y := C$ (since $x \in C \ \Leftrightarrow \ (x \in C \wedge \phi)$); (ii) ϕ does not contain x as a free variable and ϕ is false (independently of x) – then (2.2) holds with $Y := A$; (iii) ϕ does contain x as a free variable and ϕ is true for $x = B$ – then (2.2) holds with $Y := C$ (since $x \in C \ \Leftrightarrow \ (x \in C \wedge \phi)$, both sides being true for $x = B$, both sides being false for $x \neq B$); (iv) ϕ does contain x as a free variable and ϕ is false for $x = B$ – then (2.2) holds with $Y := A$ (since both sides of $x \in A \ \Leftrightarrow \ (x \in C \wedge \phi)$ are false for each x).

Axiom 2 holds in M_1: Since D_1 contains only one set, namely a, which is empty (according to E_1), (2.2) is true for $Y := a$ by Claim 1. In combination with Ex. 2.2, we see that M_1 satisfies all Axioms 0 – 2. As it also satisfies

$$\underset{x}{\forall} \left(\neg \underset{y}{\exists} \, (y \in x) \right),$$

M_1 shows that Axioms 0 – 2 do not suffice to prove the existence of nonempty sets.

Axiom 2 does not hold in M_2, M_3, M_4: We know from Ex. 2.2(a),(c) that these models satisfy Axiom 0, but violate (2.3). Thus, Lem. 2.5 yields that Axiom 2 does not hold.

Axiom 2 holds in M_5: If $X := a$, then, since a is an empty set, (2.2) holds with $Y := a$ by Claim 1. If $X := b$, then, since b contains precisely a, (2.2) holds by Claim 2 (using $A := B := a$, $C := b$). If $X := c$, then, since c contains precisely a, (2.2) holds again by Claim 2 (using $A := B := a$, $C := c$).

It is an exercise to show that Axiom 2 holds in M_7, M_8, M_9, but fails in M_6 and M_{10}.

We summarize the toy models' properties we found so far in the following table:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T
$\neg$(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F

In particular, comparing the corresponding rows in the table above, we find Axioms 1 and 2 to be mutually independent.

Remark 2.7. Comprehension alone does not provide uniqueness (for instance, we found in Ex. 2.6 that model M_8 satisfies comprehension, even though it has two distinct empty sets). However, if one also assumes Axiom 1 (extensionality) and if both Y and Y' are sets containing precisely the elements of X that have the property ϕ, then

$$\underset{x}{\forall} \Big(x \in Y \;\Leftrightarrow\; (x \in X \wedge \phi) \;\Leftrightarrow\; x \in Y' \Big),$$

and extensionality implies $Y = Y'$. Thus, due to extensionality, the set Y given by comprehension is unique, justifying the common notation

$$\{x : x \in X \wedge \phi\} := \{x \in X : \phi\} := Y. \tag{2.4}$$

Theorem 2.8. *Assuming Axioms* 0 – 2*, there exists a unique empty set (which we denote by* $\emptyset$ *or by* 0 *– it is common to identify the empty set with the number zero in axiomatic set theory).*

Proof. Axiom 0 provides the existence of a set X. Then comprehension allows us to define the empty set by

$$0 := \emptyset := \{x \in X : x \neq x\},$$

where, as explained in Rem. 2.7, extensionality guarantees uniqueness. ■

Remark 2.9. In Rem. 1.12 we said that every formula with additional symbols and notation is to be regarded as an abbreviation or transcription of a set-theoretic formula as defined in Def. 1.9(b). Thus, formulas containing symbols for defined sets (e.g. 0 or $\emptyset$ for the empty set) are to be regarded as abbreviations for formulas without such symbols. Some logical subtleties arise from the fact that there is some ambiguity in the way such abbreviations can be resolved: For example, $0 \in X$ might abbreviate

$$\psi : \underset{y}{\exists} \Big(\phi(y) \wedge y \in X\Big) \quad \text{or} \quad \chi : \underset{y}{\forall} \Big(\phi(y) \Rightarrow y \in X\Big), \quad \text{where } \phi(y) \text{ stands for } \underset{v}{\forall} (v \notin y).$$

Then ψ and χ are equivalent if

$$\underset{y}{\exists} \quad \Big(\phi(y) \wedge \underset{z}{\forall} \Big(\phi(z) \Rightarrow y = z\Big)\Big)$$

(e.g., if Axioms 0 – 2 hold), but they can be nonequivalent, otherwise: For example, in model M_8 of Def. 2.1, consider ψ and χ with $X := c$. In M_8, $\phi(y)$ is true for $y := a$ and $y := b$. Thus, ψ is true in M_8 (since $(b, c) \in E_8$), but χ is false in M_8 (since $(a, c) \notin E_8$). To avoid introducing logical ambiguities, we will only use formulas with symbols for defined sets under the assumption of extensionality.

—

At first glance, the role played by the free variables in ϕ, which are allowed to occur in Axiom 2, might seem a bit obscure. So let us consider examples to illustrate that allowing free variables (i.e. set parameters) in comprehension is quite natural:

Example 2.10. In view of Rem. 2.9, assume Axiom 1 (extensionality).

(a) If ϕ in (2.2) is the formula $x \in Z$ (having x, Z as free variables), then the set given by the resulting axiom yields precisely the intersection of X and Z:

$$X \cap Z := \{x \in X : \phi\} = \{x \in X : x \in Z\}.$$

(b) While (a) shows how Axiom 2 provides the intersection of *two* sets, with a modification, Axiom 2 also yields the existence of intersections of more than two sets (of both finitely and even infinitely many): If $\mathcal{M}$ is a nonempty set, $X \in \mathcal{M}$, and ϕ in (2.2) is the formula $\underset{M \in \mathcal{M}}{\forall} x \in M$ (having $x, \mathcal{M}$ as free variables), then the set given by the resulting axiom yields precisely the intersection of all sets that are elements of $\mathcal{M}$:

$$\bigcap \mathcal{M} := \bigcap_{M \in \mathcal{M}} M := \left\{x : \underset{M \in \mathcal{M}}{\forall} x \in M\right\} := \left\{x \in X : \underset{M \in \mathcal{M}}{\forall} x \in M\right\}. \tag{2.5}$$

It is also customary (and useful) to define intersections

$$\bigcap_{i \in I} M_i := \left\{x : \underset{i \in I}{\forall} x \in M_i\right\} := \left\{x \in M_{i_0} : \underset{i \in I}{\forall} x \in M_i\right\}, \tag{2.6}$$

where $I \neq \emptyset$ is a nonempty index set, $i_0 \in I \neq \emptyset$ is an arbitrary fixed element of I, and $(M_i)_{i \in I}$ is a so-called *family* of sets. However, conceptually (2.6) is significantly more involved than (2.5) and not justifiable from the axioms considered so far:

Formally, the family $(M_i)_{i\in I}$ is a *function* $f : I \longrightarrow \mathcal{N}$, $M_i := f(i)$ for each $i \in I$, where the existence of functions will be justified, once we have Axiom 3 (pairing, Sec. 2.5), Axiom 4 (union, Sec. 2.6), and Axiom 5 (replacement, Sec. 3.1). The definitions in (2.5) and (2.6) will be equivalent (in the sense that $\bigcap \mathcal{M} = \bigcap_{i\in I} M_i$), if we are allowed to form the set

$$\mathcal{M} := \{M_i : i \in I\}$$

(if I is a set and M_i is a set for each $i \in I$, then $\mathcal{M}$ as above will be a set by Axiom 5). It is emphasized that the sets $\mathcal{M}$ and I in (2.5) and (2.6), respectively, were required to be nonempty. If one tries to form

$$\bigcap \emptyset = \left\{x : \underset{X\in\emptyset}{\forall} x \in X\right\} = \left\{x : \underset{i\in\emptyset}{\forall} x \in M_i\right\} = \bigcap_{i\in\emptyset} M_i,$$

then one obtains the so-called *universal class* of all sets $\mathbf{V}$, which is *not* a set (cf. Sec. 2.4 below, in particular Ex. 2.13(b)).

(c) Suppose ϕ in (2.2) is the formula $x \notin Z$ (again having x, Z as free variables), then the set given by the resulting axiom yields precisely the difference X minus Z:

$$X \setminus Z := \{x \in X : \phi\} = \{x \in X : x \notin Z\}.$$

(d) Note that it is even allowed for ϕ in (2.2) to have X as a free variable, so one can let ϕ be the formula $\underset{u}{\exists}\,(x \in u \wedge u \in X)$ to define the set

$$X^* := \left\{x \in X : \underset{u}{\exists}\,(x \in u \wedge u \in X)\right\}.$$

Then, if $0 := \emptyset$, $1 := \{0\}$, $2 := \{0, 1\}$, we obtain

$$2^* = \{0\} = 1.$$

(e) It is essential that ϕ in (2.2) must not contain Y as a free variable. Otherwise, one would have a contradiction as soon as there exists any nonempty set: Suppose ϕ in (2.2) were allowed to be the formula $x \notin Y$. Then, if X is nonempty, i.e. there exists $x \in X$, (2.2) required the existence of a set Y such that $x \in Y \Leftrightarrow x \notin Y$.

Example 2.11. Another example of extensionality consequences is the important result that the mathematical universe consists of sets and only of sets: Suppose there were other objects in the mathematical universe, for example a cow C and a monkey M (or any other object without elements, other than the empty set) – this would be equivalent to allowing a cow or a monkey (or any other object without elements, other than the empty set) to be considered a set, which would mean that our set-theoretic variables v_j were allowed to be a cow or a monkey as well. However, extensionality then implies the false statement $C = M = \emptyset$, thereby excluding cows and monkeys from the mathematical universe. Similarly, $\{C\}$ and $\{M\}$ (or any other object that contains a non-set), can not be inside the mathematical universe. Indeed, otherwise we had

$$\underset{x}{\forall} \Big(x \in \{C\} \Leftrightarrow x \in \{M\} \Big)$$

(as C and M are non-sets) and, by extensionality, $\{C\} = \{M\}$ were true, in contradiction to a set with a cow inside not being the same as a set with a monkey inside. Thus, we see that all objects of the mathematical universe must be so-called *hereditary sets*, i.e. sets all of whose elements (thinking of the elements as being the children of the sets) are also sets.

2.4 Classes

As we need to avoid contradictions such as Russell's antinomy, we must not require the existence of a set $\{x : \phi\}$ for each set-theoretic formula ϕ. However, it can still be useful to think of a "collection" of all sets having the property ϕ. Such collections are commonly called *classes*:

Definition 2.12. **(a)** If ϕ is a set-theoretic formula, then we call $\{x : \phi\}$ a *class*, namely the class of all sets that have the property ϕ (typically, ϕ will have x as a free variable).

(b) If ϕ is a set-theoretic formula, then we say the class $\{x : \phi\}$ *exists* (as a set) if, and only if

$$\underset{X}{\exists} \Big(\underset{x}{\forall} \Big(x \in X \Leftrightarrow \phi \Big) \Big) \tag{2.7}$$

is true. Assuming Axiom 1 (extensionality), X is then actually unique and we identify X with the class $\{x : \phi\}$. If (2.7) is false, then $\{x : \phi\}$ is called a *proper class* (and the usual interpretation is that the class is in some sense "too large" to be a set).

Example 2.13. **(a)** Due to Russell's antinomy of Sec. 1.1, we know that

$$\mathbf{R} := \{x : x \notin x\}$$

forms a proper class.

(b) The *universal class* of all sets, $\mathbf{V} := \{x : x = x\}$, is a proper class. Once again, this is related to Russell's antinomy: If $\mathbf{V}$ were a set, then

$$\mathbf{R} = \{x : x \notin x\} = \{x : x = x \wedge x \notin x\} = \{x : x \in \mathbf{V} \wedge x \notin x\}$$

would also be a set by comprehension. However, this is in contradiction to $\mathbf{R}$ being a proper class by (a).

Remark 2.14. From the perspective of formal logic, statements involving proper classes are to be regarded as abbreviations for statements without proper classes. For example, it turns out that the class $\mathbf{G}$ of all sets forming a group is a proper class. But we might write $G \in \mathbf{G}$ as an abbreviation for the statement "The set G is a group."

2.5 Pairing

As we saw from our investigation of model M_1 in Ex. 2.6, Axioms 0 – 2 are still consistent with the empty set being the only set in existence. The next axiom will provide the existence of nonempty sets:

Axiom 3 *Pairing:*

$$\underset{x}{\forall}\,\underset{y}{\forall}\,\underset{Z}{\exists}\,(x \in Z \ \wedge \ y \in Z). \tag{2.8}$$

Thus, the pairing axiom states that, for all sets x and y, there exists a set Z that contains x and y as elements.

—

In consequence of the pairing axiom, the sets

$$0 := \emptyset, \tag{2.9a}$$

$$1 := \{0\}, \tag{2.9b}$$

$$2 := \{0, 1\} \tag{2.9c}$$

all exist. More generally, we may define:

Definition 2.15. Assume Axioms 0 – 3. If x, y are sets and Z is given by the pairing axiom, then we call

(a) $\{x, y\} := \{u \in Z : u = x \vee u = y\}$ the *unordered pair* given by x and y,

(b) $\{x\} := \{x, x\}$ the *singleton set* given by x,

(c) $(x,y) := \{\{x\},\{x,y\}\}$ the *ordered pair* given by x and y.

—

We can now show that ordered pairs behave as expected:

Lemma 2.16. *Assuming Axioms* 0 – 3*, the following holds true:*

$$\underset{x,y,x',y'}{\forall} \Big((x,y) = (x',y') \;\Leftrightarrow\; (x = x') \wedge (y = y') \Big). \tag{2.10}$$

Proof. "⇐" is merely

$$(x,y) = \{\{x\},\{x,y\}\} \overset{x=x',\,y=y'}{=} \{\{x'\},\{x',y'\}\} = (x',y').$$

"⇒" is done by distinguishing two cases: If $x = y$, then

$$\{\{x\}\} = (x,y) = (x',y') = \{\{x'\},\{x',y'\}\}.$$

Next, by extensionality, we first get $\{x\} = \{x'\} = \{x',y'\}$, followed by $x = x' = y'$, establishing the case. If $x \neq y$, then

$$\{\{x\},\{x,y\}\} = (x,y) = (x',y') = \{\{x'\},\{x',y'\}\},$$

where, by extensionality $\{x\} \neq \{x,y\} \neq \{x'\}$. Thus, using extensionality again, $\{x\} = \{x'\}$ and $x = x'$. Next, we conclude

$$\{x,y\} = \{x',y'\} = \{x,y'\}$$

and a last application of extensionality yields $y = y'$. ■

Remark 2.17. Assume Axioms 0 – 3.

(a) We now have the existence of the infinitely many different sets $0, \{0\}, \{\{0\}\}, \dots$. In particular, none of our finite toy models $M_1, \dots, M_{10}$ from Def. 2.1 can satisfy Axioms 0 – 3. While we will need the axiom of infinity of Sec. 4.1 below to formally define the notions *finite* and *infinite*, in Ex. 2.18 below, we will see that only M_2 and M_{10} satisfy pairing (and we know from Ex. 2.6 that M_2 and M_{10} do not satisfy comprehension). However, Axioms 0 – 3 do not, yet, suffice to prove the existence of sets with more than two elements (cf. Ex. 4.32; one needs recursion to construct a suitable model).

(b) At this stage, it would already be possible to introduce the notion of a *relation* by calling a set a *relation* if, and only if, all its elements are ordered pairs. However, without further axioms, this becomes cumbersome, one can not, actually, construct many interesting relations anyway, and certain definitions (such as *domain, image, function*) would depend on the particular definition of $(x,y) := \{\{x\},\{x,y\}\}$ in Def. 2.15(c), rather than merely on the key property (2.10) of ordered pairs (cf. [Kun12, Sec. I.7.1]). Thus, we postpone the definition and consideration of relations and functions to Sec. 3.2, where we can use the axioms of union and replacement to justify the existence of Cartesian products, then giving rise to relations and functions in the usual way.

(c) Once one has ordered pairs, one can proceed to define more general ordered tuples by letting

$$\begin{aligned}
(v_1) &:= v_1,\\
(v_1,v_2) &:= \big((v_1),v_2\big) := \big\{\{v_1\},\{v_1,v_2\}\big\} \quad \text{(ordered pair, same as Def. 2.15(c))},\\
(v_1,v_2,v_3) &:= \big((v_1,v_2),v_3\big) \quad \text{(ordered triple)},\\
(v_1,v_2,v_3,v_4) &:= \big((v_1,v_2,v_3),v_4\big) \quad \text{(ordered quadruple)},\\
&\dots
\end{aligned}$$

where $v_1, v_2, \dots$ are arbitrary sets. While this is less elegant than the usual definition of ordered n-tuples $(v_1,\dots,v_n)$ as the function $v : \{1,\dots,n\} \longrightarrow \{v_1,\dots,v_n\}$, $v_i := v(i)$, it has the advantage of not needing any further axioms. Once we have sufficiently many axioms to justify definition via recursion and proof via induction, we can show both definitions of ordered n-tuples to be equivalent (cf. Ex. 4.31).

Example 2.18. We check which of our toy models $M_1,\dots,M_{10}$ of Def. 2.1 satisfy Axiom 3 (pairing): Axiom 3 holds only in M_2 and M_{10}, and is violated in all the remaining models: Axiom 3 holds in M_2, since a is the only set in the model and a is an element of a. Axiom 3 does not hold in $M_1, M_3, \dots, M_8$: In M_1, there is no set containing a; in M_3, there is no set containing both a and b; in M_4, M_5, M_7, M_8, there is no set containing c; and in M_6 and M_9, there is no set containing d. Axiom 3 holds in M_{10}, since a and b are the only sets in the model and b contains both a and b. We summarize the toy models' properties we found so far in the following table:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T
$\neg$(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F
Axiom 3 (Pairing)	F	T	F	F	F	F	F	F	F	T

2.6 Union

To be able to construct sets with more than two elements, we introduce the following axiom:

Axiom 4 *Union:*

$$\underset{\mathcal{M}}{\forall}\ \underset{Y}{\exists}\ \underset{x}{\forall}\ \underset{X}{\forall} \Big((x \in X \ \wedge\ X \in \mathcal{M}) \ \Rightarrow\ x \in Y \Big). \tag{2.11}$$

Thus, the union axiom states that, for each set of sets $\mathcal{M}$, there exists a set Y containing all elements of elements of $\mathcal{M}$.

Definition 2.19. (a) If $\mathcal{M}$ is a set and Y is given by the union axiom, then define

$$\bigcup \mathcal{M} := \bigcup_{X \in \mathcal{M}} X := \Big\{ x \in Y : \underset{X \in \mathcal{M}}{\exists} \quad x \in X \Big\}. \tag{2.12}$$

(b) If X and Y are sets, then define

$$X \cup Y := \bigcup \{X, Y\}.$$

(c) If x, y, z are sets, then define

$$\{x, y, z\} := \{x, y\} \cup \{z\}.$$

Remark 2.20. (a) Analogous to (2.6) for intersections, once one has a family of sets $(M_i)_{i \in I}$, it is also useful to define set-theoretic unions as

$$\bigcup_{i \in I} M_i := \Big\{ x : \underset{i \in I}{\exists}\ x \in M_i \Big\}. \tag{2.13}$$

Analogous to the remark in Ex. 2.10(b), the definitions in (2.12) and (2.13) will be equivalent (in the sense that $\bigcup \mathcal{M} = \bigcup_{i \in I} M_i$), if we are allowed to form the set $\mathcal{M} := \{M_i : i \in I\}$.

(b) The union

$$\bigcup \emptyset = \bigcup_{X \in \emptyset} X = \bigcup_{i \in \emptyset} M_i = \emptyset$$

is the empty set – in particular, a set (this is in contrast to the situation for intersections, where $\bigcap \emptyset = \mathbf{V}$, which is a proper class and not a set, cf. Ex. 2.10(b)).

Definition 2.21. For each set x, we define its *successor* to be the set $x \cup \{x\}$. While we will define functions between *sets* in the usual way in Sec. 3.2 below, it can already be useful to think of the *successor function* as a *class function*

$$\mathbf{S} : \mathbf{V} \longrightarrow \mathbf{V}, \quad \mathbf{S}(x) := x \cup \{x\}$$

(clearly, $\mathbf{S}$ will not be a function between sets, since it is defined for each set x, that means it is defined on the proper class $\mathbf{V}$ – however each restriction to a set V will be a set function in the usual sense). Recalling (2.9), we have $1 = \mathbf{S}(0)$, $2 = \mathbf{S}(1)$; and we can define $3 := \mathbf{S}(2)$, ...

Example 2.22. We check which of our toy models $M_1, \dots, M_{10}$ of Def. 2.1 satisfy Axiom 4 (union): As it turns out, Axiom 4 holds in each M_i, except for $i = 9$:

Axiom 4 holds in M_1, since a is the only set in D_1 and a is empty. Axiom 4 holds in M_3: If $\mathcal{M} := a$ in (2.11), then the only possibility (due to E_3) is $X = b$ and, thus, $x = a$, implying (2.11) to hold with $Y = b$ (since $(a, b) \in E_3$). Switching the roles of a and b shows (2.11) to hold with $Y = a$ for $\mathcal{M} := b$. Axiom 4 holds in M_5: For $\mathcal{M} := a$, (2.11) is trivially true (with arbitrary $Y \in D_5$), since a is empty; for $\mathcal{M} := b$ or $\mathcal{M} := c$, (2.11) still holds with arbitrary Y, since, in both cases, $X = a$ and a is empty.

We leave it as an exercise to verify Axiom 4 also holds in M_2, M_4, M_6, M_7, M_8, M_{10}.

Axiom 4 does not hold in M_9: Consider (2.11) with $\mathcal{M} := e$. Since e contains b and c, b contains a, and c contains b, we would need an element Y of D_9 that contains both a and b. However, D_9 does not contain such an element.

We summarize the toy models' properties we found so far in the following table:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T
$\neg$(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F
Axiom 3 (Pairing)	F	T	F	F	F	F	F	F	F	T
Axiom 4 (Union)	T	T	T	T	T	T	T	T	F	T

3 Replacement

3.1 Replacement Scheme, Cartesian Products

As mentioned before, we desire to define relations and functions in the usual manner, making use of the Cartesian product $A \times B$ of two sets A and B, where $A \times B$ consists of all ordered pairs (x, y), where $x \in A$ and $y \in B$. However, Axioms 0 – 4 are not sufficient to justify the existence of Cartesian products. To obtain Cartesian products, we employ the following axiom of replacement. Analogous to the axiom of comprehension, the axiom of replacement actually consists of a scheme of infinitely many axioms, one for each set-theoretic formula. For the formulation of replacement, it is convenient to introduce another abbreviation:

Notation 3.1. If ϕ is a set-theoretic formula, then

$$\underset{y}{\exists!}\,\phi \quad \text{is short for} \quad \underset{y}{\exists}\Big(\phi(y) \wedge \underset{z}{\forall}\big(\phi(z) \Rightarrow y = z\big)\Big), \tag{3.1}$$

where the notation $\phi(y)$ and $\phi(z)$ is supposed to mean that, if y is free in ϕ, then this free y is replaced by z to obtain $\phi(z)$ from $\phi(y)$. Thus, $\underset{y}{\exists!}\,\phi$ holds if, and only if, there exists a *unique* set y with the property ϕ.

Axiom 5 *Replacement Scheme:* For each set-theoretic formula ϕ, not containing Y as a free variable, the universal closure of

$$\Big(\underset{x\in X}{\forall}\ \underset{y}{\exists!}\ \phi\Big) \quad \Rightarrow \quad \Big(\underset{Y}{\exists}\ \underset{x\in X}{\forall}\ \underset{y\in Y}{\exists}\ \phi\Big) \tag{3.2}$$

is an axiom. Thus, the replacement scheme states that if, for each $x \in X$, there exists a unique y having the property ϕ (where, in general, ϕ will depend on x), then there exists a set Y that, for each $x \in X$, contains this y with property ϕ. One can view this as obtaining Y by *replacing* each $x \in X$ by the corresponding $y = y(x)$.

Theorem 3.2. *Assuming Axioms* 0 – 5*, the following holds true: If A and B are sets, then the* Cartesian product *of A and B, i.e. the class*

$$A \times B := \Big\{x : \underset{a\in A}{\exists}\ \underset{b\in B}{\exists}\ x = (a, b)\Big\} \tag{3.3}$$

exists as a set.

Proof. For each $a \in A$, we can use replacement with $X := B$ and $\phi := \phi_a$ being the formula $y = (a, x)$ to obtain the existence of the set

$$\{a\} \times B := \{(a, x) : x \in B\} \tag{3.4a}$$

(in the usual way, comprehension and extensionality were used as well). Analogously, using replacement again with $X := A$ and ϕ being the formula $y = \{x\} \times B$, we obtain the existence of the set

$$\mathcal{M} := \{\{x\} \times B : x \in A\}. \tag{3.4b}$$

In a final step, the union axiom now shows

$$\bigcup \mathcal{M} = \bigcup_{a \in A} \{a\} \times B = A \times B \tag{3.4c}$$

to be a set as well. ■

Example 3.3. We check which of our toy models $M_1, \dots, M_{10}$ of Def. 2.1 satisfy Axiom 5 (replacement): We will see that Axiom 5 holds in M_1, M_2, M_3, M_{10}, but fails in $M_4, \dots, M_9$:

Axiom 5 holds in M_1: Since the only set in D_1 is the empty set a, $x \in X = a$ in (3.2) is false for each x, implying (3.2) to hold with $Y = a$.

Axiom 5 holds in M_2: Once again, $X = a$ is the only possibility in (3.2). Since a is the only element of D_1, each admissible ϕ must hold precisely for $y := a$, implying (3.2) to hold with $Y = a$.

Axiom 5 holds in M_3: The only possibilities in (3.2) are $X := a$ or $X := b$. In both cases, since a and b each have precisely one element, each admissible ϕ must either hold precisely for $y := a$ (in which case (3.2) holds with $Y = b$) or precisely for $y := b$ (in which case (3.2) holds with $Y = a$).

Axiom 5 does not hold in M_4: Consider (3.2) with $X := b$ and

$$\phi := \underset{u,v}{\exists} \Big(u \neq v \,\wedge\, u \in y \,\wedge\, v \in y \Big).$$

Then ϕ is admissible in (3.2) (since $y = c$ is the unique set in D_4 with precisely two elements). However, there does not exist a set $Y \in D_4$ such that $(c, Y) \in E_4$.

Models $M_5 - M_{10}$ are left as an exercise.

We summarize the toy models' properties we found so far in the following table[5]:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T
¬(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F
Axiom 3 (Pairing)	F	T	F	F	F	F	F	F	F	T
Axiom 4 (Union)	T	T	T	T	T	T	T	T	F	T
Axiom 5 (Replacement)	T	T	T	F	F	F	F	F	F	T

[5]In the literature, one sometimes finds the statement that the axiom of replacement plus the existence of an empty set implies the axiom of comprehension. Models M_3 and M_{10} show that with the axiom of replacement in the form (3.2), which is the version found, e.g., in [Kun12, Sec. I.2] and [Hal17, Ch. 3.7], this is *not* the case! The situation is different if the axiom of replacement requires that the set Y in (3.2) contains *precisely* those y with $\phi = \phi(x, y)$ true for some $x \in X$.

3.2 Relations and Functions

Now that we have the existence of Cartesian products according to Th. 3.2, we proceed to define relations in the usual way:

Definition 3.4. Assume Axioms 0 – 5. Given sets A and B, each subset R of $A \times B$ is called a *relation* over A and B (if $A = B$, then we call R a relation on A). If one wants to be completely precise, a relation is an ordered triple (A, B, R), where $R \subseteq A \times B$ (see Rem. 2.17(c) above for the definition of ordered triples). The set A is called the *domain* of R, denoted $\operatorname{dom}(R)$[6], B is called the *codomain* of R, denoted $\operatorname{codom}(R)$, and R is the relation's *graph* (here we commit the usual abuse of notation, referring to both the relation triple and relation's graph as R). One says that $a \in A$ and $b \in B$ are *related* according to the relation R if, and only if, $(a, b) \in R$. In this context, one often writes $a\,R\,b$ instead of $(a, b) \in R$.

Definition and Remark 3.5. Assume Axioms 0 – 5. Let A, B be sets and let $R \subseteq A \times B$ be a relation over A and B. If T is a subset of A, then call

$$R(T) := \left\{ b \in B : \underset{a \in T}{\exists} (a, b) \in R \right\}$$

the *image* of T under R; if U is a subset of B, then call

$$R^{-1}(U) := \left\{ a \in A : \underset{b \in U}{\exists} (a, b) \in R \right\}$$

the *preimage* or *inverse image* of U under R. Moreover, we call $R(A)$ the *image* of R and we call $R^{-1}(B)$ the *preimage*, *inverse image*, or *active domain* of R (cf. footnote to the definition of *domain* in Def. 3.4 above). To prove the existence of $R(T)$ and $R^{-1}(U)$ as sets, apply (3.2) with $X := R$ and

$$\phi := \underset{a}{\exists} \big(x = (a, y) \big)$$

to obtain Y to be a superset of $R(T)$ (and, then, $R(T)$ via comprehension), and with $X := R$ and

$$\phi := \underset{b}{\exists} \big(x = (y, b) \big)$$

to obtain Y to be a superset of $R^{-1}(U)$ (and, then, $R^{-1}(U)$ via comprehension).

Definition 3.6. Assume Axioms 0 – 5. Let A, B be sets and let $R \subseteq A \times B$ be a relation over A and B.

[6]As a caveat we note that the notion of *domain* varies in the literature – for example, [Kun12, Def. I.7.3] defines a relation's domain to be what we call its *preimage* or *active domain* according to Def. and Rem. 3.5.

(a) R is called *univalent* or *right-unique* or a *partial function* if, and only if,

$$\forall_{x\in A} \; \forall_{y_1,y_2\in B} \; \Big((x\,R\,y_1 \;\wedge\; x\,R\,y_2) \;\Rightarrow\; y_1 = y_2 \Big),$$

i.e. if, and only if, every element of A is related to at most one element of B.

(b) R is called *total* or *left-total* if, and only if,

$$\forall_{x\in A} \; \exists_{y\in B} \; (x\,R\,y),$$

i.e. if, and only if, in terms of Def. and Rem. 3.5, the active domain of R is all of A (i.e. $R^{-1}(B) = A$).

(c) R is called *injective* or *left-unique* if, and only if,

$$\forall_{x_1,x_2\in A} \; \forall_{y\in B} \; \Big((x_1\,R\,y \;\wedge\; x_2\,R\,y) \;\Rightarrow\; x_1 = x_2 \Big),$$

i.e. if, and only if, for every element y of B there exists at most one element of A that is related to y.

(d) R is called *one-to-one* if, and only if, it is an injective partial function.

(e) R is called *surjective* or *right-total* or *onto* if, and only if,

$$\forall_{y\in B} \; \exists_{x\in A} \; (x\,R\,y),$$

i.e. if, and only if, in terms of Def. and Rem. 3.5, the image of R is all of B (i.e. $R(A) = B$).

(f) R is called a *function* if, and only if, it is a total partial function. In this case, one usually writes $R:\ A \longrightarrow B$ and one introduces the usual notation

$$\forall_{x\in A} \; \forall_{y\in B} \; \Big(R(x) = y \quad :\Leftrightarrow \quad x \mapsto y \quad :\Leftrightarrow \quad xRy \Big)$$

(where the notation $x \mapsto y$ is only useful, if the function R is understood). One calls $x \mapsto R(x)$ the *assignment rule* of the function. Also note that, for functions, injective and surjective have their usual meanings, where, for functions, the notions injective and one-to-one coincide. Moreover, a function is called *bijective* if, and only if, it is both injective and surjective.

(g) If $A = B$, then R is called the *identity* on A if, and only if, $R: A \longrightarrow A$, $R(x) = x$. For the identity on R, one writes Id_A (or simply Id, if A is understood). Actually, the identity on A is the same as the *equality relation* "=" on A, sometimes also called the *diagonal* on A, denoted $\Delta(A)$. Thus, one has

$$\mathrm{Id}_A = \Delta(A) := \{(x,x) \in A \times A : x \in A\}$$

and

$$\underset{x,y\in A}{\forall} \Big(x = y \Leftrightarrow \mathrm{Id}_A(x) = y \Leftrightarrow (x,y) \in \Delta(A) \Big).$$

The preferred terms and notation depend on the emphasis being either on the function perspective or the relation perspective.

Definition 3.7. Assume Axioms 0 – 5 and let $R \subseteq A \times B$ be a relation over sets A and B.

(a) The relation

$$R^{-1} := \big\{(b,a) \in B \times A : a\,R\,b\big\} \subseteq B \times A$$

is called the *inverse* or *converse* relation of R (note that the notation R^{-1} is consistent with the notation introduced in Def. and Rem. 3.5).

(b) Given $U \subseteq A$, the relation $S \subseteq U \times B$ over U and B, defined by

$$S := \big\{(a,b) \in U \times B : a\,R\,b\big\}$$

is called the *restriction* of R to U; R is called an *extension* of S to A. In this situation, one also uses the notation $R\restriction_U$ for S (some authors prefer the notation $R|_U$ or $R|U$ and often one is less precise and still writes R for the restriction). If R is a relation on A (i.e. $R \subseteq A \times A$), then we also define its *strong restriction* to U, denoted $R\Lsh_U \subseteq U \times U$, to be the relation on U defined by

$$R\Lsh_U := \big\{(a,b) \in U \times U : a\,R\,b\big\}$$

(in general, one then has $R\Lsh_U \subsetneq R\restriction_U$).

(c) Given a relation $T \subseteq C \times D$ over sets C and D the *composition* of R and T is the relation over A and D defined by

$$T \circ R := \left\{(a,d) \in A \times D : \underset{b\in B\cap C}{\exists} \big(a\,R\,b \wedge b\,T\,d \big)\right\} \subseteq A \times D.$$

The expression $T \circ R$ is read as "T after R" or "T composed with R". Of course, if R and T are functions with $R(A) \subseteq C$, then $T \circ R$ is the function

$$T \circ R : A \longrightarrow D, \quad (T \circ R)(a) = T\big(R(a)\big).$$

Proposition 3.8. *Assume Axioms* 0 – 5. *Consider sets* A, B, C, D, E, F *and relations* $R \subseteq A \times B$, $S \subseteq C \times D$, $T \subseteq E \times F$.

(a) *Associativity of Compositions: It holds that*

$$T \circ (S \circ R) = (T \circ S) \circ R. \tag{3.5}$$

(b) *Properties of the Inverse Relation: One has*

$$(R^{-1})^{-1} = R. \tag{3.6a}$$

Moreover,

$$\begin{aligned}
R \text{ is a partial function} &\quad\Leftrightarrow\quad R^{-1} \text{ is injective}, && (3.6b)\\
R \text{ is injective} &\quad\Leftrightarrow\quad R^{-1} \text{ is a partial function}, && (3.6c)\\
R \text{ is one-to-one} &\quad\Leftrightarrow\quad R^{-1} \text{ is one-to-one}, && (3.6d)\\
R \text{ is surjective} &\quad\Leftrightarrow\quad R^{-1} \text{ is total}, && (3.6e)\\
R \text{ is total} &\quad\Leftrightarrow\quad R^{-1} \text{ is surjective}, && (3.6f)\\
R \text{ is a function} &\quad\Leftrightarrow\quad R^{-1} \text{ is injective and surjective}, && (3.6g)\\
R \text{ is injective and surjective} &\quad\Leftrightarrow\quad R^{-1} \text{ is a function}, && (3.6h)\\
R \text{ is a bijective function} &\quad\Leftrightarrow\quad R^{-1} \text{ is a bijective function}. && (3.6i)
\end{aligned}$$

(c) *The law for forming inverse relations reads:*

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}. \tag{3.7}$$

(d) *One has the following law for forming images and preimages:*

$$\forall_{U \subseteq A} \quad (S \circ R)(U) = S(R(U)), \tag{3.8a}$$

$$\forall_{W \subseteq D} \quad (S \circ R)^{-1}(W) = R^{-1}(S^{-1}(W)). \tag{3.8b}$$

(e) *If* R *and* S *are both partial functions (resp. both injective or both one-to-one), then so is* $S \circ R$.

(f) *Assuming* $R(A) \subseteq C$, *the following holds true: If* R *and* S *are both total (resp. both a function), then so is* $S \circ R$ *(but see Ex.* 3.9(a)*).*

(g) *Assuming* $S^{-1}(D) \subseteq B$, *the following holds true: If* R *and* S *are both surjective, then so is* $S \circ R$ *(but see Ex.* 3.9(a)*).*

(h) *Assuming $B = C$, the following holds true*[7]*: If R and S are both bijective functions, then so is $S \circ R$ (but see Ex.* 3.9(a)*).*

(i) *If R is a bijective function, then $R^{-1} \circ R = \mathrm{Id}_A$ (but see Ex.* 3.9(b)*).*

Proof. (a): According to Def. 3.7(c), both $T \circ (S \circ R)$ and $(T \circ S) \circ R$ are relations over A and F. So it just remains to prove

$$\underset{(a,f)\in A\times F}{\forall} \Big((a,f) \in T \circ (S \circ R) \;\Leftrightarrow\; (a,f) \in (T \circ S) \circ R \Big).$$

Indeed, we obtain, for each $(a,f) \in A \times F$,

$$\begin{aligned}(a,f) \in T \circ (S \circ R) \quad &\Leftrightarrow \quad \underset{d\in D\cap E}{\exists} \Big(a\,(S \circ R)\,d \;\wedge\; d\,T\,f \Big) \\ &\Leftrightarrow \quad \underset{b\in B\cap C}{\exists}\;\underset{d\in D\cap E}{\exists} \Big(a\,R\,b \;\wedge\; b\,S\,d \;\wedge\; d\,T\,f \Big) \\ &\Leftrightarrow \quad \underset{b\in B\cap C}{\exists} \Big(a\,R\,b \;\wedge\; b\,(T \circ S)\,f \Big) \quad \Leftrightarrow \quad (a,f) \in (T \circ S) \circ R,\end{aligned}$$

thereby establishing the case.

(b) – (e) are left as exercises.

(f): Assume $R(A) \subseteq C$. If R and S are total, then, given $a \in A$,

$$\underset{b\in B}{\exists} (a\,R\,b) \quad \overset{b\in C}{\Rightarrow} \quad \underset{b\in B\cap C}{\exists}\;\underset{d\in D}{\exists} \Big(a\,R\,b \;\wedge\; b\,S\,D \Big) \quad \Rightarrow \quad \underset{d\in D}{\exists}\, a\,(S \circ R)\,d,$$

proving $S \circ R$ to be total. If R and S are both functions, then they are both total partial functions, implying $S \circ R$ to be a total partial function (i.e. a function) by combining what we have just proved with (e).

(g) – (i) are left as exercises. ∎

Example 3.9. (a) To see that Prop. 3.8(f),(g),(h) are not correct without their respective assumptions $R(A) \subseteq C$, $S^{-1}(D) \subseteq B$, $B = C$, consider $A := B := \{1,2\}$, $C := D := \{2,3\}$, $R := \mathrm{Id}_A = \{(1,1),(2,2)\}$, $S := \mathrm{Id}_C = \{(2,2),(3,3)\}$. Then R and S are bijective functions, but $S \circ R = \{(2,2)\}$ is neither total nor surjective. An even simpler example is given by $A := B := \{1\}$, $C := D := \{2\}$, $R := \mathrm{Id}_A$, $S := \mathrm{Id}_C$, $S \circ R = \emptyset$.

[7] As one wants to apply (f) and (g), instead of $B = C$, one might be inclined to use the hypotheses $R(A) \subseteq C$ and $S^{-1}(D) \subseteq B$, since, at first glance, this might appear weaker. However, the also assumed surjectivity of R then yields $R(A) = B \subseteq C$ and the also assumed totality of S (i.e. surjectivity of S^{-1}) then yields $S^{-1}(D) = C \subseteq B$, and we are back to $B = C$.

(b) To see that the converse of Prop. 3.8(i) does not hold, consider $A := \{1\}$, $B := \{1, 2, 3\}$, $R := \{(1, 1), (1, 2)\}$. Then R is not a function and not surjective, but still $R^{-1} = \{(1, 1), (2, 1)\}$ and $R^{-1} \circ R = \mathrm{Id}_A$.

Definition 3.10. Assume Axioms 0 – 5 and let R be a relation on a set A, i.e. $R \subseteq A \times A$.

(a) R is called *reflexive* if, and only if,

$$\forall_{x \in A} \; x\,R\,x,$$

i.e. if, and only if, every element is related to itself.

(b) R is called *symmetric* if, and only if,

$$\forall_{x,y \in A} \; \big(x\,R\,y \;\Rightarrow\; y\,R\,x\big),$$

i.e. if, and only if, each x is related to y if, and only if, y is related to x.

(c) R is called *antisymmetric* if, and only if,

$$\forall_{x,y \in A} \; \big((x\,R\,y \;\wedge\; y\,R\,x) \;\Rightarrow\; x = y\big),$$

i.e. if, and only if, the only possibility for x to be related to y at the same time that y is related to x is in the case $x = y$.

(d) R is called *asymmetric* if, and only if,

$$\forall_{x,y \in A} \; \Big(x\,R\,y \;\Rightarrow\; \neg(y\,R\,x)\Big),$$

i.e. if, and only if, x is related to y only if y is not related to x.

(e) R is called *transitive* if, and only if,

$$\forall_{x,y,z \in A} \; \big((x\,R\,y \;\wedge\; y\,R\,z) \;\Rightarrow\; x\,R\,z\big),$$

i.e. if, and only if, the relatedness of x and y together with the relatedness of y and z implies the relatedness of x and z.

(f) R is called an *equivalence relation* if, and only if, R is reflexive, symmetric, and transitive.

(g) R satisfies *trichotomy* if, and only if,

$$\forall_{x,y\in A} \big(x\,R\,y \;\vee\; y\,R\,x \;\vee\; x=y\big),$$

i.e. if, and only if, if x and y are distinct, then x is related to y or y is related to x.

(h) R is called a *partial order* if, and only if, R is reflexive, antisymmetric, and transitive. If R is a partial order, then one usually writes $x \leq y$ instead of $x\,R\,y$. A partial order is called a *total* or *linear order* if, and only if, it also satisfies trichotomy.

(i) R is called a *strict partial order* if, and only if, R is asymmetric and transitive. If R is a partial order, then one usually writes $x < y$ instead of $x\,R\,y$. A strict partial order is called a *strict total* or *strict linear order* if, and only if, it also satisfies trichotomy.

Lemma 3.11. *If $\leq$ is a partial order on a set A, then, using the notation of Def.* 3.6(g)*, $< := \leq \setminus \Delta(A)$ is a strict partial order (called the strict partial order* corresponding *to $\leq$). Conversely, if $<$ is a strict partial order on A, then $\leq := < \cup \Delta(A)$ is a partial order (called the partial order* corresponding *to $<$).*

Proof. Let $\leq$ be a partial order on A, $< := \leq \setminus \Delta(A)$. If $x, y \in A$ with $x < y$, then $x \neq y$ (since $\neg(x < x)$). Then, also $\neg(y < x)$: Otherwise, we had $x \leq y$ and $y \leq x$, implying the contradiction $x = y$ (by the antisymmetriy of $\leq$). Thus, $<$ is asymmetric. Now suppose, we have $x < y$ and $y < z$ with $x, y, z \in A$. Then $x \leq y$ and $y \leq z$, implying $x \leq z$ by the transitivity of $\leq$. If $x = z$, then $z \leq x \leq y \leq z$, $y = z$, in contradiction to $y < z$. Thus, $x \neq z$ and $x < z$, showing $<$ to be transitive and a strict partial order. The proof that $\leq := < \cup \Delta(A)$ is a partial order, if $<$ is a given strict partial order, is left as an exercise. ∎

Proposition 3.12. *Let R be a relation on a set A and R^{-1} its inverse relation as defined in Def.* 3.7(a)*. Then*

$$R \text{ is reflexive} \;\Leftrightarrow\; R^{-1} \text{ is reflexive}, \tag{3.9a}$$

$$R \text{ is symmetric} \;\Leftrightarrow\; R^{-1} \text{ is symmetric}, \tag{3.9b}$$

$$R \text{ is antisymmetric} \;\Leftrightarrow\; R^{-1} \text{ is antisymmetric}, \tag{3.9c}$$

$$R \text{ is asymmetric} \;\Leftrightarrow\; R^{-1} \text{ is asymmetric}, \tag{3.9d}$$

$$R \text{ is transitive} \;\Leftrightarrow\; R^{-1} \text{ is transitive}, \tag{3.9e}$$

$$R \text{ is an equivalence relation} \;\Leftrightarrow\; R^{-1} \text{ is an equivalence relation}, \tag{3.9f}$$

$$R \text{ satisfies trichotomy} \;\Leftrightarrow\; R^{-1} \text{ satisfies trichotomy}, \tag{3.9g}$$

$$R \text{ is a partial (resp. total) order} \;\Leftrightarrow\; R^{-1} \text{ is a partial (resp. total) order}, \tag{3.9h}$$

$$R \text{ is a str. par. (resp. total) order} \;\Leftrightarrow\; R^{-1} \text{ is a str. par. (resp. total) order}, \tag{3.9i}$$

$$\big(R \setminus \Delta(A)\big)^{-1} = R^{-1} \setminus \Delta(A). \tag{3.9j}$$

Proof. Since $R = (R^{-1})^{-1}$, for each equivalence, it suffices to prove just one implication (the converse then follows by applying the first implication with R replaced by R^{-1}). Let $x, y, z \in A$. Then
$$x\,R\,x \quad\Rightarrow\quad x\,R^{-1}\,x,$$
proving (3.9a). If R is transitive, then
$$x\,R^{-1}\,y \,\wedge\, y\,R^{-1}\,z \quad\Rightarrow\quad z\,R\,y \,\wedge\, y\,R\,x \quad\Rightarrow\quad z\,R\,x \quad\Rightarrow\quad x\,R^{-1}\,z,$$
showing R^{-1} to be transitive and (3.9e). Also,
$$\begin{aligned}(x,y) \in R^{-1} \setminus \Delta(A) \quad&\Leftrightarrow\quad (y,x) \in R \,\wedge\, x \neq y \quad\Leftrightarrow\quad (y,x) \in R \setminus \Delta(A)\\ &\Leftrightarrow\quad (x,y) \in \big(R \setminus \Delta(A)\big)^{-1},\end{aligned}$$
thereby proving (3.9j). We leave the remaining cases (all straightforward) as exercises. ■

Definition 3.13. Let $\leq$ be a partial order on $A \neq \emptyset$, $\emptyset \neq B \subseteq A$.

(a) $x \in A$ is called *lower* (resp. *upper*) *bound* for B if, and only if, $x \leq b$ (resp. $b \leq x$) for each $b \in B$. Moreover, B is called *bounded from below* (resp. from above) if, and only if, there exists a lower (resp. upper) bound for B; B is called *bounded* if, and only if, it is bounded from above and from below.

(b) $x \in B$ is called *minimum* or just *min* (resp. *maximum* or *max*) of B if, and only if, x is a lower (resp. upper) bound for B. One writes $x = \min B$ if x is minimum and $x = \max B$ if x is maximum.

(c) A maximum of the set of lower bounds of B (i.e. a largest lower bound) is called *infimum* of B, denoted $\inf B$; a minimum of the set of upper bounds of B (i.e. a smallest upper bound) is called *supremum* of B, denoted $\sup B$.

We extend all the notions defined above to strict partial orders $<$ by applying them to the partial order corresponding to $<$, i.e. to $\leq\; :=\; < \cup\, \Delta(A)$: For example, we call $x \in A$ a lower bound of $B \subseteq A$ with respect to $<$ if, and only if, x is a lower bound of B with respect to $\leq$, and analogous for the other notions.

Lemma 3.14. *Let $\leq$ and $<$ be relations on a set A, where $\leq$ is a partial order and $<$ is a strict partial order. Let $\geq := (\leq)^{-1}$ and $> := (<)^{-1}$ be the respective inverse relations according to Def.* 3.7(a)*, i.e.*
$$\forall_{x,y \in A} \Big(\big(x \geq y \;\Leftrightarrow y \leq x\big) \quad\wedge\quad \big(x > y \;\Leftrightarrow\; y < x\big) \Big). \tag{3.10}$$

According to (3.9h) *and* (3.9i), $\geq$ *is also a partial order on* A *and* $>$ *is also a strict partial order on* A*, where* $\leq$ *(resp.* $<$*) being total on* A*, implies* $\geq$ *(resp.* $>$*) to be total on* A *as well. If* $<$ *is the strict order corresponding to* $\leq$*, then* $>$ *is the strict order corresponding to* $\geq$*. Moreover for* $A \neq \emptyset$ *and* $\emptyset \neq B \subseteq A$*, using obvious notation, we have, for each* $x \in A$*,*

$$x \leq\text{-lower bound for } B \quad\Leftrightarrow\quad x \geq\text{-upper bound for } B, \tag{3.11a}$$

$$x \leq\text{-upper bound for } B \quad\Leftrightarrow\quad x \geq\text{-lower bound for } B, \tag{3.11b}$$

$$x = \min\nolimits_{\leq} B \quad\Leftrightarrow\quad x = \max\nolimits_{\geq} B, \tag{3.11c}$$

$$x = \max\nolimits_{\leq} B \quad\Leftrightarrow\quad x = \min\nolimits_{\geq} B, \tag{3.11d}$$

$$x = \inf\nolimits_{\leq} B \quad\Leftrightarrow\quad x = \sup\nolimits_{\geq} B, \tag{3.11e}$$

$$x = \sup\nolimits_{\leq} B \quad\Leftrightarrow\quad x = \inf\nolimits_{\geq} B. \tag{3.11f}$$

All the equivalences in (3.11) *also hold if* $\leq$ *is replaced by* $<$ *and* $\geq$ *is replaced by* $>$*.*

Proof. If $<$ is the strict order corresponding to $\leq$, then

$$< \;=\; \leq \setminus \Delta(A) \quad\Rightarrow\quad > \;=\; (<)^{-1} \;=\; \big(\leq \setminus \Delta(A)\big)^{-1} \overset{(3.9j)}{=} (\leq)^{-1} \setminus \Delta(A) \;=\; \geq \setminus \Delta(A),$$

i.e. $>$ is the strict order corresponding to $\geq$. Moreover,

$$x \leq\text{-lower bound for } B \;\Leftrightarrow\; \underset{b\in B}{\forall}\; x \leq b \;\Leftrightarrow\; \underset{b\in B}{\forall}\; b \geq x \;\Leftrightarrow\; x \geq\text{-upper bound for } B,$$

proving (3.11a). Analogously, we obtain (3.11b). Next, (3.11c) and (3.11d) are implied by (3.11a) and (3.11b), respectively. Finally, (3.11e) is proved by

$$\begin{aligned} x = \inf\nolimits_{\leq} B \;&\Leftrightarrow\; x = \max\nolimits_{\leq}\{y \in A :\; y \leq\text{-lower bound for } B\} \\ &\Leftrightarrow\; x = \min\nolimits_{\geq}\{y \in A :\; y \geq\text{-upper bound for } B\} \;\Leftrightarrow\; x = \sup\nolimits_{\geq} B, \end{aligned}$$

and (3.11f) follows analogously. That all the equivalences in (3.11) also hold if $\leq$ is replaced by $<$ and $\geq$ is replaced by $>$ is now immediate from the last paragraph of Def. 3.13. ■

Proposition 3.15. *Let* $\leq$ *be a partial order on* $A \neq \emptyset$*,* $\emptyset \neq B \subseteq A$*. The elements* $\max B$*,* $\min B$*,* $\sup B$*,* $\inf B$ *are all unique, provided they exist.*

Proof. Exercise. ■

Definition 3.16. Let A, B be nonempty sets with partial orders, both denoted by $\leq$ (even though they might be different). A function $f : A \longrightarrow B$, is called (*strictly*) *isotone*, *order-preserving*, or *increasing* if, and only if,

$$\underset{x,y\in A}{\forall} \big(x < y \;\Rightarrow\; f(x) \leq f(y) \;\; (\text{resp. } f(x) < f(y))\big); \tag{3.12a}$$

f is called (*strictly*) *antitone*, *order-reversing*, or *decreasing* if, and only if,

$$\forall_{x,y\in A} \big(x < y \;\Rightarrow\; f(x) \geq f(y) \;\; (\text{resp. } f(x) > f(y))\big). \tag{3.12b}$$

Functions that are (strictly) isotone or antitone are called (strictly) *monotone*.

Proposition 3.17. *Let A, B be nonempty sets with partial orders, both denoted by $\leq$.*

(a) *A (strictly) isotone function $f : A \longrightarrow B$ becomes a (strictly) antitone function and vice versa if precisely one of the relations $\leq$ is replaced by $\geq$.*

(b) *If the order $\leq$ on A is total and $f : A \longrightarrow B$ is strictly isotone or strictly antitone, then f is injective.*

(c) *If the order $\leq$ on A is total and $f : A \longrightarrow B$ is bijective and strictly isotone (resp. antitone), then f^{-1} is also strictly isotone (resp. antitone).*

Proof. (a) is immediate from (3.12).

(b): Due to (a), it suffices to consider the case that f is strictly isotone. If f is strictly isotone and $x \neq y$, then $x < y$ or $y < x$ since the order on A is total. Thus, $f(x) < f(y)$ or $f(y) < f(x)$, i.e. $f(x) \neq f(y)$ in every case, showing f is injective.

(c): Again, due to (a), it suffices to consider the isotone case. If $u, v \in B$ such that $u < v$, then $u = f(f^{-1}(u))$, $v = f(f^{-1}(v))$, and the isotonicity of f imply $f^{-1}(u) < f^{-1}(v)$ (we are using that the order on A is total – otherwise, $f^{-1}(u)$ and $f^{-1}(v)$ need not be comparable). ■

Example 3.18. The following examples show that the assertions of Prop. 3.17(b),(c) are no longer correct if one does not assume the order on A to be total. Let

$$A := \big\{(1,1), (2,1), (1,2)\big\}.$$

Then

$$(m_1, m_2) \leq (n_1, n_2) \;\Leftrightarrow\; m_1 \leq n_1 \;\wedge\; m_2 \leq n_2, \tag{3.13}$$

defines a partial order on A that is not a total order (for example, neither $(1,2) \leq (2,1)$ nor $(2,1) \leq (1,2)$).

(a) The function

$$f : A \longrightarrow \{1,2\}, \quad \begin{cases} f(1,1) := 1, \\ f(1,2) := 2, \\ f(2,1) := 2, \end{cases}$$

is strictly isotone, but not one-to-one.

(b) The function

$$f : A \longrightarrow \{1,2,3\}, \quad \begin{cases} f(1,1) := 1, \\ f(1,2) := 2, \\ f(2,1) := 3, \end{cases}$$

is strictly isotone and bijective, however f^{-1} is not isotone (since $2 < 3$, but $f^{-1}(2) = (1,2)$ and $f^{-1}(3) = (2,1)$ are not comparable, i.e. $f^{-1}(2) \leq f^{-1}(3)$ is *not* true).

Definition 3.19. A relation R on a set A is called a *(strict) well-order* if, and only if, R is a (strict) total order and every nonempty subset of A has a min with respect to R (for example, we will see that the usual $\leq$ constitutes a well-order on $\mathbb{N}$; however, the usual $\leq$ does not constitute a well-order on $\mathbb{Z}$ (e.g., $\mathbb{Z}$ does not have a min) or on $\mathbb{R}_0^+$ (e.g., $\mathbb{R}^+$ does not have a min)).

Definition 3.20. **(a)** Let R be a relation on a set A. We define $R_{\neq} := R \setminus \Delta(A)$, i.e. $R_{\neq}$ is the relation on A defined by

$$x \, R_{\neq} \, y \quad :\Leftrightarrow \quad x \, R \, y \ \wedge \ x \neq y$$

(for example, if $\leq$ is a partial order, then $< := (\leq)_{\neq}$ is the corresponding strict partial order, cf. Lem. 3.11).

(b) Let R be a relation on a set A and let S be a relation on a set B. We define a relation $P := R \odot S$ on $A \times B$, called the *lexicographic product* of R and S, where

$$(a_1, b_1) \, P \, (a_2, b_2) \quad :\Leftrightarrow \quad (a_1, b_1) \, (R \odot S) \, (a_2, b_2) \quad :\Leftrightarrow \quad a_1 \, R_{\neq} \, a_2 \vee (a_1 = a_2 \wedge b_1 \, S \, b_2).$$

Proposition 3.21. *Let R be a relation on a set A, let S be a relation on a set B, and let $P := R \odot S$ be the lexicographic product on $A \times B$, as defined in Def.* 3.20.

(a) *If S is reflexive, then P is reflexive.*

(b) *If R and S are symmetric, then P is symmetric.*

(c) *If R and S are antisymmetric, then P is antisymmetric.*

(d) *If R and S are asymmetric, then P is asymmetric.*

(e) *If $R_{\neq}$ and S are transitive, then P is transitive (but see Ex.* 3.22*).*

(f) *If R and S satisfy trichotomy, then P satisfies trichotomy.*

(g) *If R and S are (strict) partial orders, then P is a (strict) partial order. In this situation, one calls P the* lexicographic order *given by R and S. It is also common to denote all three orders R, S, P by the same symbol $\leq$ (or all by $<$ in the strict case).*

(h) *If R and S are (strict) total orders, then P is a (strict) total order.*

(i) *If R and S are (strict) well-orders, then P is a (strict) well-order.*

Proof. Let $a, a_1, a_2, a_3 \in A$ and $b, b_1, b_2, b_3 \in B$.

(a): If S is reflexive, then $(a,b)\,P\,(a,b)$, since $a = a$ and $b\,S\,b$.

(b): Exercise.

(c): If R and S are antisymmetric, then $(a_1,b_1)\,P\,(a_2,b_2) \;\wedge\; (a_2,b_2)\,P\,(a_1,b_1)$ implies $a_1 = a_2$ (otherwise, $a_1\,R_{\neq}\,a_2$ and $a_2\,R_{\neq}\,a_1$ needed to hold, in contradiction to the antisymmetry of R). Thus,

$$\begin{aligned}(a_1,b_1)\,P\,(a_2,b_2) \;\wedge\; (a_2,b_2)\,P\,(a_1,b_1) \quad &\Rightarrow \quad a_1 = a_2 \;\wedge\; b_1\,S\,b_2 \;\wedge\; b_2\,S\,b_1\\ &\Rightarrow \quad (a_1,b_1) = (a_2,b_2),\end{aligned}$$

showing P to be antisymmetric.

(d),(e): Exercise.

(f): Assume R and S to satisfy trichotomy. Then

$$\begin{aligned}&\neg\big((a_1,b_1)\,P\,(a_2,b_2)\big) \;\wedge\; \neg\big((a_2,b_2)\,P\,(a_1,b_1)\big) \quad \Rightarrow \quad \neg(a_1\,R_{\neq}\,a_2) \;\wedge\; \neg(a_2\,R_{\neq}\,a_1)\\ &\Rightarrow \quad a_1 = a_2 \quad \Rightarrow \quad \neg(b_1\,S\,b_2) \;\wedge\; \neg(b_2\,S\,b_1) \quad \Rightarrow \quad b_1 = b_2 \quad \Rightarrow \quad (a_1,b_1) = (a_2,b_2),\end{aligned}$$

proving P to satisfy trichotomy.

(g),(h): Exercise.

(i): Assume R and S are (strict) well-orders. Then P is a (strict) total order by (h). If $\emptyset \neq C \subseteq A \times B$, then, letting

$$A_1 := \left\{a \in A:\; \underset{b\in B}{\exists}\; (a,b) \in C\right\},$$

we have $\emptyset \neq A_1 \subseteq A$. Since R is a (strict) well-order, there exists $\alpha := \min A_1$. Now, letting

$$B_1 := \big\{b \in B:\; (\alpha,b) \in C\big\},$$

we have $\emptyset \neq B_1 \subseteq B$. Since S is a (strict) well-order, there exists $\beta := \min B_1$. We show $(\alpha,\beta) = \min C$: Let $(a,b) \in C$. If $a \neq \alpha$, then $\alpha\,R_{\neq}\,a$, since $a, \alpha \in A_1$ and $\alpha = \min A_1$.

If $a = \alpha$, then $b = \beta$ or $\beta\, S_{\neq}\, b$, since $b, \beta \in B_1$ and $\beta = \min B_1$. Thus, $(a, b) = (\alpha, \beta)$ or $(\alpha, \beta)\, P\, (a, b)$ (both hold if P is a total order, but not if P is a strict total order), showing (α, β) to be a lower bound for C. Since, also, $(\alpha, \beta) \in C$, we have shown $(\alpha, \beta) = \min C$ and P is a (strict) well-order. ■

Example 3.22. To see that the lexicographic product of transitive relations need not be transitive and that the lexicographic product of equivalence relations need not be an equivalence relation, consider $A := \{1, 2\}$ with $R := \{(1, 1), (2, 2), (1, 2), (2, 1)\}$, and $S := \{(1, 1), (2, 2)\}$. It is an exercise to show R and S are both equivalence relations, but $R \odot S$ is not transitive (in particular, not an equivalence relation).

Lemma 3.23. *Let R be a relation on a set A, $U \subseteq A$, and let $R\!\upharpoonleft\!\upharpoonright_U$ denote its strong restriction to U as defined in Def.* 3.7(b)*.*

(a) *If R is reflexive, then $R\!\upharpoonleft\!\upharpoonright_U$ is reflexive.*

(b) *If R is symmetric, then $R\!\upharpoonleft\!\upharpoonright_U$ is symmetric.*

(c) *If R is antisymmetric, then $R\!\upharpoonleft\!\upharpoonright_U$ is antisymmetric.*

(d) *If R is asymmetric, then $R\!\upharpoonleft\!\upharpoonright_U$ is asymmetric.*

(e) *If R is transitive, then $R\!\upharpoonleft\!\upharpoonright_U$ is transitive.*

(f) *If R is an equivalence relation, then $R\!\upharpoonleft\!\upharpoonright_U$ is an equivalence relation.*

(g) *If R satisfies trichotomy, then $R\!\upharpoonleft\!\upharpoonright_U$ satisfies trichotomy.*

(h) *If R is a (strict) partial order, then $R\!\upharpoonleft\!\upharpoonright_U$ is a (strict) partial order.*

(i) *If R is a (strict) total order, then $R\!\upharpoonleft\!\upharpoonright_U$ is a (strict) total order.*

(j) *If R is a (strict) well-order, then $R\!\upharpoonleft\!\upharpoonright_U$ is a (strict) well-order.*

(k) $R_{\neq}\!\upharpoonleft\!\upharpoonright_U = (R\!\upharpoonleft\!\upharpoonright_U)_{\neq}$.

Proof. Let $x, y, z \in U$. Since $U \subseteq A$, we then have $x, y, z \in A$, which is the key ingredient to the proofs below.

(a): If R is reflexive, then $x\, R\, x$, showing $R\!\upharpoonleft\!\upharpoonright_U$ to be reflexive.

(b): If R is symmetric, then $x\, R\, y$ implies $y\, R\, x$, showing $R\!\upharpoonleft\!\upharpoonright_U$ to be symmetric.

(c): If R is antisymmetric, then $x\, R\, y$ and $y\, R\, x$ implies $x = y$, showing $R\!\upharpoonleft\!\upharpoonright_U$ to be antisymmetric.

(d): If R is asymmetric, then $x\, R\, y$ implies $\neg(y\, R\, x)$, showing $R\!\upharpoonleft\!\upharpoonright_U$ to be asymmetric.

(e): If R is transitive, then $x\,R\,y$ and $y\,R\,z$ implies $x\,R\,z$, showing $R{\upharpoonright\upharpoonright}_U$ to be transitive.

(f) follows by combining (a), (b), and (e).

(g): If R satisfies trichotomy, then

$$x\,R\,y \;\vee\; y\,R\,x \;\vee\; x = y$$

holds, showing $R{\upharpoonright\upharpoonright}_U$ to satisfy trichotomy.

(h) follows by combining (a), (c), and (e) (resp. (d) and (e) in the strict case).

(i) follows by combining (h) with (g).

(j): Due to (h), it merely remains to show that every nonempty subset $V \subseteq U$ has a min. However, since $V \subseteq A$ and R is a well-order on A, there is $m \in V$ such that m is a min for R on A, implying m to be a min for R on U as well.

(k): Since

$$(x,y) \in R_{\neq}{\upharpoonright\upharpoonright}_U \quad\Leftrightarrow\quad x\,R\,y \;\wedge\; x \neq y \quad\Leftrightarrow\quad (x,y) \in (R{\upharpoonright\upharpoonright}_U)_{\neq},$$

the proof is complete. ■

3.3 Ordinals

In preparation for our official definition of $\mathbb{N}$ in Def. 4.5 below, we will study so-called ordinals, which are special sets also of further interest to the field of set theory (the natural numbers will turn out to be precisely the finite ordinals).

Definition 3.24. A set X is called *transitive* if, and only if, every element of X is also a subset of X:

$$\forall_{x \in X} \quad x \subseteq X. \tag{3.14a}$$

Clearly, (3.14a) is equivalent to

$$\forall_{x,y} \quad \Big(x \in y \;\wedge\; y \in X \;\Rightarrow\; x \in X\Big). \tag{3.14b}$$

Lemma 3.25. **(a)** *Intersections of transitive sets are transitive: If $\mathcal{M}$ is a nonempty set, then*

$$\Big(\forall_{X \in \mathcal{M}} \; X \textit{ is transitive}\Big) \quad\Rightarrow\quad \bigcap \mathcal{M} \textit{ is transitive}$$

(in particular, if X, Y are transitive sets, then $X \cap Y$ is a transitive set).

(b) *Unions of transitive sets are transitive: If $\mathcal{M}$ is a set, then*

$$\left(\forall_{X\in\mathcal{M}} X \text{ is transitive}\right) \quad\Rightarrow\quad \bigcup\mathcal{M} \text{ is transitive}$$

(in particular, if X, Y are transitive sets, then $X \cup Y$ is a transitive set).

Proof. (a): If $x \in \bigcap\mathcal{M}$ and $X \in \mathcal{M}$, then $x \in X$. Thus, if $y \in x$, then $y \in X$, since X is transitive, showing $y \in \bigcap\mathcal{M}$. In consequence, $\bigcap\mathcal{M}$ is transitive.

(b): If $x \in \bigcup\mathcal{M}$, then there exists $X \in \mathcal{M}$ such that $x \in X$. Thus, if $y \in x$, then $y \in X$, since X is transitive. This, in turn, implies $y \in \bigcup\mathcal{M}$, since $X \subseteq \mathcal{M}$, showing $\bigcup\mathcal{M}$ to be transitive. ■

Definition 3.26. (a) A set α is called an *ordinal number* or just an *ordinal* if, and only if, α is transitive and $\in$ constitutes a strict well-order on α. An ordinal α is called a *successor ordinal* if, and only if, there exists an ordinal β such that $\alpha = \mathbf{S}(\beta)$, where $\mathbf{S}$ is the successor function of Def. 2.21. An ordinal $\alpha \neq 0$ is called a *limit ordinal* if, and only if, it is not a successor ordinal. We denote the class of all ordinals by $\mathbf{ON}$ (it is a proper class by Cor. 3.34 below).

(b) We define

$$\forall_{\alpha,\beta\in\mathbf{ON}} \quad (\alpha < \beta \;:\Leftrightarrow\; \alpha \in \beta), \tag{3.15a}$$

$$\forall_{\alpha,\beta\in\mathbf{ON}} \quad (\alpha \leq \beta \;:\Leftrightarrow\; \alpha < \beta \,\vee\, \alpha = \beta). \tag{3.15b}$$

Notation 3.27. Given a set A, we define the *element relation* $R_\in$ on A by

$$R_\in := \{(x, y) \in A \times A :\, x \in y\}, \tag{3.16a}$$

i.e.

$$\forall_{x,y\in A} \quad (x, y) \in R_\in \quad\Leftrightarrow\quad x \in y. \tag{3.16b}$$

Example 3.28. (a) Using (2.9), $0 = \emptyset$ is an ordinal, and $1 = \mathbf{S}(0)$, $2 = \mathbf{S}(1)$ are both successor ordinals (in Prop. 4.7, we will identify $\mathbb{N}_0$ as the smallest limit ordinal). Even though $X := \{1\}$ and $Y := \{0, 2\}$ are well-ordered by $\in$, they are not ordinals, since they are not transitive sets: $1 \in X$, but $1 \not\subseteq X$ (since $0 \in 1$, but $0 \notin X$); similarly, $1 \in 2 \in Y$, but $1 \notin Y$.

(b) As a caveat, we point out that, in genereal, saying that a set A is transitive is *not* equivalent to saying that $R_\in$ is transitive on A: Actually, in general, neither implication is true: In (a) we saw that $R_\in$ was a transitive relation on the nontransitive sets X and Y. To see that the converse implication can fail, consider

$$A := \big\{0, 1, 2, \{1\}\big\}.$$

Recalling $1 = \{0\}$ and $2 = \{0, 1\}$, we observe A to be a transitive set. However, $R_\in$ is not transitive on A, since $0 \in 1$ and $1 \in \{1\}$, but $0 \notin \{1\}$.

Lemma 3.29. *No ordinal contains itself, i.e.*

$$\forall_{\alpha \in \mathbf{ON}} \quad \alpha \notin \alpha.$$

Proof. If α is an ordinal, then $\in$ is a strict order on α. Due to asymmetry of strict orders, $x \in x$ can not be true for any element of α, implying that $\alpha \in \alpha$ can not be true. ■

Proposition 3.30. *Every element of an ordinal is an ordinal, i.e.*

$$\forall_{\alpha \in \mathbf{ON}} \quad \Big(X \in \alpha \;\Rightarrow\; X \in \mathbf{ON} \Big)$$

(in other words, **ON** *is a transitive class).*

Proof. Let $\alpha \in \mathbf{ON}$ and $X \in \alpha$. Since α is transitive, we have $X \subseteq \alpha$. As $\in$ is a strict well-order on α, it must also be a strict well-order on X by Lem. 3.23(j). In consequence, it only remains to prove that X is transitive as well. To this end, let $x \in X$. Then $x \in \alpha$, as α is transitive. If $y \in x$, then, using transitivity of α again, $y \in \alpha$. Now $y \in X$, as $\in$ is transitive on α, proving $x \subseteq X$, i.e. X is transitive. ■

Proposition 3.31. *If $\alpha, \beta \in \mathbf{ON}$, then $X := \alpha \cap \beta \in \mathbf{ON}$ (we will see in Th.* 3.36(a) *below that, actually, $\alpha \cap \beta = \min\{\alpha, \beta\}$ and, moreover, the result extends to arbitrary intersections and, analogously, to arbitrary unions).*

Proof. The set X is transitive by Lem. 3.25(a), and, since $X \subseteq \alpha$, $\in$ is a strict well-order on X by Lem. 3.23(j). ■

Proposition 3.32. *On the class* **ON**, *the relation $\leq$ (as defined in* (3.15)*) is the same as the relation $\subseteq$, i.e.*

$$\forall_{\alpha, \beta \in \mathbf{ON}} \quad \Big(\alpha \leq \beta \;\Leftrightarrow\; \alpha \subseteq \beta \;\Leftrightarrow\; (\alpha \in \beta \,\vee\, \alpha = \beta) \Big). \tag{3.17}$$

Proof. Exercise. ■

Theorem 3.33. *The class* **ON** *is strictly well-ordered by $\in$, i.e.*

(i) *$\in$ is transitive on* **ON**:

$$\forall_{\alpha, \beta, \gamma \in \mathbf{ON}} \quad \Big(\alpha < \beta \,\wedge\, \beta < \gamma \;\Rightarrow\; \alpha < \gamma \Big).$$

(ii) $\in$ *is asymmetric on* **ON***:*

$$\forall_{\alpha,\beta\in\mathbf{ON}} \Big(\alpha < \beta \;\Rightarrow\; \neg(\beta < \alpha)\Big).$$

(iii) *Ordinals are always comparable:*

$$\forall_{\alpha,\beta\in\mathbf{ON}} \Big(\alpha < \beta \,\vee\, \beta < \alpha \,\vee\, \alpha = \beta\Big).$$

(iv) *Every nonempty set of ordinals has a min.*

Proof. (i) is clear, as γ is a transitive set.

(ii): If $\alpha, \beta \in \mathbf{ON}$, then $\alpha \in \beta \in \alpha$ implies $\alpha \in \alpha$ by (i), which is a contradiction to Lem. 3.29.

(iii): Let $\gamma := \alpha \cap \beta$. Then $\gamma \in \mathbf{ON}$ by Prop. 3.31. Thus

$$\gamma \subseteq \alpha \,\wedge\, \gamma \subseteq \beta \quad \overset{\text{Lem. 3.32}}{\Rightarrow} \quad (\gamma \in \alpha \,\vee\, \gamma = \alpha) \,\wedge\, (\gamma \in \beta \,\vee\, \gamma = \beta). \tag{3.18}$$

If $\gamma \in \alpha$ and $\gamma \in \beta$, then $\gamma \in \alpha \cap \beta = \gamma$, in contradiction to Lem. 3.29. Thus, by (3.18), $\gamma = \alpha$ or $\gamma = \beta$. If $\gamma = \alpha$, then $\alpha \subseteq \beta$. If $\gamma = \beta$, then $\beta \subseteq \alpha$, completing the proof of (iii).

(iv): Let X be a nonempty set of ordinals and consider $\alpha \in X$. If $\alpha = \min X$, then we are already done. Otherwise, $Y := \alpha \cap X = \{\beta \in X : \beta \in \alpha\} \neq \emptyset$. Since α is well-ordered by $\in$, there is $m := \min Y$. If $\beta \in X$, then either $\beta < \alpha$ or $\alpha \leq \beta$ by (iii). If $\beta < \alpha$, then $\beta \in Y$ and $m \leq \beta$. If $\alpha \leq \beta$, then $m < \alpha \leq \beta$. Thus, $m = \min X$, proving (iv). ■

Corollary 3.34. **ON** *is a proper class (i.e. there is no set containing all the ordinals).*

Proof. If there is a set X containing all ordinals, then, by comprehension, $\beta := \mathbf{ON} = \{\alpha \in X : \alpha \text{ is an ordinal}\}$ must be a set as well. But then Prop. 3.30 says that the *set* β is transitive and Th. 3.33 yields that the *set* β is well-ordered by $\in$, implying β to be an ordinal, i.e. $\beta \in \beta$ in contradiction to Lem. 3.29. ■

Corollary 3.35. *For each set X of ordinals, we have:*

(a) *X is well-ordered by $\in$.*

(b) *X is an ordinal if, and only if, X is transitive. Note: A transitive set of ordinals X is sometimes called an* initial segment *of* **ON***, since, here, transitivity can be restated in the form*

$$\forall_{\alpha\in\mathbf{ON}} \; \forall_{\beta\in X} \Big(\alpha < \beta \;\Rightarrow\; \alpha \in X\Big). \tag{3.19}$$

Proof. (a) is a simple consequence of Th. 3.33(i)-(iv).

(b) is immediate from (a). ■

Theorem 3.36. *Let X be a nonempty set of ordinals.*

(a) *Then $\gamma := \bigcap X$ is an ordinal, namely $\gamma = \min X$. In particular, if $\alpha, \beta \in \mathbf{ON}$, then $\min\{\alpha, \beta\} = \alpha \cap \beta$.*

(b) *Then $\delta := \bigcup X$ is an ordinal, namely $\delta = \sup X$. In particular, if $\alpha, \beta \in \mathbf{ON}$, then $\max\{\alpha, \beta\} = \alpha \cup \beta$.*

Proof. (a): Let $m := \min X$. Then $\gamma \subseteq m$, since $m \in X$. Conversely, if $\alpha \in X$, then $m \leq \alpha$ implies $m \subseteq \alpha$ by Prop. 3.32, i.e. $m \subseteq \gamma$. Thus, $m = \gamma$, proving (a).

(b): Exercise. ■

Next, we obtain some results regarding the successor function $\mathbf{S}$ of Def. 2.21 in the context of ordinals.

Lemma 3.37. *We have*

$$\underset{\alpha \in \mathbf{ON}}{\forall} \Big(x, y \in \mathbf{S}(\alpha) \wedge x \in y \quad \Rightarrow \quad x \neq \alpha \Big).$$

Proof. Seeking a contradiction, we reason as follows:

$$x = \alpha \overset{\alpha \notin \alpha}{\Rightarrow} y \neq \alpha \overset{y \in \mathbf{S}(\alpha)}{\Rightarrow} y \in \alpha \overset{\alpha \text{ transitive}}{\Rightarrow} y \subseteq \alpha \overset{x \in y}{\Rightarrow} \alpha \in \alpha.$$

This contradiction to $\alpha \notin \alpha$ yields $x \neq \alpha$, concluding the proof. ■

Proposition 3.38. *For each $\alpha \in \mathbf{ON}$, the following holds:*

(a) $\mathbf{S}(\alpha) \in \mathbf{ON}$.

(b) $\alpha < \mathbf{S}(\alpha)$.

(c) *For each ordinal β, $\beta < \mathbf{S}(\alpha)$ holds if, and only if, $\beta \leq \alpha$.*

(d) *For each ordinal β, if $\beta < \alpha$, then $\mathbf{S}(\beta) \leq \alpha < \mathbf{S}(\alpha)$.*

(e) *For each ordinal β, if $\mathbf{S}(\beta) < \mathbf{S}(\alpha)$, then $\beta < \alpha$.*

(f) *If α is a limit ordinal, then $\alpha = \sup \alpha$. If α is a successor ordinal, then $\alpha = \mathbf{S}(\sup \alpha)$.*

Proof. (a): Due to Prop. 3.30, $\mathbf{S}(\alpha)$ is a set of ordinals. Thus, by Cor. 3.35(b), it merely remains to prove that $\mathbf{S}(\alpha)$ is transitive. Let $x \in \mathbf{S}(\alpha)$. If $x = \alpha$, then $x = \alpha \subseteq \alpha \cup \{\alpha\} = \mathbf{S}(\alpha)$. If $x \neq \alpha$, then $x \in \alpha$ and, since α is transitive, this implies $x \subseteq \alpha \subseteq \mathbf{S}(\alpha)$, showing $\mathbf{S}(\alpha)$ to be transitive, thereby completing the proof of (a).

(b) holds, as $\alpha \in \mathbf{S}(\alpha)$ holds by the definition of $\mathbf{S}(\alpha)$.

(c) is clear, since, for each ordinal β,

$$\beta < \mathbf{S}(\alpha) \;\Leftrightarrow\; \beta \in \mathbf{S}(\alpha) \;\Leftrightarrow\; \beta \in \alpha \vee \beta = \alpha \;\Leftrightarrow\; \beta \leq \alpha.$$

(d): If $\beta < \alpha$, then $\mathbf{S}(\beta) = \beta \cup \{\beta\} \subseteq \alpha$, i.e. $\mathbf{S}(\beta) \leq \alpha < \mathbf{S}(\alpha)$.

(e) follows from (d) using contraposition: If $\neg(\beta < \alpha)$, then $\beta = \alpha$ or $\alpha < \beta$, implying $\mathbf{S}(\beta) = \mathbf{S}(\alpha)$ or $\mathbf{S}(\alpha) < \mathbf{S}(\beta)$, i.e. $\neg(\mathbf{S}(\beta) < \mathbf{S}(\alpha))$.

(f): Suppose $\alpha > 0$. Since $\beta < \alpha$ means $\beta \in \alpha$, α is an upper bound for α, showing $\sup \alpha \leq \alpha$. Since α is a set of ordinals by Prop. 3.30, Th. 3.36(b) yields $\sup \alpha = \bigcup \alpha$. If $\beta \in \alpha$, then (b) and (d) imply $\beta < \mathbf{S}(\beta) \leq \alpha$. If α is a limit ordinal, then $\mathbf{S}(\beta) < \alpha$, i.e. $\beta \in \mathbf{S}(\beta) \in \alpha$ and $\beta \in \bigcup \alpha$, proving $\alpha = \sup \alpha$. If α is a successor ordinal, then there exists $\beta \in \alpha$ with $\mathbf{S}(\beta) = \alpha$. We still show $\beta = \max \alpha$: If $\beta < \gamma \in \mathbf{ON}$, then, by (d), $\alpha = \mathbf{S}(\beta) \leq \gamma$, showing $\gamma \notin \alpha$. Since $\beta \in \alpha$, this shows $\beta = \max \alpha = \sup \alpha$. ∎

In Th. 3.44 below, we will show that, up to isomorphism, ordinals are the only strictly well-ordered sets. While we are mostly interested in order isomorphisms, it seems to make sense to introduce homomorphism for relations in general:

Definition 3.39. Let A, B be sets, let R be a relation on A, and let S be a relation on B. A function $f : A \longrightarrow B$ is called a *homomorphism* between (A, R) and (B, S) if, and only if,

$$\underset{x,y \in A}{\forall} \Big(x \, R \, y \;\Rightarrow\; f(x) \, S \, f(y) \Big). \tag{3.20}$$

If f is a homomorphism, then it is called *monomorphism* if, and only if, it is injective; *epimorphism* if, and only if, it is surjective; *isomorphism* if, and only if, it is bijective and $f^{-1} : B \longrightarrow A$ is a homomorphism as well; *endomorphism* if, and only if, $(A, R) = (B, S)$; *automorphism* if, and only if, it is both endomorphism and isomorphism. Moreover, (A, R) and (B, S) are called *isomorphic* (denoted $(A, R) \cong (B, S)$) if, and only if, there exists an isomorphism $f : A \longrightarrow B$. In this case, we also write $f : (A, R) \cong (B, S)$.

Lemma 3.40. *Let A, B be sets with total orders, both denoted by $\leq$ and the respective corresponding strict total orders both denoted by $<$. Given a function $f : A \longrightarrow B$, the following statements are equivalent:*

(i) *f is an isomorphism with respect to the total orders, i.e.* $f : (A, \le) \cong (B, \le)$.

(ii) *f is an isomorphism with respect to the strict total orders, i.e.* $f : (A, <) \cong (B, <)$.

(iii) *f is strictly isotone and surjective.*

Proof. Exercise. ∎

Definition 3.41. Let R be a relation on a set A. Using the notation of Def. and Rem. 3.5, we define

$$\forall_{a \in A} \quad a_\downarrow := \text{pred}(A, a) := \text{pred}(A, a, R) := R^{-1}(\{a\}) = \{x \in A : x\, R\, a\},$$

where we use the notation $\text{pred}(A, a)$ and $a_\downarrow$ if R or both R and A are understood. One can think of $\text{pred}(A, a, R)$ as the set of *predecessors* of a in A with respect to the relation R (which is especially useful, if R constitutes an order relation on A). If R well-orders A, then one can also think of $\text{pred}(A, a, R)$ as an *initial segment* of A with respect to the well-order.

Lemma 3.42. *Isomorphisms between well-ordered sets map initial segments to initial segments: If A, B are sets with strict well-orders, both denoted by $<$, and $f : (A, <) \cong (B, <)$ is an isomorphism, then*

$$f : (A, <) \cong (B, <) \quad \Rightarrow \quad \forall_{a \in A} \; f(a_\downarrow) = (f(a))_\downarrow.$$

Proof. If $y \in f(a_\downarrow)$, then there exists $x \in A$ with $x < a$ such that $y = f(x)$. Then, as f is strictly isotone by Lem. 3.40, $y = f(x) < f(a)$, i.e. $y \in (f(a))_\downarrow$, proving $f(a_\downarrow) \subseteq (f(a))_\downarrow$. We can now apply what we just proved with a replaced by $f(a)$ and f replaced by f^{-1} to obtain $f^{-1}\big((f(a))_\downarrow\big) \subseteq (f^{-1}(f(a)))_\downarrow = a_\downarrow$. Applying f to both sides of this inclusion yields $(f(a))_\downarrow \subseteq f(a_\downarrow)$, thereby completing the proof of the lemma. ∎

In Th. 4.4 and Th. 4.10 below, we will justify the proof method of *induction* on the set of natural numbers and, subsequently, we will generalize induction proofs such that they can be applied on general well-ordered sets and even on well-ordered classes (like **ON**) and still more general ojects. The basic idea of induction proofs is as follows: To proof an assertion $P(x)$ holds for all $x \in \mathbf{C}$, $\mathbf{C}$ being a suitable class, one first establishes that $P(x)$ holds for all "small" $x \in \mathbf{C}$, then assumes the existence of a smallest $x \in \mathbf{C}$ with $\neg P(x)$, showing this to provide a contradiction. We will see first examples of this strategy in the proofs of Prop. 3.43 and Th. 3.44 below.

Proposition 3.43. *If $\alpha, \beta \in \mathbf{ON}$ and $f : (\alpha, <) \cong (\beta, <)$, then $\alpha = \beta$ and $f = \text{Id}_\alpha$ (in particular, the identity is the unique automorphism on an ordinal).*

Proof. If $\alpha = 0$ or $\beta = 0$, then there is nothing to prove. Thus, let $\xi \in \alpha$. Then $f(\xi) \in \beta$. Since $\xi \in \mathbf{ON}$, we have $\xi = \xi_\downarrow$ and Lem. 3.42 implies

$$f(\xi) = f(\xi_\downarrow) = (f(\xi))_\downarrow = \{f(\mu) : \mu < \xi\} = \{f(\mu) : \mu \in \xi\}. \tag{3.21}$$

Now let $X := \{\xi \in \alpha : f(\xi) \neq \xi\}$ and, seeking a contradiction, assume $X \neq \emptyset$. Since $<$ is a strict well-order on α, there exists $m := \min X \in X$. Thus, for each $\mu \in m$, we have $\mu < m$ and $f(\mu) = \mu$, implying

$$f(m) \overset{(3.21)}{=} \{f(\mu) : \mu \in m\} = \{\mu \in \alpha : \mu \in m\} = m,$$

in contradiction to $m \in X$. In consequence, we have shown $X = \emptyset$ and $f = \mathrm{Id}_\alpha$. ■

Theorem 3.44. *If A is a set and $<$ is a strict well-order on A, then there exists a unique $\alpha \in \mathbf{ON}$ such that $(A, <) \cong (\alpha, \in)$ (we then define* $\mathrm{type}(A) := \mathrm{type}(A, <) := \alpha$ *and call α the* order type *of the strict well-order $(A, <)$; we write* $\mathrm{type}(A)$*, if the strict well-order $<$ on A is understood). Moreover, the isomorphism $f : (A, <) \cong (\alpha, \in)$ is unique.*

Proof. Uniqueness of α is clear due to Prop. 3.43. If $f, g : (A, <) \cong (\alpha, \in)$ are both isomorphisms, then Prop. 3.43 yields

$$\mathrm{Id}_\alpha = f \circ g^{-1} \quad \Rightarrow \quad g = f,$$

proving uniqueness of the isomorphism f. It remains to prove existence. The idea is to show that the theorem's claim holds for each initial segment of A and then, in consequence, for A. If $A = \emptyset$, then the empty function $f := \emptyset$ provides the isomorphism $f : (A, <) \cong (0, \in)$ (recall $0 = \emptyset$ as well). Thus, we now assume $A \neq \emptyset$ and we call $a \in A$ *good* if, and only if,

$$\underset{f(a) := \xi \in \mathbf{ON}}{\exists} \quad (a_\downarrow, <) \cong (\xi, \in).$$

Letting $G := \{a \in A : a \text{ good}\}$, we know $G \neq \emptyset$, since, for $m := \min A$, we have $m_\downarrow = \emptyset$ and $f(m) = 0$. Due to the uniqueness of $f(a)$ for each $a \in A$, we can use Axiom 5 (replacement) to obtain

$$\underset{B}{\exists} \ \underset{a \in G}{\forall} \ \underset{\xi \in B}{\exists} \ \Big(\xi \in \mathbf{ON} \ \wedge \ (a_\downarrow, <) \cong (\xi, \in)\Big),$$

justifying the function definition $f : G \longrightarrow B$, $a \mapsto f(a)$. Let f_a denote the corresponding unique isomorphism $f_a : a_\downarrow \longrightarrow f(a)$. If $c \in a_\downarrow$, then $g_c := f_a{\restriction}_{c_\downarrow} : c_\downarrow \longrightarrow g_c(c_\downarrow)$ is surjective and strictly isotone (as f_a is strictly isotone), implying g_c to be an isomorphism by Lem. 3.40(iii). Moreover, Lem. 3.42 yields $g_c(c_\downarrow) = f_a(c_\downarrow) = (f_a(c))_\downarrow = f_a(c)$, as $f_a(c)$ is an ordinal. The uniqueness of isomorphisms now implies $g_c = f_c$. Thus, we have shown

$$\underset{a\in G}{\forall}\ \underset{c\in a_\downarrow}{\forall}\ \Big(c \in G \ \wedge\ f_c = f_a{\restriction}_{c_\downarrow}\ \wedge f(c) = f_a(c)\Big). \tag{3.22}$$

Thus, $c, a \in G$ with $c < a$ implies $f(c) = f_a(c) \in f(a)$, showing $f : G \longrightarrow f(G) \subseteq \mathbf{ON}$ to be strictly isotone. As it is also surjective, it is an isomorphism by Lem. 3.40(iii). We finish the proof by showing $f(G) \in \mathbf{ON}$ and $G = A$. To verify $f(G)$ being transitive, consider $a \in G$ and $\nu \in f(a)$. Since $f_a : a_\downarrow \cong f(a)$, there exists $c \in a_\downarrow$ with $\nu = f(c)$. Then $c \in G$ (cf. (3.22)) and $\nu = f(c) \in f(G)$, showing $f(G)$ to be transitive and, thus, $f(G) \in \mathbf{ON}$ by Cor. 3.35(b). Finally, seeking a contradiction, assume $G \neq A$ and let $m := \min(A \setminus G)$. Then $m_\downarrow = G$: For $a \in m_\downarrow$, one has $a < m$ and, thus, $a \in G$ (since $m := \min(A \setminus G)$), showing $m_\downarrow \subseteq G$; conversely, if $a \in G$, then $a < m$ (otherwise $m < a$, implying $m \in G$ by (3.22)) and, thus, $a \in m_\downarrow$, showing $G \subseteq m_\downarrow$. Now $(m_\downarrow, <) = (G, <) \cong (f(G), \in)$, i.e. $m \in G$ in contradiction to $m \in A \setminus G$. Summarizing, we have proved $f : (A, <) \cong (\alpha, \in)$ with $\alpha := f(G)$. ■

4 Infinity

4.1 Natural Numbers

The following axiom of infinity guarantees the existence of infinite sets (e.g., it will allow us to define the set of natural numbers $\mathbb{N}$, which is infinite by Th. 4.13 below).

Axiom 6 *Infinity:*

$$\underset{X}{\exists}\ \left(0 \in X\ \wedge\ \underset{x\in X}{\forall}\ \big(x \cup \{x\} \in X\big)\right). \tag{4.1}$$

Thus, the infinity axiom states the existence of a set X containing $\emptyset$ (identified with the number 0), and, for each of its elements x, its successor $\mathbf{S}(x) = x \cup \{x\}$.

Example 4.1. We would like to check which of our toy models $M_1, \dots, M_{10}$ of Def. 2.1 satisfy Axiom 6 (one might expect that the answer is "none", since the models all are finite, however M_{10} will show that Axiom 6 does not guarantee the existence of infinite sets in the absence of comprehension). There is a slight complication arising from the fact that the formulation of (4.1) already makes use of Axiom 1 (extensionality) and Axiom 4 (union). Therefore, for the purpose of this example only, we replace (4.1) by

$$\underset{X}{\exists}\ \left(\left(\underset{Y\in X}{\exists}\ \underset{y}{\forall}\ \ y \notin Y\right)\ \wedge\ \underset{x\in X}{\forall}\ \underset{Z\in X}{\exists}\ \underset{u}{\forall}\ \Big(u \in Z\ \Leftrightarrow\ \big(u = x\ \vee\ u \in x\big)\Big)\right). \tag{4.2}$$

Indeed, each $M_1, \dots, M_9$ violates (4.2): In D_1, there exists no set containing a; in D_2, D_3, and D_4 there exists no empty set. To abbreviate the following arguments, we say that X is x-bad, if $x \in X$ and there does not exist $Z \in X$ with $x \in Z$. Note that X violates (4.2), if there exists x such that X is x-bad. In D_5, a is empty and b, c are both a-bad; in D_6, a is empty, b is a-bad, c is b-bad, and d is c-bad; in D_7, a is empty, b is a-bad, c is b-bad; in D_8, b is empty, c is b-bad; in D_9, a is empty, b is a-bad, c is b-bad, d is c-bad, e is c-bad. In M_{10}, (4.2) does hold: Indeed, (4.2) is satisfied with $X := b$: b contains the empty set a and the second part of (4.2) is also satisfied: For both $x := a$ and $x := b$, one can choose $Z := b$, since both a and b are in b, and each element of either a (there is none) or b (namely a and b) are also in b. Once again, this example shows that Axiom 6 does not guarantee the existence of infinite sets in the absence of comprehension.

We summarize the toy models' properties we found so far in the following table:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T
¬(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F
Axiom 3 (Pairing)	F	T	F	F	F	F	F	F	F	T
Axiom 4 (Union)	T	T	T	T	T	T	T	T	F	T
Axiom 5 (Replacement)	T	T	T	F	F	F	F	F	F	T
Axiom 6 (Infinity)	F	F	F	F	F	F	F	F	F	T

—

We now proceed to define the natural numbers:

Definition 4.2. An ordinal n is called a *natural number* if, and only if,

$$n \neq 0 \ \wedge \ \forall_{m \in \mathbf{ON}} \Big(m \leq n \ \Rightarrow \ m = 0 \ \vee \ m \text{ is successor ordinal} \Big).$$

Proposition 4.3. *If $n = 0$ or n is a natural number, then $\mathbf{S}(n)$ is a natural number and every element of n is a natural number or 0.*

Proof. Suppose n is 0 or a natural number. If $m \in n$, then m is an ordinal by Prop. 3.30. Suppose $m \neq 0$ and $k \in m$. Then $k \in n$, since n is transitive. Since n is a natural number, $k = 0$ or k is a successor ordinal. Thus, m is a natural number. It remains to show that $\mathbf{S}(n)$ is a natural number. By definition, $\mathbf{S}(n) = n \cup \{n\} \neq 0$. Moreover, $\mathbf{S}(n) \in \mathbf{ON}$ by Prop. 3.38(a), and, thus, $\mathbf{S}(n)$ is a successor ordinal. If $m \in \mathbf{S}(n)$, then $m \leq n$, implying $m = 0$ or m is a successor ordinal, completing the proof that $\mathbf{S}(n)$ is a natural number. ■

Theorem 4.4 (Principle of (Ordinary) Induction)**.** *If X is a set satisfying*

$$0 \in X \ \wedge \ \forall_{x\in X} \quad \mathbf{S}(x) \in X, \tag{4.3}$$

then X contains 0 and all natural numbers.

Proof. Let X be a set satisfying (4.3). Then $0 \in X$ is immediate. Let n be a natural number and, seeking a contradiction, assume $n \notin X$. Consider $N := \mathbf{S}(n) \setminus X$. According to Prop. 4.3, $\mathbf{S}(n)$ is a natural number and all nonzero elements of $\mathbf{S}(n)$ are natural numbers. Since $N \subseteq \mathbf{S}(n)$ and $0 \in X$, $0 \notin N$ and all elements of N must be natural numbers. As $n \in N$, $N \neq 0$. Since $\mathbf{S}(n)$ is well-ordered by $\in$ and $0 \neq N \subseteq \mathbf{S}(n)$, N must have a min $m \in N$, $0 \neq m \leq n$. Since m is a natural number, there must be k such that $m = \mathbf{S}(k)$. Then $k < m$, implying $k \notin N$. On the other hand

$$k < m \wedge m \leq n \ \Rightarrow \ k \leq n \ \Rightarrow \ k \in \mathbf{S}(n).$$

Thus, $k \in X$, implying $m = \mathbf{S}(k) \in X$, in contradiction to $m \in N$. This contradiction proves $n \in X$, thereby establishing the case. ■

Definition 4.5. If the set X is given by the axiom of infinity, then we use comprehension to define the set

$$\omega := \mathbb{N}_0 := \{n \in X : \ n = 0 \ \vee \ n \text{ is a natural number}\}$$

and note $\omega = \mathbb{N}_0$ to be unique by extensionality. We also denote $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$. In set theory, it is very common to use the symbol ω for the set $\mathbb{N}_0$.

Corollary 4.6. *$\omega = \mathbb{N}_0$ is the set of all natural numbers and 0, i.e.*

$$\forall_{n} \quad \Big(n \in \mathbb{N}_0 \ \Leftrightarrow \ n = 0 \ \vee \ n \textit{ is a natural number} \Big).$$

Proof. "$\Rightarrow$" is clear from Def. 4.5 and "$\Leftarrow$" is due to Th. 4.4. ■

Proposition 4.7. *$\omega = \mathbb{N}_0$ is the smallest limit ordinal.*

Proof. Exercise. ■

Lemma 4.8. *If $\alpha \in \mathbf{ON}$ and $\beta \in \alpha$, then*

$$\beta = \min(\alpha \setminus \beta).$$

Proof. Exercise. ■

Theorem 4.9. *For each $n \in \omega$, the set $N := N_n := \omega \setminus n$ (in particular $N_0 = \omega$ and $N_1 = \mathbb{N}$) satisfies the following so-called* Peano axioms P1 – P3*, where*

P1: *N contains a special element μ.*

P2: *There exists an injective function $S: N \longrightarrow N \setminus \{\mu\}$.*

P3: *If a subset A of N has the property that $\mu \in A$ and $S(k) \in A$ for each $k \in A$, then A is equal to N. Written as a formula, the third axiom reads:*

$$\forall_{A} \Big(A \subseteq N \ \wedge \ \mu \in A \ \wedge \ S(A) \subseteq A \ \Rightarrow \ A = N \Big).$$

Proof. Letting $\mu := n$, we know $\mu = \min N$ from Lem. 4.8. Define

$$S: N \longrightarrow N \setminus \{n\}, \quad S(k) := \mathbf{S}(k).$$

For P1 and P2, we have to show that, for each $k \in N$, one, indeed, has $S(k) \in N \setminus \{n\}$, and that $S(m) \neq S(k)$ for each $m, k \in N$, $m \neq k$. Let $k \in N$. Then $S(k) \in \omega$ by Prop. 4.3. Moreover, using Prop. 3.38(b), $j < n \leq k < S(k)$ for each $j \in n$, showing $S(k) \in N \setminus \{n\}$. If $m, k \in N$ with $m \neq k$, then $S(m) \neq S(k)$ is due to Prop. 3.38(d). To prove P3, suppose $A \subseteq N$ has the property that $n \in A$ and $S(k) \in A$ for each $k \in A$. We need to show $A = N$ (i.e. $N \subseteq A$, as $A \subseteq N$ is assumed). Let $X := A \cup n$. Then X satisfies (4.3), since $0 \in n$ or $0 = n \in A$ (i.e. $0 \in X$), $S(k) \in A \subseteq X$ for $k \in A$, and

$$k \in n \quad \Rightarrow \quad k < n \quad \overset{\text{Prop. 3.38(d)}}{\Rightarrow} \quad S(k) \leq n,$$

i.e. $S(k) \in n \subseteq X$ or $S(k) = n \in A \subseteq X$. As X satisfies (4.3), Th. 4.4 yields $\omega \subseteq X$. Thus, if $k \in N$, then $k \in X \setminus n = A$, showing $N \subseteq A$. ■

Theorem 4.10 ((Ordinary) Induction)**.** *Let (N, μ, S) be a so-called* Peano structure*, i.e. N is a set satisfying the Peano axioms* P1 – P3 *of Th.* 4.9 *with $\mu \in N$ and $S : N \longrightarrow N \setminus \{\mu\}$, and let ϕ be a set-theorectic formula. Then*

$$\forall_{k \in N} \phi \tag{4.4}$$

is true (i.e. ϕ holds for each $k \in N$) if, and only if, (a) *and* (b) *both hold, where*

(a) *$\phi(\mu)$ is true,*

(b) *$\forall_{k \in N} \Big(\phi(k) \ \Rightarrow \ \phi\big(S(k)\big) \Big)$ is true,* *using $\phi(k) := \phi$ and the notation $\phi(\mu)$ (resp. $\phi(S(k))$) means that, if k is free in ϕ, then one obtains $\phi(\mu)$ (resp. $\phi(S(k))$) by substituting the free k by μ (resp. by $S(k)$)*[8].

Proof. It is immediate that (4.4) implies (a) and (b). For the converse, let

$$A := \{k \in N : \phi\}.$$

We have to show $A = N$. Since $\mu \in A$ by (a), and

$$k \in A \Rightarrow \phi \overset{\phi(k)=\phi}{\Rightarrow} \phi(k) \overset{\text{(b)}}{\Rightarrow} \phi\big(S(k)\big) \Rightarrow S(k) \in A,$$

i.e. $S(A) \subseteq A$, the Peano axiom P3 implies $A = N$. ∎

In Th. 4.13 below, we want to show that ω (and $\mathbb{N}$ and all the sets N_n of Th. 4.9) are infinite. In preparation, we define the *cardinality* of sets:

Definition 4.11. (a) The sets A, B are defined to have the same *cardinality* or the same *size* (notation: $A \approx B$) if, and only if, there exists a bijective function $\varphi : A \longrightarrow B$. According to Th. 4.12 below, this defines an equivalence relation on every set of sets.

(b) The *cardinality* of a set A is $n \in \mathbb{N}$ (denoted $\#A = n$) if, and only if, there exists a bijective function $\varphi : A \longrightarrow \{1, \dots, n\}$. The cardinality of $\emptyset$ is defined as 0, i.e. $\#\emptyset := 0$. A set A is called *finite* if, and only if, there exists $n \in \omega = \mathbb{N}_0$ such that $\#A = n$; A is called *infinite* if, and only if, A is not finite, denoted $\#A = \infty$ (in the strict sense, this is an abuse of notation, since ∞ is *not* a cardinality – we will see subsequently that, in general, infinite sets do *not* have the same cardinality. If there exists a strict well-order $<$ on the set A, then we use Th. 3.44 to define its cardinality by

$$\#A := \min\big\{\alpha \in \mathbf{ON} : \alpha \approx \operatorname{type}(A, <)\big\} \tag{4.5}$$

(for finite sets, (4.5) is, clearly, consistent with the previous cardinality definition above; using the axiom of choice (AC), Axiom 9 of Sec. 7 below, we will prove that *every* set can be well-ordered; however, without AC, in general, not every infinite set is assigned a cardinality by (4.5)).

(c) The set A is called *countable* if, and only if, A is finite or A has the same cardinality as ω. Otherwise, A is called *uncountable*.

---

[8] As usual, when conducting an induction proof based on Th. 4.10, we call the proof of (a) the *base case* and the proof of (b) the *induction step*.

Theorem 4.12. *Let $\mathcal{M}$ be a set. Then the relation $\approx$ on $\mathcal{M}$, defined by*

$$A \approx B \;:\Leftrightarrow\; A \text{ and } B \text{ have the same cardinality},$$

constitutes an equivalence relation on $\mathcal{M}$.

Proof. According to Def. 3.10(f), we have to prove that $\approx$ is reflexive, symmetric, and transitive. According to Def. 4.11(a), $A \approx B$ holds for $A, B \in \mathcal{M}$ if, and only if, there exists a bijective function $f : A \longrightarrow B$. Thus, since the identity $\mathrm{Id} : A \longrightarrow A$ is bijective, $A \approx A$, showing $\approx$ is reflexive. If $A \approx B$, then there exists a bijective function $f : A \longrightarrow B$, and f^{-1} is a bijective function $f^{-1} : B \longrightarrow A$, showing $B \approx A$ and that $\approx$ is symmetric. If $A \approx B$ and $B \approx C$, then there are bijective functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Then, according to Prop. 3.8(h), the composition $(g \circ f) : A \longrightarrow C$ is also bijective, proving $A \approx C$ and that $\approx$ is transitive. ■

Theorem 4.13. *For each $n \in \omega$, the set $N := N_n := \omega \setminus n$ is infinite (in particular $N_0 = \omega$ and $N_1 = \mathbb{N}$ are infinite).*

Proof. Since $n \notin n$, we have $n \in N \neq \emptyset$. Thus, if N were finite, then there were a bijection $f : N \longrightarrow A_m := \{1, \dots, m\} = \{k \in \mathbb{N} : k \leq m\}$ for some $m \in \mathbb{N}$. However, we will show by induction on $m \in \mathbb{N}$ that there is no injective function $f : N \longrightarrow A_m$. Since $\mathbf{S}(n) \notin n$, we have $\mathbf{S}(n) \in N$. Thus, if $f : N \longrightarrow A_1 = \{1\}$, then $f(n) = f(\mathbf{S}(n)) = 1$, showing that f is not injective and proving the base case $m = 1$. For the induction step, we proceed by contraposition and show that the existence of an injective function $f : N \longrightarrow A_{\mathbf{S}(m)}$, $m \in \mathbb{N}$, implies the existence of an injective function $g : N \longrightarrow A_m$. To this end, let $m \in \mathbb{N}$ and $f : N \longrightarrow A_{\mathbf{S}(m)}$ be injective. If $\mathbf{S}(m) \notin f(N)$, then f itself is an injective function into A_m. If $\mathbf{S}(m) \in f(N)$, then there is a unique $a \in N$ such that $f(a) = \mathbf{S}(m)$. Define

$$g : N \longrightarrow A_m, \quad g(k) := \begin{cases} f(k) & \text{for } k < a, \\ f\big(\mathbf{S}(k)\big) & \text{for } a \leq k. \end{cases}$$

Then g is well-defined: If $k \in N$ and $a \leq k$, then $\mathbf{S}(k) \in N \setminus \{a\}$, and, since f is injective, g does, indeed, map into A_m. We verify g to be injective: If $k, l \in N$, $k < l$, then also $k < \mathbf{S}(l)$ and $\mathbf{S}(k) < \mathbf{S}(l)$ (by Prop. 3.38(c),(d)). In each case, $g(k) \neq g(l)$, proving g to be injective. ■

We conclude this section with some basic results on finite cardinalities:

Theorem 4.14. *If $m, n \in \mathbb{N}$ and the function $f : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$ is bijective, then $m = n$.*

Proof. We conduct the proof via induction on $m \in \mathbb{N}$.

Base Case ($m = 1$): If $m = 1$, then the surjectivity of f implies $n = 1$.

Induction Step: Let $m, n \in \mathbb{N}$ and assume $f : \{1, \dots, \mathbf{S}(m)\} \longrightarrow \{1, \dots, n\}$ to be bijective. Define the auxiliary function

$$g : \{1, \dots, \mathbf{S}(m)\} \longrightarrow \{1, \dots, n\}, \quad g(x) := \begin{cases} n & \text{for } x = \mathbf{S}(m), \\ f(\mathbf{S}(m)) & \text{for } x = f^{-1}(n), \\ f(x) & \text{otherwise.} \end{cases}$$

Then g is bijective by Prop. 3.8(h), since it is the composition $g = h \circ f$ of the bijective function f with the bijective function

$$h : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}, \quad h(x) := \begin{cases} n & \text{for } x = f(\mathbf{S}(m)), \\ f(\mathbf{S}(m)) & \text{for } x = n, \\ x & \text{otherwise.} \end{cases}$$

Thus, the restriction $g\restriction_{\{1,\dots,m\}}: \{1, \dots, m\} \longrightarrow N := \{1, \dots, n\} \setminus \{n\}$ must also be bijective. Since $N = \{1, \dots, k\}$ with $n = \mathbf{S}(k)$, the induction hypothesis yields $m = k$, which, in turn, implies $\mathbf{S}(m) = \mathbf{S}(k) = n$, as desired. ■

Corollary 4.15. *Let $m, n \in \mathbb{N}$ and let A be a set. If $\#A = m$ and $\#A = n$, then $m = n$.*

Proof. If $\#A = m$, then, according to Def. 4.11(b), there exists a bijective function $f : A \longrightarrow \{1, \dots, m\}$. Analogously, if $\#A = n$, then there exists a bijective function $g : A \longrightarrow \{1, \dots, n\}$. In consequence, we have the bijective function $(g \circ f^{-1}) : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$, such that Th. 4.14 yields $m = n$. ■

Theorem 4.16. *Let $A \neq \emptyset$ be a finite set.*

(a) *If $B \subseteq A$ with $A \neq B$, then B is finite with $\#B < \#A$.*

(b) *If $a \in A$, then $\#A = \mathbf{S}\big(\#\big(A \setminus \{a\}\big)\big)$.*

Proof. For $\#A = \mathbf{S}(n)$, $n \in \omega$, we use induction on n to prove (a) and (b) simultaneously, i.e. we show

$$\forall_{n \in \omega} \underbrace{\left(\#A = \mathbf{S}(n) \;\Rightarrow\; \forall_{B \subsetneq A} \; \forall_{a \in A} \; \#B \leq n \;\wedge\; \#\big(A \setminus \{a\}\big) = n \right)}_{\phi(n)}.$$

Base Case ($n = 0$): In this case, A has precisely one element, i.e. $B = A \setminus \{a\} = \emptyset$, and $\#\emptyset = 0$ proves $\phi(0)$.

Induction Step: For the induction hypothesis, we let $n \in \omega$ and assume $\phi(n)$ to be true, i.e. we assume (a) and (b) hold for each A with $\#A = \mathbf{S}(n)$. We have to prove $\phi(\mathbf{S}(n))$, i.e., we consider A with $\#A = \mathbf{S}(\mathbf{S}(n))$. From $\#A = \mathbf{S}(\mathbf{S}(n))$, we conclude the existence of a bijective function $\varphi : A \longrightarrow \{1, \dots, \mathbf{S}(\mathbf{S}(n))\}$. We have to construct a bijective function $\psi : A \setminus \{a\} \longrightarrow \{1, \dots, \mathbf{S}(n)\}$. To this end, set $k := \varphi(a)$ and define the auxiliary bijective function

$$f : \{1, \dots, \mathbf{S}(\mathbf{S}(n))\} \longrightarrow \{1, \dots, \mathbf{S}(\mathbf{S}(n))\}, \quad f(x) := \begin{cases} \mathbf{S}(\mathbf{S}(n)) & \text{for } x = k, \\ k & \text{for } x = \mathbf{S}(\mathbf{S}(n)), \\ x & \text{for } x \notin \{k, \mathbf{S}(\mathbf{S}(n))\}. \end{cases}$$

Then $f \circ \varphi : A \longrightarrow \{1, \dots, \mathbf{S}(\mathbf{S}(n))\}$ is bijective by Prop. 3.8(h), and

$$(f \circ \varphi)(a) = f(\varphi(a)) = f(k) = \mathbf{S}(\mathbf{S}(n)).$$

Thus, the restriction $\psi := (f \circ \varphi)\restriction_{A \setminus \{a\}}$ is the desired bijective function $\psi : A \setminus \{a\} \longrightarrow \{1, \dots, \mathbf{S}(n)\}$, proving $\#\big(A \setminus \{a\}\big) = \mathbf{S}(n)$. It remains to consider the strict subset B of A. Since B is a strict subset of A, there exists $a \in A \setminus B$. Thus, $B \subseteq A \setminus \{a\}$ and, as we have already shown $\#\big(A \setminus \{a\}\big) = \mathbf{S}(n)$, the induction hypothesis applies and yields B is finite with $\#B \leq \#\big(A \setminus \{a\}\big) = \mathbf{S}(n)$, i.e. $\#B \in \{0, \dots, \mathbf{S}(n)\}$, proving $\phi(\mathbf{S}(n))$, thereby completing the induction. ■

4.2 Transfinite Induction on Well-Founded Relations

After the proof of Lem. 3.42, it was already remarked that the principle of induction generalizes to quite general situations. Instead of the well-ordered set ω (or $\mathbb{N}$), where induction is based on the fact that every nonempty subset has a minimum, one can, in general, conduct induction proofs over (even proper) classes endowed with so-called *well-founded* relations, which, in general, do not need to be partial orders or satisfy tichotomy, but still have the property that every nonempty subset has a so-called *minimal element* (cf. Def. 4.19(a),(b) below): Important examples are **ON** with $\in$ (which is a strict well-order on **ON** by Th. 3.33) and even **V** with $\in$ (which is well-founded on **V**, if one assumes the axiom of foundation, Axiom 8 below, cf. Prop. 6.1(a)). Moreover, on ω (or $\mathbb{N}$), one can make use of the well-order to define functions via so-called *recursion*, cf., e.g., [Phi19a, Th. 3.7]. In a similar way to induction generalizing to classes with well-founded relations, definition via recursion generalizes to classes with well-founded relations as well.

In preparation for the main results of this section, we need to continue the presentation of classes that was started in Sec. 2.4: Many concepts are straight-forward to generalize from sets to classes, but one always needs to bear in mind that each class concept must be interpretable as an abbreviation for a concept formulatable without classes (e.g., it usually makes no immediate sense to speak of all classes with a certain property). For example, it makes sense to define the class

$$\mathbf{V} \times \mathbf{V} := \Big\{ p : \underset{x,y}{\exists} \; p = (x,y) \Big\},$$

and even general Cartesian products of classes (see Def. 4.17(c) below).

Definition 4.17. Let ϕ_A and ϕ_B be set-theoretic formulas, $\mathbf{A} := \{x : \phi_A\}$, $\mathbf{B} := \{x : \phi_B\}$.

(a) We call the class
$$\mathbf{A} \cap \mathbf{B} := \big\{ x : \; x \in \mathbf{A} \; \wedge \; x \in \mathbf{B} \big\}$$
the *intersection* of **A** and **B**.

(b) We call the class
$$\mathbf{A} \cup \mathbf{B} := \big\{ x : \; x \in \mathbf{A} \; \vee \; x \in \mathbf{B} \big\}$$
the *union* of **A** and **B**.

(c) The class
$$\mathbf{A} \times \mathbf{B} := \Big\{ p : \underset{x,y}{\exists} \, p = (x,y) \; \wedge \; x \in \mathbf{A} \; \wedge \; y \in \mathbf{B} \Big\}$$
is called the *Cartesian product* of **A** and **B**.

(d) We call **A** a *subclass* of **B** (denoted $\mathbf{A} \subseteq \mathbf{B}$) if, and only if,
$$\underset{x}{\forall} \; \Big(x \in \mathbf{A} \; \Rightarrow \; x \in \mathbf{B} \Big).$$

(e) We call a class **R** a *relation* over **A** and **B** if, and only if, **R** is a subclass of $\mathbf{A} \times \mathbf{B}$.

(f) All definitions for relations of Def. 3.6, Def. 3.7, Def. 3.10, Def. 3.13, and Def. 3.16, including the notions *injective*, *surjective*, *function*, *reflexive*, *transitive*, *partial order*, *total order*, *restriction* make sense for relations over classes and will subsequently be used in such situations, where it seems useful.

(g) Definition 3.19 of a (strict) well-order also makes sense for a relation **R** on a class **A**. However, it is emphasized that a relation **R** on a class **A** is called a *(strict) well-order* if, and only if, **R** is a (strict) total order and every nonempty sub*set* of **A** has a min with respect to **R**.

Remark 4.18. (a) While, structurally, definitions involving classes have the same form as analogous definitions for sets, logically they are considerably more sophisticated: For example, in Def. 4.17(d), if A and B are *sets*, then $A \subseteq B$ is equivalent to the *single* set-theoretic formula

$$\forall_x \Big(x \in A \;\Rightarrow\; x \in B \Big).$$

However, if $\mathbf{A}$ and $\mathbf{B}$ are allowed to be proper classes, then there is no single set-theoretic formula representing $\mathbf{A} \subseteq \mathbf{B}$ – it, rather, represents *infinitely many* different set-theoretic formulas: For example, if $\mathbf{A}$ is the class of singleton sets and $\mathbf{B}$ is the class of sets with at most two elements, i.e.

$$\mathbf{A} := \{x : \#x = 1\}, \quad \mathbf{B} := \{x : \#x \leq 2\},$$

then $\mathbf{A} \subseteq \mathbf{B}$ is equivalent to

$$\forall_x \Big(\#x = 1 \;\Rightarrow\; \#x \leq 2 \Big);$$

if $\mathbf{A}$ is the class of all limit ordinals and $\mathbf{B}$ is the class of all infinite sets, then $\mathbf{A} \subseteq \mathbf{B}$ is equivalent to the set-theoretic formula expressing "every limit ordinal is an infinite set". In consequence, in the literature, a definition involving classes, such as Def. 4.17(d), is often called a *definition scheme* in the so-called metatheory (the theory of set-theoretic formulas), in contrast to a definition via a single set-theoretic formula.

(b) Analogous to definitions involving proper classes representing definition schemes in the metatheory as described in (a), whereas theorems about sets constitute a single set-theoretic formula, theorems involving proper classes constitute *theorem schemes* in the metatheory, representing infinitely many statements, each expressible by a set-theoretic formula: For example, if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are classes, then

$$\big(\mathbf{A} \subseteq \mathbf{B} \;\wedge\; \mathbf{B} \subseteq \mathbf{C}\big) \quad \Rightarrow \quad \mathbf{A} \subseteq \mathbf{C} \tag{4.6}$$

is equivalent to

$$\forall_x \Big(\big((x \in \mathbf{A} \;\Rightarrow\; x \in \mathbf{B}) \;\wedge\; (x \in \mathbf{B} \;\Rightarrow\; x \in \mathbf{C})\big) \quad \Rightarrow \quad (x \in \mathbf{A} \;\Rightarrow\; x \in \mathbf{C}) \Big),$$

which is true by the transitivity of implication (Th. 1.7(a)). If $\mathbf{A}$ is the class of sets of singleton sets, $\mathbf{B}$ is the class of sets with at most two elements, and $\mathbf{C}$ is the class of sets with at most three elements, then (4.6) expresses the formula

$$\forall_x \Big(\big((\#x = 1 \;\Rightarrow\; \#x \leq 2) \wedge (\#x \leq 2 \;\Rightarrow\; \#x \leq 3)\big) \;\Rightarrow\; (\#x = 1 \;\Rightarrow\; \#x \leq 3) \Big).$$

On the other hand, if $\mathbf{A}$ is the class of rings, $\mathbf{B}$ is the class of principle ideal domains, and $\mathbf{C}$ is the class of fields, then (4.6) expresses the formula that states that "since each field is a principle ideal domain and each principle ideal domain is a ring, each field must be a ring".

—

In preparation for Th. 4.25 on transfinite induction on well-founded relations, we need to introduce such relations as well as a relation's transfinite closure.

Definition 4.19. Let $\mathbf{A}$ be a class and $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ a relation on $\mathbf{A}$.

(a) If $\mathbf{X}$ is a subclass of $\mathbf{A}$, then we call $m \in \mathbf{X}$ a *minimal (resp. maximal) element* (with respect to $\mathbf{R}$) in $\mathbf{X}$ if, and only if,

$$\neg \underset{x \in \mathbf{X}}{\exists}\; x \,\mathbf{R}\, m \qquad \left(\text{resp. } \neg \underset{x \in \mathbf{X}}{\exists}\; m \,\mathbf{R}\, x\right). \tag{4.7}$$

(b) We call $\mathbf{R}$ *well-founded* if, and only if, each nonempty sub*set* of $\mathbf{A}$ contains a minimal element.

(c) If $x, y \in \mathbf{A}$, then we call a function $\pi : D \longrightarrow \pi(D)$ an $\mathbf{R}$-path from x to y if, and only if,

$$\underset{n \in \mathbb{N}}{\exists} \left(\begin{array}{l} D = \mathbf{S}(n) = \{0, \dots, n\} \;\wedge\; \pi(D) \subseteq \mathbf{A} \\ \wedge\; \pi(0) = x \;\wedge\; \pi(n) = y \;\wedge\; \underset{i \in n}{\forall}\; \pi(i) \,\mathbf{R}\, \pi\big(\mathbf{S}(i)\big) \end{array} \right).$$

We call the relation $\mathbf{R}^*$ on $\mathbf{A}$, defined by

$$\underset{x,y \in \mathbf{A}}{\forall} \Big(x \,\mathbf{R}^*\, y \;\Leftrightarrow\; \underset{\pi}{\exists}\; (\pi \text{ is an } \mathbf{R}\text{-path from } x \text{ to } y) \Big),$$

the *transitive closure* of $\mathbf{R}$.

(d) $\mathbf{R}$ is called *acyclic* if, and only if, $\mathbf{R}^*$ is *irreflexive*, i.e.

$$\underset{x \in \mathbf{A}}{\forall} \;\neg\, x \,\mathbf{R}^*\, x.$$

Example 4.20. (a) In general, minimal and maximal elements are not unique (cf. Prop. 4.21(a) below). In fact, if $\mathbf{A}$ is a class and $R := \emptyset$ is the empty relation on $\mathbf{A}$, then every element of every subclass of $\mathbf{A}$ is both minimal and maximal.

(b) If $\mathbf{R}$ is a strict partial order on $\mathbf{A}$, $\emptyset \neq \mathbf{B} \subseteq \mathbf{A}$, and $m = \min \mathbf{B}$, then m is the unique minimal element in $\mathbf{B}$ (exercise).

(c) If $\mathbf{R}$ is a strict total order on $\mathbf{A}$, then $\mathbf{R}$ is well-founded if, and only if, $\mathbf{R}$ is a strict well-order (for example, $\in$ is well-founded on $\mathbf{ON}$ as well as on every $\alpha \in \mathbf{ON}$, but the usual $<$ is neither well-founded on $\mathbb{Z}$ nor on real intervals with more than one element): Indeed, since every minimum is a minimal element by (b), it is immediate that every strict well-order is well-founded. Conversely, if $\mathbf{R}$ is a strict total order and well-founded and $\emptyset \neq X$ is a subset of $\mathbf{A}$, then X has a minimal element $m \in X$. As a strict total order satisfies tichotomy, if $x \in X$ and $m \neq x$, then $m\,\mathbf{R}\,x$ must hold, showing $m = \min X$, i.e. $\mathbf{R}$ is a strict well-order.

(d) One has to use care when forming transitive closures after restricting relations to subclasses: Consider $\in$ as a relation on $\mathbf{V}$ and define $a := 0$ and $b := \{\{\{0\}\}\}$. Then $a \in^* b$, since $a \in \{0\} \in \{\{0\}\} \in b$ yields an $\in$-path from a to b in $\mathbf{V}$. However, if we let $A := \{a, b, \{0\}\}$ and $\in_A := \in\restriction_A$ (cf. Def. 3.7(b)), then $\neg(a\,(\in_A)^*\,b)$ since $\{\{0\}\} \notin A$, i.e. there is no $\in_A$-path from a to b.

Proposition 4.21. *Let $\mathbf{A}$ be a class and $\mathbf{R} \subseteq \mathbf{A} \times \mathbf{A}$ a relation on $\mathbf{A}$.*

(a) *Uniqueness of minimal elements holds if, and only if, $\mathbf{R}$ satisfies trichotomy, i.e.*

$$\mathbf{R} \text{ satisfies trichotomy}$$
$$\Leftrightarrow \quad \neg\left(\underset{\mathbf{B}\subseteq\mathbf{A}}{\exists} \quad \underset{m_1,m_2\in\mathbf{B}}{\exists} \quad \Big(m_1 \neq m_2 \ \wedge\ m_1, m_2 \text{ both minimal in } \mathbf{B}\Big)\right)$$

(note that the above is not a set-theoretic formula, if one allows $\mathbf{B}$ to stand for a proper class; it is then, rather, a theorem scheme in the metatheory – the statement is, actually, also valid, if $\mathbf{B}$ is only allowed to be a set, but, here, the restriction to sets seems artificial, since the proof yields the stated stronger version and we are dealing with theorem schemes, anyway).

(b) *The transitive closure $\mathbf{R}^*$ is transitive.*

(c) *If $\mathbf{R}$ is well-founded, then $\mathbf{R}$ is acyclic; if $R := \mathbf{R}$ is acyclic on the finite set $A := \mathbf{A}$, then R is well-founded (e.g. $\mathbb{Z}$ with its usual order shows that, on infinite sets, a relation can be acyclic without being well-founded (as $\mathbb{Z}$ has no minimal element)).*

(d) *If $\mathbf{R}$ is well-founded, then $\mathbf{R}^*$ is a strict partial order.*

(e) *If $R := \mathbf{R}$ is a strict partial order on the finite set $A := \mathbf{A}$, then R is well-founded (again, $(\mathbb{Z}, <)$ shows the result does not extend to infintie sets).*

(f) *If one can define a (class) function $\mathbf{f} : \mathbf{A} \longrightarrow \mathbf{ON}$ such that*

$$\underset{x,y\in\mathbf{A}}{\forall} \quad \Big(x\,\mathbf{R}\,y \ \Rightarrow\ \mathbf{f}(x) < \mathbf{f}(y)\Big), \tag{4.8}$$

then $\mathbf{R}$ is well-founded.

Proof. (a): If $\mathbf{R}$ satisfies trichotomy, $\mathbf{B} \subseteq \mathbf{A}$ and $m_1, m_2 \in \mathbf{B}$ with $m_1 \neq m_2$, then $m_1 \,\mathbf{R}\, m_2$ or $m_2 \,\mathbf{R}\, m_1$. If $m_1 \,\mathbf{R}\, m_2$, then m_2 is not minimal in $\mathbf{B}$; if $m_2 \,\mathbf{R}\, m_1$, then m_1 is not minimal in $\mathbf{B}$. Conversely, if $\mathbf{R}$ does not satisfy trichotomy, then there exist sets $m_1, m_2 \in \mathbf{A}$ such that

$$m_1 \neq m_2 \;\wedge\; \neg(m_1 \,\mathbf{R}\, m_2) \;\wedge\; \neg(m_2 \,\mathbf{R}\, m_1).$$

Thus, letting $B := \{m_1, m_2\}$, both m_1 and m_2 are minimimal elements in B.

(b): To show $\mathbf{R}^*$ is transitive, let $x, y, z \in \mathbf{A}$ such that $x \,\mathbf{R}^*\, y$ and $y \,\mathbf{R}^*\, z$. We have to show $x \,\mathbf{R}^*\, z$. From $x \,\mathbf{R}^*\, y$ and $y \,\mathbf{R}^*\, z$, we have $\mathbf{R}$-paths $\pi_1 : D_1 \longrightarrow \pi_1(D_1)$ and $\pi_2 : D_2 \longrightarrow \pi_2(D_2)$ with $m, n \in \mathbb{N}$, $D_1 = \mathbf{S}(m)$, $D_2 = \mathbf{S}(n)$, $\pi_1(0) = x$, $\pi_1(m) = y = \pi_2(0)$, $\pi_2(n) = z$. We need to show the existence of an $\mathbf{R}$-path $\pi : D \longrightarrow \pi(D)$, $D = \mathbf{S}(k)$, $k \in \mathbb{N}$, $\pi(0) = x$, and $\pi(k) = z$. We conduct the proof[9] via induction on $n \in \mathbb{N}$: If $n = 1$, then let $k := \mathbf{S}(m)$ and define

$$\pi : \mathbf{S}(k) \longrightarrow \pi\big(\mathbf{S}(k)\big) \subseteq \mathbf{A}, \quad \pi(i) := \begin{cases} \pi_1(i) & \text{for } i < k = \mathbf{S}(m), \\ \pi_2(1) & \text{for } i = k. \end{cases}$$

Then $\pi(0) = \pi_1(0) = x$, $\pi(k) = \pi_2(1) = z$, and π is an $\mathbf{R}$-path, since

$$\begin{aligned} i \in m \;&\Rightarrow\; \pi(i) = \pi_1(i) \,\mathbf{R}\, \pi_1\big(\mathbf{S}(i)\big) = \pi\big(\mathbf{S}(i)\big), \\ i = m \;&\Rightarrow\; \pi(i) = \pi_1(m) = y = \pi_2(0) \,\mathbf{R}\, \pi_2(1) = \pi(k) = \pi\big(\mathbf{S}(i)\big). \end{aligned}$$

Now assume there is $l \in \mathbb{N}$ such that $n = \mathbf{S}(l)$. By induction, there is an $\mathbf{R}$-path $\pi_0 : D_0 \longrightarrow \pi_0(D_0)$, with $D_0 = \mathbf{S}(n_0)$, $n_0 \in \mathbb{N}$, $\pi_0(0) = x$, $\pi_0(n_0) = \pi_2(l)$. We can now proceed analogous to the case $n = 1$: Let $k := \mathbf{S}(n_0)$ and define

$$\pi : \mathbf{S}(k) \longrightarrow \pi\big(\mathbf{S}(k)\big) \subseteq \mathbf{A}, \quad \pi(i) := \begin{cases} \pi_0(i) & \text{for } i < k = \mathbf{S}(n_0), \\ \pi_2(n) & \text{for } i = k. \end{cases}$$

Then $\pi(0) = \pi_0(0) = x$, $\pi(k) = \pi_2(n) = z$, and π is an $\mathbf{R}$-path, since

$$\begin{aligned} i \in n_0 \;&\Rightarrow\; \pi(i) = \pi_0(i) \,\mathbf{R}\, \pi_0\big(\mathbf{S}(i)\big) = \pi\big(\mathbf{S}(i)\big), \\ i = n_0 \;&\Rightarrow\; \pi(i) = \pi_0(n_0) = \pi_2(l) \,\mathbf{R}\, \pi_2\big(\mathbf{S}(l)\big) = \pi_2(n) = \pi(k) \overset{k=\mathbf{S}(n_0)}{=} \pi\big(\mathbf{S}(i)\big), \end{aligned}$$

thereby completing the induction and the proof of (b).

(c): If $\mathbf{R}$ is not acyclic, then there exists $x \in \mathbf{A}$ such that $x \,\mathbf{R}^*\, x$. Thus, there exists an $\mathbf{R}$-path $\pi : D \longrightarrow \pi(D)$, $D = \mathbf{S}(n)$ with $n \in \mathbb{N}$, such that $\pi(0) = \pi(n) = x$.

[9]The proof could be simplified if we had arithmetic on $\mathbb{N}$ already available. However, this would mean to, first, develop recursion on $\mathbb{N}$, which is not the route pursued in our treatment.

Then $X := \pi(D) = \{\pi(i) : 0 < i \in \mathbf{S}(n)\}$ does not have a minimal element: Indeed, if $0 < i \in \mathbf{S}(n)$, then, by Def. 4.2, there exists $j \in i$ such that $i = \mathbf{S}(j)$, implying $\pi(j) \,\mathbf{R}\, \pi\big(\mathbf{S}(j)\big) = \pi(i)$, showing $\pi(i)$ is not a minimal element of X. We leave the other direction for finite sets as an exercise.

(e): As a strict partial order, R is transitive and irreflexive (as it is asymmetric). Thus, R is acyclic. As A is finite, acyclic implies well-founded by (c).

(d): If $\mathbf{R}$ is well-founded, then $\mathbf{R}$ is acyclic by (c), i.e. $\mathbf{R}^*$ is irreflexive. As $\mathbf{R}^*$ is transitive by (b), it merely remains to verify that $\mathbf{R}^*$ is asymmetric. However, if there existed $x, y \in \mathbf{A}$ with $x \,\mathbf{R}^*\, y$ and $y \,\mathbf{R}^*\, x$, then transitivity implied $x \,\mathbf{R}^*\, x$, in contradiction to $\mathbf{R}^*$ being irreflexive.

(f): If $\mathbf{f} : \mathbf{A} \longrightarrow \mathbf{ON}$ satisfies (4.8) and $\emptyset \neq X$ is a nonempty subset of $\mathbf{A}$, then $\emptyset \neq \mathbf{f}(X) \subseteq \mathbf{ON}$ (note that X being a set implies $\mathbf{f}(X)$ to be a set by the axiom of replacement, Axiom 5). As $\mathbf{ON}$ is well-ordered, there exists $\alpha = \min \mathbf{f}(X)$. Then each $m \in \mathbf{f}^{-1}(\{\alpha\})$ is a minimal element of X: Indeed, if $x \in X$ with $x \,\mathbf{R}\, m$, then $\mathbf{f}(x) \in \mathbf{f}(X)$ and (4.8) implies $\mathbf{f}(x) < \mathbf{f}(m) = \alpha$, in contradiction to $\alpha = \min \mathbf{f}(X)$. ■

As mentioned before, Th. 4.25 below will be provided in a form that allows us to conduct induction proofs over well-founded relations that constitute proper classes. For this reason, we need another, somewhat technical preparation:

Definition 4.22. Let $\mathbf{R}$ be a relation on a class $\mathbf{A}$ and recall the notation for predecessors from Def. 3.41 (which extends to classes in the usual way): Letting, for each $a \in \mathbf{A}$, $a_\downarrow := \operatorname{pred}(\mathbf{A}, a, \mathbf{R}) := \{x \in \mathbf{A} : x \,\mathbf{R}\, a\}$, we call $\mathbf{R}$ *set-like* if, and only if, $a_\downarrow$ is a set for each $a \in \mathbf{A}$.

Example 4.23. (a) If the relation R on the class $\mathbf{A}$ is a set, then it is set-like, since each $\{x \in \mathbf{A} : x \,\mathbf{R}\, a\}$ is a set by replacement (Axiom 5).

(b) While the element relation $\in$ is a proper class on $\mathbf{V}$ (since $\mathbf{V}$ is a proper class), it is set-like due to the fact that

$$\underset{x \in \mathbf{V}}{\forall} \quad x_\downarrow = \{y \in \mathbf{V} : y \in x\} = x$$

(by the same argument, $\in$ is set-like on *every* class $\mathbf{A}$).

(c) As an example of a relation that is not set-like (even though it is well-founded and even a strict well-order, cf. Prop. 3.21(g),(i)), consider the lexicographic order $<$ on $\mathbf{ON} \times \mathbf{ON}$, defined by

$$(\alpha, \beta) < (\gamma, \delta) \quad :\Leftrightarrow \quad \alpha < \gamma \;\vee\; (\alpha = \gamma \wedge \beta < \delta):$$

For each $(\alpha, \beta) \in \mathbf{ON} \times \mathbf{ON}$ with $0 < \alpha$, the class $(\alpha, \beta)_\downarrow$ is a proper class, since the proper class $\{0\} \times \mathbf{ON}$ is a subclass: $\{0\} \times \mathbf{ON} \subseteq (\alpha, \beta)_\downarrow$

Proposition 4.24. *Let* $\mathbf{R}$ *be a set-like relation on a class* $\mathbf{A}$*. We define, for each* $n \in \omega$ *and each* $y \in \mathbf{A}$*, the class* $D_n(y)$ *(which will turn out to be a set) of all* $x \in \mathbf{A}$ *such that there exists an* $\mathbf{R}$*-path of* n *steps from* x *to* y*, i.e.*

$$D_0(y) := \{y\},$$
$$\underset{y\in\mathbf{A}}{\forall} \quad \underset{n\in\mathbb{N}}{\forall} \quad D_n(y) := \left\{ x \in \mathbf{A} : \underset{\pi:\, D\longrightarrow\pi(D)}{\exists} \begin{pmatrix} \pi \text{ is } \mathbf{R}\text{-path} \wedge D = \mathbf{S}(n) \\ \wedge\ \pi(0) = x \wedge \pi(n) = y \end{pmatrix} \right\}.$$

(a) *Using the notation* $a_\downarrow = \operatorname{pred}(\mathbf{A}, a, \mathbf{R})$ *for each* $a \in \mathbf{A}$*, we have*

$$\underset{a\in\mathbf{A}}{\forall} \quad a_\downarrow = \operatorname{pred}(\mathbf{A}, a, \mathbf{R}) = D_1(a).$$

(b) *For each* $n \in \omega$ *and each* $y \in \mathbf{A}$*,* $D_n(y)$ *is a set and, moreover,*

$$D_{\mathbf{S}(n)}(y) = \bigcup \mathcal{E}, \quad \text{where} \quad \mathcal{E} := \big\{D_n(z) : z \in D_1(y)\big\}. \tag{4.9}$$

(c) *For each* $a \in \mathbf{A}$*,* $\mathcal{F} := \{D_n(a) : n \in \mathbb{N}\}$ *is a set,* $\operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*) = \bigcup \mathcal{F}$ *is a set, and* $\mathbf{R}^*$ *is set-like.*

(d) *Defining*

$$\underset{x\in\mathbf{A}}{\forall} \quad d_x := \{x\} \cup \operatorname{pred}(\mathbf{A}, x, \mathbf{R}^*),$$

it holds that

$$\underset{a\in\mathbf{A}}{\forall} \quad \operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*) = \bigcup \{d_x : x \in a_\downarrow\}. \tag{4.10}$$

(e) *Here, we do not even need to assume* $\mathbf{R}$ *to be set-like. Instead, we assume* d *to be a* set *such that* $d \subseteq \mathbf{A}$*. Then*

$$\left(\underset{a\in d}{\forall} \ a_\downarrow \subseteq d \right) \quad \Rightarrow \quad \left(\underset{a\in d}{\forall} \ \operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*) \subseteq d \right).$$

Proof. (a) is immediate from the definition of $\mathbf{R}$-paths in Def. 4.19(c).

(b): We prove (b) via induction on $n \in \omega$: Let $y \in \mathbf{A}$. For $n = 0$, $D_0(y) = \{y\}$ is a set and

$$D_1(y) = \bigcup \big\{D_0(z) : z \in D_1(y)\big\} = \bigcup \big\{\{z\} : z \in D_1(y)\big\}$$

is true. Moreover, $D_1(y) = \operatorname{pred}(\mathbf{A}, y, \mathbf{R})$ is a set by our assumption that $\mathbf{R}$ be set-like. For the induction step, let $n \in \mathbb{N}$ and assume $D_n(z)$ to be a set for each $z \in \mathbf{A}$ by induction hypothesis. To show (4.9), let $x \in D_{\mathbf{S}(n)}(y)$. Then there exists an $\mathbf{R}$-path $\pi : D \longrightarrow \pi(D)$, $D = \mathbf{S}(\mathbf{S}(n))$ from x to y. Letting $z := \pi(n)$, $\pi_0 := \pi\restriction_{\mathbf{S}(n)}$ is an

$\mathbf{R}$-path from x to z, showing $x \in D_n(z)$, whereas $\pi_1 : 2 \longrightarrow \{z, y\}$, $\pi_1(0) := z = \pi(n)$, $\pi_1(1) := y = \pi(\mathbf{S}(n))$, is an $\mathbf{R}$-path from z to y, showing $z \in D_1(y)$ and $x \in \bigcup \mathcal{E}$. Conversely, if $x \in \bigcup \mathcal{E}$, then there exists $z \in D_1(y)$ such that $x \in D_n(z)$. Thus, there exists $\mathbf{R}$-paths $\pi_0 : \mathbf{S}(n) \longrightarrow \pi_0(\mathbf{S}(n))$ and $\pi_1 : 2 \longrightarrow \{z, y\}$ such that $\pi_0(0) = x$, $\pi_0(n) = z = \pi_1(0)$, and $\pi_1(1) = y$. Then

$$\pi : D \longrightarrow \pi(D), \quad D := \mathbf{S}(\mathbf{S}(n)), \quad \pi(u) := \begin{cases} \pi_1(u) & \text{for } u \in \mathbf{S}(n), \\ y & \text{for } u = \mathbf{S}(n), \end{cases}$$

defines an $\mathbf{R}$-path from x to y, proving $x \in D_{\mathbf{S}(n)}(y)$ and completing the proof of (4.9). Since $D_1(y)$ is a set and each $D_n(z)$ is a set, $\mathcal{E}$ is a set by replacement (Axiom 5) and, thus, $D_{\mathbf{S}(n)}(y)$ is a set via the union axiom (Axiom 4), completing the induction proof that each $D_n(y)$ is a set.

(c): Since each $D_n(a)$ is uniquely defined by a and n, we obtain $\mathcal{F} := \{D_n(a) : n \in \mathbb{N}\}$ to be a set by replacement. Then, as before, $\operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*) = \bigcup \mathcal{F}$ by the union axiom, proving $\mathbf{R}^*$ to be set-like.

(d): Exercise.

(e): From the proof of (b), if each $D_1(y) \subseteq d$, then, inductively, each $D_n(y) \subseteq d$ as well (in particular, as subsets of d, the $D_n(y)$ are sets as well), implying $\operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*) = \bigcup \{D_n(a) : n \in \mathbb{N}\} \subseteq d$ as claimed. ■

Theorem 4.25 (Principle of Transfinite Induction on Well-Founded Relations)**.** *Suppose that* $\mathbf{R}$ *is a well-founded and set-like relation on the class* $\mathbf{A}$*. If* $\mathbf{X}$ *is a nonempty subclass of* $\mathbf{A}$*, then* $\mathbf{X}$ *has a minimal element.*

Proof. If $a \in \mathbf{X}$, then, as $\mathbf{R}$ is set-like, $\operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*)$ is a set by Prop. 4.24(c). Then $Y := \{a\} \cup (\mathbf{X} \cap \operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*))$ is a set as well and, as $a \in Y$, Y is nonempty. Since $\mathbf{R}$ is well-founded, Y must have a minimal element. We show that each minimal element y of Y is also a minimal element of $\mathbf{X}$: As $a \in \mathbf{X}$, the definition of Y yields that $y \in Y$ implies $y \in \mathbf{X}$. If $x \in \mathbf{X}$ with $x \, \mathbf{R} \, y$, then $x \, \mathbf{R}^* \, y$ and $y \, \mathbf{R}^* \, a$, implying $x \, \mathbf{R}^* \, a$, i.e. $x \in Y$, in contradiction to y being minimal in Y. ■

Corollary 4.26 (Induction on Well-Founded Relations)**.** *Let* $\mathbf{R}$ *be a well-founded and set-like relation on the class* $\mathbf{A}$ *and* ϕ *a set-theoretic formula. Consider the statement*

$$\underset{x \in \mathbf{A}}{\forall} \phi. \tag{4.11}$$

If ϕ *in* (4.11) *contains* x *as a free variable, than formulas of the form* $\phi(\dots)$ *below mean that* x *in* ϕ *is replaced by the expression between the parentheses.*

(a) Transfinite Induction on Well-Founded Relations*: Statement* (4.11) *is true if, and only if,* (i) *and* (ii) *both hold, where*

(i) *$\phi(m)$ is valid for each minimal element m of $\mathbf{A}$.*

(ii) *Using the notation of Def.* 3.41*:*

$$\underset{a\in\mathbf{A}}{\forall}\quad\left(\left(a \textit{ not minimal in } \mathbf{A} \;\wedge\; \underset{p\in a_\downarrow}{\forall}\phi(p)\right)\quad\Rightarrow\quad\phi(a)\right).$$

Moreover, (a) *remains true if* (ii) *is replaced by*

(ii)′

$$\underset{p,a\in\mathbf{A}}{\forall}\quad\Big(\big(\phi(p)\;\wedge\;p\,\mathbf{R}\,a\big)\;\;\Rightarrow\;\;\phi(a)\Big).$$

(b) Transfinite Induction on the Ordinals*: If $(\mathbf{A},\mathbf{R}) = (\mathbf{ON},\in)$, then* (4.11) *is true if, and only if,* (i),(ii),(iii) *hold, where*

(i) *$\phi(0)$ is valid.*

(ii)

$$\underset{\alpha\in\mathbf{ON}}{\forall}\quad\Big(\phi(\alpha)\;\Rightarrow\;\phi\big(\mathbf{S}(\alpha)\big)\Big).$$

(iii)

$$\underset{\alpha\in\mathbf{ON}}{\forall}\quad\left(\left(\alpha \textit{ is limit ordinal }\;\wedge\;\underset{\beta\in\alpha}{\forall}\phi(\beta)\right)\quad\Rightarrow\quad\phi(\alpha)\right).$$

(c) Bounded Transfinite Induction on the Ordinals*: Suppose $\gamma\in\mathbf{ON}$, $\gamma>0$. If $(\mathbf{A},\mathbf{R}) = (\gamma,\in)$, then* (4.11) *is true if, and only if,* (i),(ii),(iii) *hold, where*

(i) *$\phi(0)$ is valid.*

(ii)

$$\underset{\alpha\in\gamma}{\forall}\quad\Big(\big(\mathbf{S}(\alpha)\in\gamma\;\wedge\;\phi(\alpha)\,\big)\;\;\Rightarrow\;\;\phi\big(\mathbf{S}(\alpha)\big)\Big).$$

(iii)

$$\underset{\alpha\in\gamma}{\forall}\quad\left(\left(\alpha \textit{ is limit ordinal }\;\wedge\;\underset{\beta\in\alpha}{\forall}\phi(\beta)\right)\quad\Rightarrow\quad\phi(\alpha)\right).$$

(d) Finite Induction*: If $\mathbf{A} = A$, where A is a finite set and $f:\{1,\dots,n\}\longrightarrow A$ is bijective with $n\in\mathbb{N}$, then* (4.11) *is true if, and only if,* (i),(ii) *hold, where*

(i) *$\phi\big(f(1)\big)$ is valid.*

(ii)

$$\forall_{i\in\{1,\dots,n\}} \Big(\big(i < n \wedge \phi\big(f(i)\big)\ \big) \ \Rightarrow \ \phi\Big(f\big(\mathbf{S}(i)\big)\Big)\Big).$$

When conducting proofs based on (a) – (d) *above, in generalization of the situation one has for induction proofs on Peano structures (cf. Th.* 4.10*), one often still calls proofs of cases* (i) *the* base case *and proofs of the remaining cases the* induction step.

Proof. (a): It is immediate that (4.11) implies (i), (ii), and (ii)′. Next, we note that (ii)′ implies (ii): Assume (ii)′ and $a \in \mathbf{A}$. We have to show that, if a is not minimal and $\phi(p)$ holds for each $p \in a_\downarrow$, then $\phi(a)$ holds as well. If a is not minimal, then there exists $p \in a_\downarrow$. Then $\phi(p)$ holds as well as $p\,\mathbf{R}\,a$ and (ii)′ implies $\phi(a)$. It remains to show that (i) and (ii) imply (4.11). Seeking a contradiction, assume (i) and (ii) and $\mathbf{X} := \{x \in \mathbf{A} : \neg\phi(x)\} \neq \emptyset$. By Th. 4.25, $\mathbf{X}$ contains a minimal element $a \in \mathbf{X}$. According to (i), a is not minimal in $\mathbf{A}$, i.e. there exists $p \in a_\downarrow$. However, if $p \in a_\downarrow$, then $p \notin \mathbf{X}$ (as a is minimal in $\mathbf{X}$), i.e. $\phi(p)$ is true. Thus, (ii) applies, implying $\phi(a)$ to hold, in contradiction to $a \in \mathbf{X}$.

(b), (c), (d): Exercise. ■

While induction proofs are often particularly useful to establish results in connexion with recursion, and we will see many examples in Sec. 4.3 below, the following Ex. 4.27 will provide a first application of transfinite induction on **ON** via Cor. 4.26(b) and the proof of Prop. 4.28(b) will provide a first example of a finite induction via Cor. 4.26(d).

Example 4.27. As an example of transfinite induction on **ON** (an application of Cor. 4.26(b)), we will show that each infinite ordinal $\alpha \geq \omega$ can be decomposed into the sum of a limit ordinal λ and a finite ordinal n, i.e. $\alpha = \lambda + n$. As we do not have ordinal arithmetic available, yet (since we will need to develop the theory of recursion first), we need to state our assertion in the less elegant form

$$\forall_{\alpha\in\mathbf{ON}} \underbrace{\left(\alpha \geq \omega \ \Rightarrow \ \exists_{\substack{\lambda \text{ limit ordinal},\\ n\in\omega,\\ A}} \ \exists_{f:\,A\to n} \Big(f \text{ is bijective} \ \wedge \ A := \{\beta \in \alpha : \lambda < \beta\}\Big)\right)}_{=:\,\phi(\alpha)}. \tag{4.12}$$

We prove (4.12) by applying Cor. 4.26(b): We need to establish Cor. 4.26(b)(i),(ii),(iii). (i): $\phi(0)$ holds, since $0 < \omega$. (ii): Let $\alpha \in \mathbf{ON}$ and assume $\phi(\alpha)$. If $\alpha < \omega$, then $\mathbf{S}(\alpha) < \omega$ as well, proving $\phi\big(\mathbf{S}(\alpha)\big)$. If $\alpha \geq \omega$, then $\phi(\alpha)$ yields the existence of a limit ordinal λ, of an $n \in \omega$, and of a bijective $f : A \longrightarrow n$, $A = \{\beta \in \alpha : \lambda < \beta\}$. Letting

$A_0 := \{\beta \in \mathbf{S}(\alpha) :\ \lambda < \beta\} = \{\alpha\} \cup A$, the function

$$f_0 :\ A_0 \longrightarrow \mathbf{S}(n), \quad f_0(\gamma) := \begin{cases} f(\gamma) & \text{for } \gamma \in A, \\ n & \text{for } \gamma = \alpha, \end{cases}$$

is, clearly, bijective, proving $\phi(\mathbf{S}(\alpha))$ and (ii). (iii): If α is a limit ordinal, then we can let $\lambda := \alpha$ and $n := 0$: Then $A = \emptyset$ and $f := \emptyset$ is bijective between $\emptyset$ and 0, i.e. $\phi(\alpha)$ holds. Having verified Cor. 4.26(b)(i),(ii),(iii), the proof of (4.12) is complete.

Proposition 4.28. **(a)** *If A is a set and $\mathcal{R}$ is a set of transitive relations on A, then $\bigcap \mathcal{R}$ is a transitive relation on A.*

(b) *Let A be a set and assume $\mathcal{P}(A \times A) := \{S :\ S \subseteq A \times A\}$ to be a set as well (this holds, e.g., when assuming the power set axiom, Axiom 7 below). If R is a relation on A, then the transitive closure R^* is the intersection of all transitive relations on A that contain R as a subset.*

Note that the above results do not extend to proper classes, as there is no direct way to form the intersection of all classes with a certain property.

Proof. (a): Suppose $\mathcal{R}$ is a set of transitive relations on the set A and let $x, y, z \in A$. Then

$$(x,y) \in \bigcap \mathcal{R} \ \wedge\ (y,z) \in \bigcap \mathcal{R} \quad \Rightarrow \quad \forall_{R \in \mathcal{R}} \Big((x,y) \in R \ \wedge\ (y,z) \in R \Big)$$
$$\stackrel{\text{each } R \text{ tr.}}{\Rightarrow} \quad \forall_{R \in \mathcal{R}} \ (x,z) \in R \quad \Rightarrow \quad (x,z) \in \bigcap \mathcal{R},$$

showing $\bigcap \mathcal{R}$ to be transitive.

(b): Let R be a relation on the set A and define

$$\mathcal{R} := \big\{ S \in \mathcal{P}(A \times A) :\ R \subseteq S \ \wedge\ S \text{ is transitive} \big\}$$

(which is a set by our assumption plus comprehension). We have to show $\bigcap \mathcal{R} = R^*$. Since $R \subseteq R^*$ and R^* is transitive by Prop. 4.21(b), we already know $R^* \in \mathcal{R}$ and $\bigcap \mathcal{R} \subseteq R^*$. To prove the remaining inclusion $R^* \subseteq \bigcap \mathcal{R}$, let $S \in \mathcal{R}$, i.e. $R \subseteq S$ and S is transitive. Suppose $(x,y) \in R^*$. Then there exists an R-path $\pi :\ D \longrightarrow \pi(D)$, $D = \mathbf{S}(n)$, $n \in \mathbb{N}$, $\pi(0) = x$, $\pi(n) = y$. We show

$$\forall_{i \in \mathbf{S}(n) \setminus \{0\}} \quad \big(x, \pi(i)\big) \in S \tag{4.13}$$

via finite induction on $i \in \mathbf{S}(n) \setminus \{0\}$ (we apply Cor. 4.26(d) with $A := \mathbf{S}(n) \setminus \{0\} = \{1, \dots, n\}$ and $f := \mathrm{Id}_A$). For the base case ($i = 1$), note

$$\big(x, \pi(1)\big) = \big(\pi(0), \pi(1)\big) \stackrel{\pi\ R\text{-path}}{\in} R \subseteq S.$$

For the induction step, let $i \in \mathbf{S}(n) \setminus \{0\}$ with $i < n$ and assume $(x, \pi(i)) \in S$ by induction hypothesis. Since π is an R-path, we also have

$$\big(\pi(i), \pi(\mathbf{S}(i))\big) \in R \subseteq S \quad \stackrel{S \text{ tr.}}{\Rightarrow} \quad \big(x, \pi(\mathbf{S}(i))\big) \in S,$$

thereby completing the induction proof of (4.13). Since $y = \pi(n)$, (4.13) yields $(x, y) \in S$, proving $R^* \subseteq S$ and $R^* \subseteq \bigcap \mathcal{R}$. ■

4.3 Transfinite Recursion on Well-Founded Relations

As mentioned at the beginning of the previous Sec. 4.2, in Th. 4.29 below, we will provide a result that allows to define functions via *recursion* on classes with well-founded relations (generalizing the simpler result [Phi19a, Th. 3.7], providing recursive definitions on $\mathbb{N}$). To recursively define a function $\mathbf{F}$ on a class $\mathbf{A}$ with a well-founded and set-like relation $\mathbf{R}$, the idea is to define $\mathbf{F}(a)$ by applying another function $\mathbf{G}$ to the pair $(a, \mathbf{F}\restriction_{a_\downarrow})$, i.e., in general, $\mathbf{F}(a)$ is allowed to depend on a and all "previously" defined values of $\mathbf{F}$ (where the assumption that $\mathbf{R}$ be set-like and Axiom 5 (replacement) guarantee $\mathbf{F}\restriction_{a_\downarrow}$ to be a set) – in typical applications, $\mathbf{F}(a)$ will, actually, depend only on a much smaller (possibly singleton) subset of $\mathbf{F}\restriction_{a_\downarrow}$. Moreover, in the statement of Th. 4.29, we will, actually, assume $\mathbf{G}$ to be defined on all of $\mathbf{V} \times \mathbf{V}$: There does not seem to be a simple formula, characterizing the subclass we need $\mathbf{G}$ to be defined on and, on the other hand, we will always be able to assume $\mathbf{G}$ to have some default value (namely 0) on all arguments not relevant for the definition of $\mathbf{F}$ on $\mathbf{A}$ (in consequence, $\mathbf{F}$ will also be defined on all of $\mathbf{V}$, taking the value 0 on $\mathbf{V} \setminus \mathbf{A}$).

We also, once again, emphasize that theorems involving proper classes (such as Th. 4.29 below) are, actually, providing theorem schemes in the metatheory: Theorem 4.29 will show, given set-theorectic formulas defining classes $\mathbf{A}$, $\mathbf{R}$, and $\mathbf{G}$, how to formulate another set-theoretic formula, defining the function $\mathbf{F}$, which will have the desired property $\mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}\restriction_{a_\downarrow}\big)$ for each $a \in \mathbf{A}$.

Theorem 4.29 (Principle of Transfinite Recursion on Well-Founded Relations)**.** *Let* $\mathbf{R}$ *be a well-founded and set-like relation on the class* $\mathbf{A}$ *and let* ϕ *be a set-theoretic formula, satisfying*

$$\mathop{\forall}_{x,s} \; \mathop{\exists!}_{y} \; \phi(x, s, y), \tag{4.14}$$

i.e., for each pair of sets (x, s)*, there exists a unique set* y *having the property* $\phi(x, s, y)$ *(note the similarity to the assumption in the replacement scheme (Axiom* 5*)), i.e. the class*

$$\mathbf{G} := \big\{((x, s), y) : \phi(x, s, y)\big\} \tag{4.15}$$

constitutes a class function $\mathbf{G} : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$*. Then one can formulate a set-theoretic formula* ψ*, satisfying* (i) *and* (ii)*, where*

(i) $\forall_x \; \exists!_y \; \psi(x,y)$, *i.e. the class*

$$\mathbf{F} := \big\{(x,y) : \psi(x,y)\big\} \tag{4.16}$$

constitutes a class function $\mathbf{F} : \mathbf{V} \longrightarrow \mathbf{V}$.

(ii) $\forall_{a \in \mathbf{A}} \quad \mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}\restriction_{a_\downarrow}\big)$ *(note that each function* $\mathbf{F}\restriction_{a_\downarrow} : a_\downarrow \longrightarrow \mathbf{F}(a_\downarrow)$ *is, actually, a* set, *since* $a_\downarrow$ *is a set by the assumption that* $\mathbf{R}$ *be set-like, and, thus,* $\{(x,y) : x \in a_\downarrow \wedge \psi(x,y)\}$ *is a set by replacement (Axiom 5)).*

Moreover, the function on $\mathbf{A}$*, recursively defined via* $\mathbf{G}$ *and* $\mathbf{R}$ *as above, is unique: If* $\mathbf{F}' : \mathbf{V} \longrightarrow \mathbf{V}$ *also satisfies* (ii)*, i.e.* $\forall_{a \in \mathbf{A}} \quad \mathbf{F}'(a) = \mathbf{G}\big(a, \mathbf{F}'\restriction_{a_\downarrow}\big)$*, then*

$$\forall_{a \in \mathbf{A}} \quad \mathbf{F}'(a) = \mathbf{F}(a). \tag{4.17}$$

Proof. We prove the uniqueness statement first, using Cor. 4.26(a): Suppose $\mathbf{F}, \mathbf{F}' : \mathbf{V} \longrightarrow \mathbf{V}$ both satisfy (ii). If $a \in \mathbf{A}$ is a minimal element, then $a_\downarrow = \emptyset$ and, thus,

$$\mathbf{F}'(a) = \mathbf{G}\big(a, \mathbf{F}'\restriction_{a_\downarrow}\big) = \mathbf{G}(a, \emptyset) = \mathbf{G}\big(a, \mathbf{F}\restriction_{a_\downarrow}\big) = \mathbf{F}(a),$$

thereby proving Cor. 4.26(a)(i). If $a \in \mathbf{A}$ is not minimal and $\mathbf{F}'(p) = \mathbf{F}(p)$ holds for each $p \in a_\downarrow$, then $\mathbf{F}'\restriction_{a_\downarrow} = \mathbf{F}\restriction_{a_\downarrow}$, implying

$$\mathbf{F}'(a) = \mathbf{G}\big(a, \mathbf{F}'\restriction_{a_\downarrow}\big) = \mathbf{G}\big(a, \mathbf{F}\restriction_{a_\downarrow}\big) = \mathbf{F}(a),$$

thereby proving Cor. 4.26(a)(ii). Thus, Cor. 4.26(a) yields (4.17).

To prove the existence of $\mathbf{F}$, we will mostly follow [Kun13, p. 49,50]: The idea is to consider *approximations* to $\mathbf{F}$, defined on subsets of $\mathbf{A}$, and to then use transfinite induction to show that the approximations must extend to all of $\mathbf{A}$. Thus, this end, for each set d and each set h, we let $\mathrm{app}(d,h)$ be an abbreviation of the set-theoretic formula that states[10]

$$h \text{ is function} \quad \wedge \quad \mathrm{dom}(h) = d \subseteq \mathbf{A} \quad \wedge \quad \forall_{y \in d} \Big(y_\downarrow \subseteq d \;\wedge\; h(y) = \mathbf{G}\big(y, h\restriction_{y_\downarrow}\big)\Big)$$

(in the above definition of $\mathrm{app}(d,h)$ as well as in the following, we identify a function with its graph, i.e. "h is a function" means h is a subset of $\mathrm{dom}(h) \times h(\mathrm{dom}(h))$ that satisfies Def. 3.6(f)).

[10]This definition of $\mathrm{app}(d,h)$ is, actually, quite natural, since, if an $\mathbf{F}$ satisfying (i) and (ii) does exist, then $\mathrm{app}(d,h)$ implies $h = \mathbf{F}\restriction_d$ via the uniqueness statement proved above ((4.17) applied with $\mathbf{A}$ replaced by d).

Using app(d,h), we are now in a position to formulate the formula $\psi(x,y)$, that will provide the class function $\mathbf{F}$ as stated in the theorem. We let

$$\psi(x,y) := \Big(x\notin\mathbf{A}\ \wedge\ y=\emptyset\Big)\quad\vee\quad\Big(x\in\mathbf{A}\ \wedge\ \exists_{d,h}\ \Big(\operatorname{app}(d,h)\ \wedge\ x\in d\ \wedge\ h(x)=y\Big)\Big).$$

The core of the proof will now have two components: A uniqueness part to show that, for each set x, there is at most one set y, satisfying $\psi(x,y)$; and an existence part to show that, given a set x, a set y, satisfying $\psi(x,y)$ does always exist. The uniqueness part will be shown in the form of the following statement (4.18a), that says that approximations to $\mathbf{F}$ have to agree, if they are defined on the same domain:

$$\forall_{d,\tilde d,h,\tilde h}\ \Big(\operatorname{app}(d,h)\ \wedge\ \operatorname{app}(\tilde d,\tilde h)\ \Rightarrow\ \operatorname{app}(d\cap\tilde d,h\cap\tilde h)\Big). \tag{4.18a}$$

The existence part will be shown in the form of the following statement:

$$\forall_{x\in\mathbf{A}}\ \exists_{d,h}\ \Big(\operatorname{app}(d,h)\ \wedge\ x\in d\Big). \tag{4.18b}$$

Before proving (4.18), we show that (4.18), indeed, implies the existence of $\mathbf{F}$: As a consequence of (4.18b), for each set x, there exists a set y, such that $\psi(x,y)$ holds. Moreover, applying (4.18a), shows y to be unique, i.e. ψ satisfies (i) and $\mathbf{F}$ is well-defined by (4.16). Now, if $a\in\mathbf{A}$, then $\psi(a,\mathbf{F}(a))$ implies the existence of d,h such that app(d,h), $a\in d$ and $h(a)=\mathbf{F}(a)$. From the definition of app(d,h), we then obtain

$$\mathbf{F}(a)=h(a)=\mathbf{G}\big(a,h\restriction_{a_\downarrow}\big)\overset{(4.17),\ \mathbf{A}\text{ repl. by }a_\downarrow}{=}\mathbf{G}\big(a,\mathbf{F}\restriction_{a_\downarrow}\big),$$

showing $\mathbf{F}$ to satisfy (ii).

Thus, it merely remains to prove (4.18).

Proof of (4.18a): Assume app(d,h) and app$(\tilde d,\tilde h)$. If $y\in d\cap\tilde d$, then $y_\downarrow\subseteq d$ and $y_\downarrow\subseteq\tilde d$, i.e. $y_\downarrow\subseteq d\cap\tilde d$. Next, note $f:=h\restriction_{d\cap\tilde d}=\tilde f:=\tilde h\restriction_{d\cap\tilde d}$: This is due to the uniqueness statement shown at the outset of the proof: Both f and $\tilde f$ satisfy (ii) with $\mathbf{A}$ replaced by $d\cap\tilde d$, such that (4.17) yields

$$\forall_{y\in d\cap\tilde d}\ \ h(y)=\tilde h(y),$$

also implying $h\restriction_{y_\downarrow}=\tilde h\restriction_{y_\downarrow}$. Thus, $h\cap\tilde h$ is a function with domain $d\cap\tilde d\subseteq d\subseteq\mathbf{A}$, satisfying

$$\forall_{y\in d\cap\tilde d}\ \ (h\cap\tilde h)(y)=h(y)=\mathbf{G}\big(y,h\restriction_{y_\downarrow}\big)=\mathbf{G}\big(y,(h\cap\tilde h)\restriction_{y_\downarrow}\big),$$

thereby proving $\operatorname{app}(d \cap \tilde{d}, h \cap \tilde{h})$.

Proof of (4.18b): We will show (4.18b), using transfinite induction on $\mathbf{R}$ according to Th. 4.25: If (4.18b) is false, then

$$\mathbf{X} := \left\{ x \in \mathbf{A} : \neg \underset{d,h}{\exists} \Big(\operatorname{app}(d,h) \ \wedge \ x \in d \Big) \right\} \neq \emptyset$$

and Th. 4.25 yields a minimal element $a \in \mathbf{X}$. Define

$$\tilde{d} := \operatorname{pred}(\mathbf{A}, a, \mathbf{R}^*) \overset{(4.10)}{=} \bigcup \{ d_x : \ x \in a_\downarrow \},$$

where, as in Prop. 4.24(d), $d_x = \{x\} \cup \operatorname{pred}(\mathbf{A}, x, \mathbf{R}^*)$. If $x \in a_\downarrow$, then $x \notin \mathbf{X}$, since a is a minimal element of $\mathbf{X}$. Thus, there exist d, h such that $\operatorname{app}(d,h)$ and $x \in d$. We now note that $d_x \subseteq d$ as a consequence of $\operatorname{app}(d,h)$ and Prop. 4.24(e) and define $h_x := h\restriction_{d_x}$. While d and h can not be expected to be uniquely determined by x, d_x is unique by its definition and h_x is unique by (4.18a). In consequence, by replacement (Axiom 5) and union (Axiom 4), we know

$$\tilde{h} := \bigcup \{ h_x : \ x \in a_\downarrow \}$$

to be a set. We claim $\operatorname{app}(\tilde{d}, \tilde{h})$: If $x \in \tilde{d}$ and there exist $x_1, x_2 \in a_\downarrow$ with $x \in d_{x_1} \cap d_{x_2}$, then $h_{x_1}(x) = h_{x_2}(x)$ by (4.18a) showing $\tilde{h}$ to be a function with $\operatorname{dom}(\tilde{h}) = \tilde{d}$. Moreover, if $y \in \tilde{d}$, then there exists $x \in a_\downarrow$ with $y \in d_x$, i.e. there exist d, h with $\operatorname{app}(d,h)$, $y \in d_x \subseteq d$. Thus, $y_\downarrow \subseteq d_x \subseteq \tilde{d} \cap d$ and

$$\tilde{h}(y) = h_x(y) = h(y) = \mathbf{G}\big(y, h\restriction_{y_\downarrow}\big) = \mathbf{G}\big(y, h_x\restriction_{y_\downarrow}\big) = \mathbf{G}\big(y, \tilde{h}\restriction_{y_\downarrow}\big),$$

completing the proof of $\operatorname{app}(\tilde{d}, \tilde{h})$. However, we can now extend $\tilde{h}$ to $d := \{a\} \cup \tilde{d}$, which will lead to a contradiction to $a \in \mathbf{X}$: Since $a_\downarrow \subseteq \tilde{d}$ (as $x \in a_\downarrow$ implies $x \in d_x \subseteq \tilde{d}$), we can define $h := \tilde{h} \cup \big\{ \big(a, \mathbf{G}(a, \tilde{h}\restriction_{a_\downarrow})\big) \big\}$. Then h is a function (as $a \notin \tilde{d}$) with $\operatorname{dom}(h) = d$ and $\operatorname{app}(d,h)$ follows, since $\operatorname{app}(\tilde{d}, \tilde{h})$ and $h(a) = \mathbf{G}(a, \tilde{h}\restriction_{a_\downarrow})$ by definition of h. In consequence, $a \notin \mathbf{X}$ and this contradiction to $a \in \mathbf{X}$ completes the proof of (4.18b) as well as the proof of the theorem. ■

Corollary 4.30. **(a)** Transfinite Recursion on the Ordinals*: Given a set* x_0 *and a function* $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$*, there exists a unique function* $\mathbf{F} : \mathbf{ON} \longrightarrow \mathbf{V}$ *such that* (i) *and* (ii) *hold:*

(i) $\mathbf{F}(0) = x_0$.

(ii) $\underset{\alpha \in \mathbf{ON} \setminus \{0\}}{\forall} \quad \mathbf{F}(\alpha) = \mathbf{H}\big(\mathbf{F}\restriction_\alpha\big)$.

(b) Bounded Transfinite Recursion on the Ordinals*: Given a set* x_0*, an ordinal* $\gamma \in \mathbf{ON} \setminus \{0\}$*, and a function* $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$*, there exists a unique function* $F : \gamma \longrightarrow F(\gamma)$ *such that* (i) *and* (ii) *hold:*

(i) $F(0) = x_0$.

(ii) $\underset{\alpha\in\gamma\setminus\{0\}}{\forall} \quad F(\alpha) = \mathbf{H}\big(F{\restriction}_\alpha\big)$.

(c) *Given sets x_0 and A, $\gamma \in \mathbf{ON}\setminus\{0\}$, $f : \gamma \longrightarrow A$ bijective, and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$, there exists a unique function $F : A \longrightarrow F(A)$ such that* (i) *and* (ii) *hold:*

(i) $F(f(0)) = x_0$.

(ii) $\underset{\alpha\in\gamma\setminus\{0\}}{\forall} \quad F(f(\alpha)) = \mathbf{H}\big((F\circ f){\restriction}_\alpha\big)$.

Proof. (a): We apply Th. 4.29 with $(\mathbf{A},\mathbf{R}) := (\mathbf{ON},<)$: As $<$ is a strict well-order on $\mathbf{ON}$ as well as set-like (recall $\alpha_\downarrow = \alpha$ for each $\alpha \in \mathbf{ON}$), Th. 4.29 applies if we let

$$\mathbf{G} : \mathbf{V}\times\mathbf{V} \longrightarrow \mathbf{V}, \quad \mathbf{G}(x,s) := \begin{cases} x_0 & \text{for } x = 0,\\ \mathbf{H}(s) & \text{for } x \in \mathbf{ON}\setminus\{0\},\\ 0 & \text{otherwise.}\end{cases}$$

Then Th. 4.29 yields $\mathbf{F} : \mathbf{V} \longrightarrow \mathbf{V}$ such that $\underset{\alpha\in\mathbf{ON}}{\forall} \quad \mathbf{F}(\alpha) = \mathbf{G}\big(\alpha, \mathbf{F}{\restriction}_{\alpha_\downarrow}\big)$. If $\alpha = 0$, this yields

$$\mathbf{F}(0) = \mathbf{G}\big(0, \mathbf{F}{\restriction}_{0_\downarrow}\big) = \mathbf{G}(0,0) = x_0;$$

whereas, for $\alpha \in \mathbf{ON}\setminus\{0\}$, one obtains

$$\mathbf{F}(\alpha) = \mathbf{G}\big(\alpha, \mathbf{F}{\restriction}_{\alpha_\downarrow}\big) = \mathbf{H}(\mathbf{F}{\restriction}_{\alpha_\downarrow}) = \mathbf{H}\big(\mathbf{F}{\restriction}_\alpha\big),$$

as desired. Thus, we can restrict $\mathbf{F}$ to $\mathbf{ON}$ to complete the proof of (a).

(b) is clear from (a), as one obtains $F := \mathbf{F}{\restriction}_\gamma$.

(c): Exercise. ■

Example 4.31. (a) Let $\mathbf{A}$ be a class, $f : \mathbf{A} \longrightarrow \mathbf{A}$, and $a \in \mathbf{A}$. As a first example of definition via recursion, we define the elements of what is sometimes called the *orbit* of a in $\mathbf{A}$ under f: We obtain $F : \omega \longrightarrow \mathbf{A}$ via Cor. 4.30(b) such that $f^n(a) := F(n)$ satisfies

$$f^0(a) := a, \quad \underset{n\in\omega}{\forall} \; f^{\mathbf{S}(n)}(a) := f(f^n(a)). \tag{4.19}$$

To define $F : \omega \longrightarrow \mathbf{A}$ using Cor. 4.30(b), let $x_0 := a$, $\gamma := \omega$, and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} f(x(n)) & \text{if } x \text{ is a function with } \mathrm{dom}(x) = \mathbf{S}(n),\ n\in\omega,\ x(n)\in\mathbf{A},\\ 0 & \text{otherwise.}\end{cases}$$

Then Cor. 4.30(b) provides a unique function $F : \omega \longrightarrow F(\omega) \subseteq \mathbf{V}$ with $F(0) = x_0 = a$ and $\underset{k\in\omega\setminus\{0\}}{\forall} F(k) = \mathbf{H}\big(F{\restriction_k}\big)$. Then, for each $k, n \in \omega$, $k = \mathbf{S}(n)$, assuming $F(n) \in \mathbf{A}$ by induction hypothesis,

$$f^{\mathbf{S}(n)}(a) = F\big(\mathbf{S}(n)\big) = F(k) = \mathbf{H}\big(F{\restriction_k}\big) = f(F(n)) = f(f^n(a)) \in \mathbf{A},$$

thereby proving $F(\omega) \subseteq \mathbf{A}$ and (4.19). As we have now defined $f^n(a)$ for each $a \in \mathbf{A}$ and $n \in \omega$, this also yields the *functions*

$$f^n : \mathbf{A} \longrightarrow \mathbf{A}, \quad a \mapsto f^n(a).$$

For an alternative approach, defining the functions f^n recursively, in case $A := \mathbf{A}$ is a set, without defining the elements $f^n(a) \in A$ first, see (d) below (if $\mathbf{A}$ is a set, then both approaches are equivalent).

(b) If $\mathbf{C}$ is a class and $\circ : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$, $(A, B) \mapsto A \circ B$, a function (which we think of as a composition map such as addition or multiplication), then we define the product symbol $\prod$ with respect to $\circ$ via recursion as follows: For each given function $V : \mathbb{N} \longrightarrow V(\mathbb{N}) \subseteq \mathbf{C}$, we denote $V_i := V(i)$ for each $i \in \mathbb{N}$ (considering the *sequence* $(V_i)_{i\in\mathbb{N}}$ in $\mathbf{C}$), we seek to obtain $F : \mathbb{N} \longrightarrow \mathbf{C}$ via Cor. 4.30(c) such that $\prod_{i=1}^n V_i := F(n)$ satisfies

$$\prod_{i=1}^{1} V_i := V_1 \quad \wedge \quad \underset{n\in\mathbb{N}}{\forall} \prod_{i=1}^{\mathbf{S}(n)} V_i = \left(\prod_{i=1}^{n} V_i\right) \circ V_{\mathbf{S}(n)}. \tag{4.20}$$

To define $F : \mathbb{N} \longrightarrow \mathbf{C}$ using Cor. 4.30(c), let $x_0 := V_1$, $A := \mathbb{N}$, $\gamma := \omega$, $f : \omega \longrightarrow \mathbb{N}$, $f(n) := \mathbf{S}(n)$, and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} x(k) \circ V_{\mathbf{S}(\mathbf{S}(k))} & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \mathbf{S}(k),\ k \in \omega,\ x(k) \in \mathbf{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Then Cor. 4.30(c) provides a unique function $F : \mathbb{N} \longrightarrow F(\mathbb{N}) \subseteq \mathbf{C}$ with $\prod_{i=1}^1 V_i = F(1) = F(f(0)) = x_0 = V_1$ and $\underset{n\in\omega\setminus\{0\}}{\forall} F(f(n)) = \mathbf{H}\big((F \circ f){\restriction_n}\big)$. Then, for each $n \in \mathbb{N}$, $n = \mathbf{S}(k)$ with $k \in \omega$, assuming $F(n) \in \mathbf{C}$ by induction hypothesis,

$$\begin{aligned} \prod_{i=1}^{\mathbf{S}(n)} V_i &= F\big(\mathbf{S}(n)\big) = F(f(n)) = \mathbf{H}\big((F \circ f){\restriction_n}\big) = F(f(k)) \circ V_{\mathbf{S}(\mathbf{S}(k))} \\ &= F(n) \circ V_{\mathbf{S}(n)} = \left(\prod_{i=1}^{n} V_i\right) \circ V_{\mathbf{S}(n)} \in \mathbf{C}, \end{aligned}$$

thereby proving $F(\mathbb{N}) \subseteq \mathbf{C}$ and (4.20). Note that, if $\mathbf{C}$ is a set with addition (respectively, multiplication), then our definition yields the usual summation symbol $\sum$ (respectively, the usual product symbol $\prod$).

(c) If we apply (b) with $\mathbf{C} := \mathbf{V}$ and $\circ$ being the Cartesian product $\times : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$, $(A, B) \mapsto A \times B$, then we obtain Cartesian products of finitely many arbitrary sets

$$\forall_{n\in\mathbb{N}} \quad \prod_{i=1}^{n} V_i = V_1 \times \cdots \times V_n,$$

where, as in (b), $V : \mathbb{N} \longrightarrow V(\mathbb{N})$ is a given function, $V_i := V(i)$. If $v : \mathbb{N} \longrightarrow v(\mathbb{N})$ is another function, $v_i := v(i)$, then, as in Rem. 2.17(c), we can define the ordered n-tuple with entries $(v_1, \dots, v_n)$, now using recursion to set

$$(v_1) := v_1, \quad \forall_{n\in\mathbb{N}} \ (v_1, \dots, v_n, v_{\mathbf{S}(n)}) := \big((v_1, \dots, v_n), v_{\mathbf{S}(n)}\big) : \tag{4.21}$$

Once again, we apply (b) with $\mathbf{C} := \mathbf{V}$, where, this time, $\circ$ is the pairing operation, $\circ : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$, $(x, y) \mapsto (x, y)$. Then, if we define the ordered n-tuple $(v_1, \dots, v_n) := \prod_{i=1}^{n} v_i$ via the product symbol with respect to $\circ$, then (4.21) holds, as it is precisely (4.20) applied with our $\circ$. It is now an exercise to verify

$$\forall_{n\in\mathbb{N}} \quad V_1 \times \cdots \times V_n = \left\{ (v_1, \dots, v_n) : \forall_{i\in\{1,\dots,n\}} \ v_i \in V_i \right\} : \tag{4.22}$$

As mentioned in Rem. 2.17(c), the advantage of defining ordered n-tuples via iterated pairing lies in its feasability without the axiom of replacement. Once one has the axiom of replacement and, in consequence, functions, it seems more natural and elegant to define the ordered n-tuple $[v_1, \dots, v_n]$ as the function $v : \{1, \dots, n\} \longrightarrow \{v_1, \dots, v_n\}$, $v(i) := v_i$ (or one could say $[v_1, \dots, v_n] := v{\restriction}_{\{1,\dots,n\}}$ if $v : \mathbb{N} \longrightarrow v(\mathbb{N})$ is already given as above). Indeed, to each $(v_1, \dots, v_n) \in \prod_{i=1}^{n} V_i$, we can assign the unique function $v : \{1, \dots, n\} \longrightarrow \{v_1, \dots, v_n\}$, $v(i) := v_i$, such that the axiom of replacement (Axiom 5) shows the class of functions

$$\left\{ \Big(f : \{1, \dots, n\} \longrightarrow \bigcup \big\{V_i : i \in \{1, \dots, n\}\big\}\Big) : \forall_{i\in\{1,\dots,n\}} \ f(i) \in V_i \right\}$$

to be a set. In particular, considering the case, where $V_i = A$ for each $i \in \{1, \dots, n\}$ and some set A, this proves the existence of the *set* of functions $A^n := \mathcal{F}(n, A)$ from n into A (without the power set axiom (Axiom 7 below), one can not show that the class of functions $\mathcal{F}(S, A)$ from a set S into a set A is a set, if S is infinite).

(d) We apply (b) to composition of functions: Let A be a set and let $\mathbf{C} := \mathcal{F}(A, A)$ denote the class of functions from A into A (as mentioned above, at the end of (c), without the power set axiom, $\mathbf{C}$ might be a proper class for A infinite). Now, in the usual way, let $\circ : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ denote composition of functions, $(f, g) \mapsto g \circ f$. Then, fiven a sequence of functions $(f_i)_{i\in\mathbb{N}}$ in $\mathbf{C}$, (b) yields

$$\underset{n\in\mathbb{N}}{\forall} \quad \prod_{i=1}^{n} f_i = \big(f_n \circ \cdots \circ f_1\big) : A \longrightarrow A.$$

If $f : A \longrightarrow A$ and $f_i = f$ for each $i \in \mathbb{N}$, then we also define

$$\underset{n\in\mathbb{N}}{\forall} \quad f^n := \prod_{i=1}^{n} f$$

(thus, in the usual way, f^n then stands for "f iterated n times"). Clearly, the $f^n : A \longrightarrow A$ are then the same functions that were already defined in (a) (however, the present appoach does not extend to the case, where A is replaced by a proper class $\mathbf{A}$, since, then, $\mathbf{f} : \mathbf{A} \longrightarrow \mathbf{A}$ is a proper class as well, and, in general, there then seems to be no way of handling the "collection of classes" $\mathcal{F}(\mathbf{A}, \mathbf{A})$ within our framework.

Example 4.32. In Rem. 2.17(a), it was stated that Axioms 0 – 3 do not suffice to prove the existence of sets with more than two elements. Using recursion, we can now produce a model $M_{11} = (D, \in)$ that satisfies Axioms 0 – 3 plus

$$\underset{X}{\forall} \quad \Big(X = \emptyset \quad \vee \quad \underset{x,y}{\exists} \quad X = \{x, y\}\Big). \tag{4.23}$$

We construct the domain D of the model as follows: We start by, recursively, defining a sequence $(S_n)_{n\in\omega}$ of sets by letting

$$S_0 := \emptyset, \quad \underset{n\in\omega}{\forall} \quad S_{\mathbf{S}(n)} := \{\emptyset\} \cup \big\{\{x, y\} : x, y \in S_n\big\}. \tag{4.24}$$

It is an exercise to justify (4.24) by using Cor. 4.30(b) to show the existence of $F : \omega \longrightarrow \mathbf{V}$ such that, letting, for each $n \in \omega$, $S_n := F(n)$, (4.24) holds. Next, we note that

$$\underset{n\in\omega}{\forall} \quad S_n \subseteq S_{\mathbf{S}(n)} : \tag{4.25}$$

Indeed, this follows inductively, as it is clear for $n = 0$ and, if $S_n \subseteq S_{\mathbf{S}(n)}$ and $\{x, y\} \in S_{\mathbf{S}(n)}$ with $x, y \in S_n \subseteq S_{\mathbf{S}(n)}$, then $\{x, y\} \in S_{\mathbf{S}(\mathbf{S}(n))}$ follows from (4.24).

We now let $D := \bigcup\{S_n : n \in \omega\}$ and $M_{11} := (D, \in)$.

It is now an exercise to verify that M_{11} satisfies (4.23), ¬(2.1), and Axioms 0 – 3 and Axiom 5, but M_{11} does not satisfy Axiom 4 (union) and Axiom 6 (infinity).

—

The proof of the following theorem also employs recursion:

Theorem 4.33. *Any two Peano structures (as defined in Th.* 4.10*) are isomorphic: If* (M,μ,S) *and* (N,ν,T) *are Peano structures, then there exists a unique isomorphism* $F: M \longrightarrow N$*, i.e. a unique bijective function* $F: M \longrightarrow N$ *such that*

$$F(\mu)=\nu \quad \wedge \quad \forall_{m\in M} \; F\big(S(m)\big)=T\big(F(m)\big). \tag{4.26}$$

Proof. First, consider the case $(M,\mu,S)=(\omega,0,\mathbf{S})$. To define F using Cor. 4.30(b), let $x_0 := \nu$, $\gamma := \omega$ and $\mathbf{H}: \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} T(x(k)) & \text{if } x \text{ is a function with } \operatorname{dom}(x)=\mathbf{S}(k),\ k\in\omega,\ x(k)\in N, \\ 0 & \text{otherwise.} \end{cases}$$

Then Cor. 4.30(b) provides a unique function $F: \omega \longrightarrow F(\omega)$ with $F(0)=x_0=\nu\in N$ and $\forall_{n\in\omega\setminus\{0\}} \; F(n)=\mathbf{H}\big(F{\restriction_n}\big)$. Then, for each $n\in\omega$, $n=\mathbf{S}(m)$ with $m\in n$, inductively assuming $F(m)\in N$, we obtain

$$F\big(\mathbf{S}(m)\big)=F(n)=\mathbf{H}\big(F{\restriction_n}\big)=T\big(F(m)\big)\in N,$$

thereby proving $F(\omega)\subseteq N$ and (4.26). Moreover, since $\nu=F(0)\in F(\omega)$, and $F(m)\in F(\omega)$ implies $T(F(m))=F(\mathbf{S}(m))\in F(\omega)$, Peano axiom P3 for N yields $F(\omega)=N$, i.e. F is surjective. However, F is also injective: We show

$$\forall_{m,n\in\omega} \; \Big(m\neq n \;\Rightarrow\; F(m)\neq F(n)\Big)$$

via induction on n: If $n=0$ and $m\neq 0$, then $m=\mathbf{S}(k)$ for $k\in\omega$. Then $\nu=F(n)$ and $F(m)=F(\mathbf{S}(k))=T(F(k))\neq\nu=F(0)$, proving the base case. Now assume $F(m)\neq F(n)$ to hold for $m\neq n$ for fixed $n\in\omega$ (and each $m\in\omega$) and let $m\neq\mathbf{S}(n)$. If $m=0$, then $F(m)\neq F(\mathbf{S}(n))$ follows from the base case. If $m=\mathbf{S}(k)$ for $k\in\omega$, then $m=\mathbf{S}(k)\neq\mathbf{S}(n)$ implies $k\neq n$, i.e. $F(k)\neq F(n)$. As T is injective by Peano axiom P2, the yields $F(m)=F(\mathbf{S}(k))=T(F(k))\neq T(F(n))=F(\mathbf{S}(n))$, completing the induction.

Now, if (M,μ,S) is arbitrary, let $F_M: \omega \longrightarrow M$ and $F_N: \omega \longrightarrow N$ be isomorphisms as constructed above. Then $F: M \longrightarrow N$, $F := F_N \circ F_M^{-1}$ is also an isomorphism: Indeed, F is bijective, $F(\mu)=F_N\big(F_M^{-1}(\mu)\big)=F_N(0)=\nu$ and, to show (4.26), note, for each

$k \in \omega$, $\mathbf{S}(k) = F_M^{-1}\big(S(F_M(k))\big)$, implying, for each $m \in M$, $\mathbf{S}\big(F_M^{-1}(m)\big) = F_M^{-1}\big(S(m)\big)$ and

$$T\big(F(m)\big) = T\big(F_N\big(F_M^{-1}(m)\big)\big) = F_N\big(\mathbf{S}\big(F_M^{-1}(m)\big)\big) = F_N\big(F_M^{-1}(S(m))\big) = F\big(S(m)\big),$$

thereby proving (4.26).

Uniqueness: If $G : M \longrightarrow N$ is also an isomorphism, then $G(m) = F(m)$ for each $m \in M$ follows via a simple induction on m: $G(\mu) = \nu = F(\mu)$, and, if $G(m) = F(m)$ holds for $m \in M$, then $G(S(m)) = S(G(m)) = S(F(m)) = F(S(m))$. ∎

Example 4.34. *Ordinal Addition:* While we will study ordinal arithmetic more systematically in Sec. 4.4 below, we already define ordinal addition to provide a first example of definition via transfinite recursion on **ON**: For each $\alpha, \beta \in \mathbf{ON}$ and each limit ordinal λ, define

$$\alpha + 0 := \alpha, \tag{4.27a}$$

$$\alpha + \mathbf{S}(\beta) := \mathbf{S}(\alpha + \beta), \tag{4.27b}$$

$$\alpha + \lambda := \bigcup\{\alpha + \gamma : \gamma < \lambda\} \overset{\text{Th. 3.36(b)}}{=} \sup\{\alpha + \gamma : \gamma < \lambda\} \tag{4.27c}$$

(note that this includes the definition of addition on $\omega = \mathbb{N}_0$). To justify, using Cor. 4.30(a), that this, for each $\alpha \in \mathbf{ON}$, defines a unique function $+ : \mathbf{ON} \longrightarrow \mathbf{ON}$, $\xi \mapsto \alpha + \xi := +(\xi)$, let $x_0 := \alpha$ and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} \mathbf{S}(x(\beta)) & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \mathbf{S}(\beta),\ \beta \in \mathbf{ON},\\ & x(\beta) \in \mathbf{ON},\\ \bigcup\{x(\gamma) : \gamma < \lambda\} & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \lambda,\ \lambda \text{ a limit ordinal},\\ 0 & \text{otherwise.} \end{cases}$$

Then Cor. 4.30(a) provides a unique function $+ : \mathbf{ON} \longrightarrow \mathbf{V}$ with $\alpha + 0 = +(0) = x_0 = \alpha$ and $\underset{\xi \in \mathbf{ON}\setminus\{0\}}{\forall} +(\xi) = \mathbf{H}\big(+\restriction_\xi\big)$. We use transfinite induction, using Cor. 4.26(b), to show that $+$ maps into **ON** and satisfies (4.27): $+(0) = \alpha \in \mathbf{ON}$ provides the base case. If $\beta \in \mathbf{ON}$ and $+(\beta) = \alpha + \beta \in \mathbf{ON}$, then

$$\alpha + \mathbf{S}(\beta) = +(\mathbf{S}(\beta)) = \mathbf{H}\big(+\restriction_{\mathbf{S}(\beta)}\big) = \mathbf{S}(+(\beta)) = \mathbf{S}(\alpha + \beta) \in \mathbf{ON},$$

yielding (4.27b); and, for each limit ordinal λ, assuming $+(\gamma) = \alpha + \gamma \in \mathbf{ON}$ for each $\gamma \in \lambda$,

$$\alpha + \lambda = +(\lambda) = \mathbf{H}\big(+\restriction_\lambda\big) = \bigcup\{+(\gamma) : \gamma < \lambda\} = \bigcup\{\alpha + \gamma : \gamma < \lambda\} \overset{\text{Th. 3.36(b)}}{\in} \mathbf{ON},$$

yielding (4.27c), and completing the induction. We establish some basic results regarding ordinal addition, also providing more examples of induction proofs on **ON**:

(a) $\alpha+1=\mathbf{S}(\alpha)$ holds for each $\alpha\in\mathbf{ON}$: Indeed, $\alpha+1=\alpha+\mathbf{S}(0)=\mathbf{S}(\alpha+0)=\mathbf{S}(\alpha)$.

(b) 0 is neutral for ordinal addition, i.e.

$$\forall_{\alpha\in\mathbf{ON}} \quad \alpha+0=0+\alpha=\alpha:$$

While $\alpha+0=\alpha$ is immediate, we show $0+\alpha=\alpha$ via induction on $\alpha\in\mathbf{ON}$: $0+0=0$ yields the base case. For the induction step, note that $0+\alpha=\alpha$ implies

$$0+\mathbf{S}(\alpha)=\mathbf{S}(0+\alpha)=\mathbf{S}(\alpha)$$

and, for each limit ordinal α, $0+\gamma=\gamma$ for each $\gamma<\alpha$ implies

$$0+\alpha=\bigcup\{0+\gamma:\,\gamma<\alpha\}=\bigcup\{\gamma:\,\gamma<\alpha\}=\bigcup\alpha=\sup\alpha\overset{\text{Prop. 3.38(f)}}{=}\alpha.$$

(c) Right addition is strictly isotone[11]:

$$\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \Big(\beta<\gamma \quad\Rightarrow\quad \alpha+\beta<\alpha+\gamma\Big):$$

We conduct the proof via transfinite induction on γ: For the base case ($\gamma=0$), there is nothing to show, since, in this case, $\beta<\gamma$ is false. Now assume $\beta<\gamma$ implies $\alpha+\beta<\alpha+\gamma$ and assume $\beta<\mathbf{S}(\gamma)$. Then $\beta\leq\gamma$ by Prop. 3.38(c). If $\beta=\gamma$, then $\alpha+\beta=\alpha+\gamma<\mathbf{S}(\alpha+\gamma)=\alpha+\mathbf{S}(\gamma)$, as needed. If $\beta<\gamma$, then, by induction, $\alpha+\beta<\alpha+\gamma<\mathbf{S}(\alpha+\gamma)=\alpha+\mathbf{S}(\gamma)$. If γ is a limit ordinal and $\beta<\gamma$, then $\mathbf{S}(\beta)<\gamma$ as well. Thus,

$$\alpha+\beta<\mathbf{S}(\alpha+\beta)=\alpha+\mathbf{S}(\beta)\subseteq\bigcup\{\alpha+\xi:\,\xi<\gamma\}=\alpha+\gamma.$$

—

The following result on strictly isotone functions on ordinals will sometimes be useful and it can be proved using transfinite induction and ordinal addition:

Proposition 4.35. (a) *If $f:\mathbf{ON}\longrightarrow\mathbf{ON}$ is strictly isotone, then*

$$\forall_{\alpha\in\mathbf{ON}} \quad \alpha\leq f(\alpha). \tag{4.28}$$

(b) *If $\beta\in\mathbf{ON}$ and $f:\beta\longrightarrow\mathbf{ON}$ is strictly isotone, then* (4.28) *holds with* $\mathbf{ON}$ *replaced by* β.

[11] Left addition is only isotone, cf. Th. 4.43(f).

Proof. (a): We conduct the proof via transfinite induction on $\alpha \in \mathbf{ON}$: The base case ($\alpha = 0$) holds, as $0 \leq f(0) \in \mathbf{ON}$. Now fix $\alpha \in \mathbf{ON}$ and assume $\alpha \leq f(\alpha)$. Using $\alpha < \mathbf{S}(\alpha)$ and the strict isotonicity of f, we obtain

$$\alpha \leq f(\alpha) < f(\mathbf{S}(\alpha)) \overset{\text{Prop. 3.38(d)}}{\Rightarrow} \mathbf{S}(\alpha) \leq f(\mathbf{S}(\alpha)),$$

as needed. If α is a limit ordinal and $\xi \leq f(\xi)$ holds for each $\xi \in \alpha$, then

$$\alpha = \sup\{\xi : \xi \in \alpha\} \leq \sup\{f(\xi) : \xi \in \alpha\} \overset{f \text{ str. iso.}}{\leq} f(\alpha),$$

thereby establishing the case.

(b): One can conduct the proof via transfinite induction on $\alpha \in \beta$, completely analogous to the proof of (a). However, on can, alternatively, obtain (b) from (a) by extending $f : \beta \longrightarrow \mathbf{ON}$ to all of $\mathbf{ON}$ by defining, for $\alpha \geq \beta$,

$$f(\alpha) := \delta + \alpha, \quad \delta := \sup\big\{f(\xi) : \xi \in \beta\big\}$$

(note $\delta \in \mathbf{ON}$ by Th. 3.36(b)). Then f is strictly isotone on $\mathbf{ON}$: If $\beta = 0$, then $f = \mathrm{Id}$ on $\mathbf{ON}$. Otherwise, we have $0 < \beta, \delta$ and, for each $\alpha_1, \alpha_2 \in \mathbf{ON}$:

$$\alpha_1 < \alpha_2 \quad \Rightarrow \quad f(\alpha_1) < f(\alpha_2) \quad \begin{cases} \text{for } \alpha_1, \alpha_2 < \beta, \text{ since } f \text{ strictly isotone on } \beta, \\ \text{for } \alpha_1 < \beta \leq \alpha_2, \text{ since } f(\alpha_1) \leq \delta < f(\alpha_2), \\ \text{for } \beta \leq \alpha_1 \text{ by Ex. 4.34(c).} \end{cases}$$

Thus, (b) now follows from (a). ■

Definition 4.36. *Ordinal Multiplication:* For each $\alpha, \beta \in \mathbf{ON}$ and each limit ordinal λ, define

$$\alpha \cdot 0 := 0, \tag{4.29a}$$
$$\alpha \cdot \mathbf{S}(\beta) := \alpha \cdot \beta + \alpha, \tag{4.29b}$$
$$\alpha \cdot \lambda := \bigcup\{\alpha \cdot \gamma : \gamma < \lambda\} \overset{\text{Th. 3.36(b)}}{=} \sup\{\alpha \cdot \gamma : \gamma < \lambda\} \tag{4.29c}$$

(note that this includes the definition of multiplication on $\omega = \mathbb{N}_0$). It is left as an exercise to show, using Cor. 4.30(a), that this, for each $\alpha \in \mathbf{ON}$, defines a unique function $\cdot : \mathbf{ON} \longrightarrow \mathbf{ON}$, $\xi \mapsto \alpha \cdot \xi := \cdot(\xi)$ (provide a suitable function $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$, similar to the one provided for ordinal addition in Ex. 4.34 above).

Proposition 4.37. **(a)** Multiplication by 0 Yields 0:

$$\underset{\alpha \in \mathbf{ON}}{\forall} \quad \alpha \cdot 0 = 0 \cdot \alpha = 0.$$

(b) 1 Is Neutral for Ordinal Multiplication:

$$\forall_{\alpha\in\mathbf{ON}} \quad \alpha \cdot 1 = 1 \cdot \alpha = \alpha.$$

(c) Right Multiplication is Strictly Isotone[12] for $\alpha > 0$:

$$\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \Big(\alpha > 0 \ \wedge \ \beta < \gamma \quad \Rightarrow \quad \alpha \cdot \beta < \alpha \cdot \gamma\Big).$$

Proof. The proofs can be conducted similar to the proofs in Ex. 4.34(b),(c) above and are left as exercises. ■

So-called *rank functions* play an important role in set theory (see, e.g., [Kun13, p. 50ff]). Theorem 4.29 allows one to define a rank function on every class $\mathbf{A}$ with a well-founded and set-like relation $\mathbf{R}$:

Definition and Remark 4.38. Let $\mathbf{R}$ be a well-founded and set-like relation on the class $\mathbf{A}$. We use transfinite recursion to define a function[13] $\mathrm{rk} : \mathbf{V} \longrightarrow \mathbf{V}$ such that

$$\forall_{y\in\mathbf{V}} \quad \mathrm{rk}(y) := \mathrm{rk}(\mathbf{A}, y, \mathbf{R}) = \begin{cases} \bigcup \big\{\mathbf{S}(\mathrm{rk}(x)) : x \in y_\downarrow\big\} & \text{for } y \in \mathbf{A}, \\ 0 & \text{otherwise :} \end{cases} \tag{4.30}$$

To obtain the rk function via Th. 4.29, we use Th. 4.29 to define $\mathbf{F} : \mathbf{V} \longrightarrow \mathbf{V}$ such that (4.30) holds with $\mathrm{rk}{\restriction}_{\mathbf{A}} := \mathbf{F}{\restriction}_{\mathbf{A}}$. To this end, let

$$\mathbf{G} : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}, \quad \mathbf{G}(x, s) := \begin{cases} \bigcup \big\{\mathbf{S}(s(t)) : t \in \mathrm{dom}(s)\big\} & \text{for } s \text{ a function,} \\ 0 & \text{otherwise.} \end{cases}$$

Then Th. 4.29 yields $\mathbf{F} : \mathbf{V} \longrightarrow \mathbf{V}$ such that

$$\forall_{a\in\mathbf{A}} \ \mathrm{rk}(a) = \mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}{\restriction}_{a_\downarrow}\big) = \bigcup \big\{\mathbf{S}(\mathbf{F}(t)) : t \in a_\downarrow\big\} = \bigcup \big\{\mathbf{S}(\mathrm{rk}(t)) : t \in a_\downarrow\big\},$$

thereby proving (4.30). In particular, if one assumes the axiom of foundation, Axiom 8 below, then $\in$ is well-founded on $\mathbf{V}$ (cf. Prop. 6.1(a)) and $\mathrm{rk}(\mathbf{V}, y, \in)$, according to (4.30), then assigns a rk to every set y. Coming back to the general case of $\mathrm{rk}(\mathbf{A}, y, \mathbf{R})$, if $a \in \mathbf{A}$ is minimal, then $a_\downarrow = \emptyset$, i.e. $\bigcup \big\{\mathbf{S}(\emptyset(t)) : t \in \mathrm{dom}(\emptyset)\big\} = \bigcup \emptyset = \emptyset$, yielding $\mathrm{rk}(a) = \mathbf{F}(a) = \mathbf{G}\big(a, \emptyset\big) = 0$. Thus,

$$\forall_{a\in\mathbf{A}} \quad \Big(a \text{ minimal} \ \Rightarrow \ \mathrm{rk}(a) = \mathrm{rk}(\mathbf{A}, y, \mathbf{R}) = 0\Big). \tag{4.31}$$

[12] Left multiplication is only isotone, cf. Th. 4.44(f).

[13] To avoid overloading pages with boldface, unlike $\mathbf{S}$, we will not typeset the class function rk in boldface.

Example 4.39. To compute some ranks for small sets, we consider the example from [Kun13, p. 47],

$$A := \Big\{0, 1, 2, \{1\}, \{1, 2\}, \big\{\{1\}, \{1, 2\}\big\}\Big\}, \quad \forall_{y \in A} \ \operatorname{rk}(y) := \operatorname{rk}(A, y, \in).$$

It is an exercise to verify

$$\operatorname{rk}(0) = 0, \quad \operatorname{rk}(1) = 1, \quad \operatorname{rk}(2) = \operatorname{rk}(\{1\}) = 2,$$
$$\operatorname{rk}(\{1, 2\}) = 3, \quad \operatorname{rk}\Big(\big\{\{1\}, \{1, 2\}\big\}\Big) = 4.$$

Proposition 4.40. *Let* $\mathbf{R}$ *be a well-founded and set-like relation on the class* $\mathbf{A}$*, and* rk *according to Def. and Rem.* 4.38*. Then every rank is an ordinal, i.e.*

$$\operatorname{rk}(\mathbf{A}) \subseteq \mathbf{ON} \quad \wedge \quad \forall_{y \in \mathbf{A}} \ \operatorname{rk}(y) = \sup\big\{\mathbf{S}(\operatorname{rk}(x)) : x \in y_\downarrow\big\}. \tag{4.32}$$

Moreover, rk *is strictly isotone, i.e.*

$$\forall_{x,y \in \mathbf{A}} \ \Big(x \,\mathbf{R}\, y \quad \Rightarrow \quad \operatorname{rk}(x) < \operatorname{rk}(y)\Big).$$

Proof. We prove $\operatorname{rk}(\mathbf{A}) \subseteq \mathbf{ON}$, using transfinite induction according to Cor. 4.26(a): From (4.31), we know $\operatorname{rk}(y) = 0 \in \mathbf{ON}$ for each minimal element $y \in \mathbf{A}$. Moreover, if $y \in \mathbf{A}$ is not minimal and $\operatorname{rk}(p) \in \mathbf{ON}$ for each $p \in y_\downarrow$, then, by Th. 3.36(b),

$$\operatorname{rk}(y) = \bigcup \big\{\mathbf{S}(\operatorname{rk}(x)) : x \in y_\downarrow\big\} \in \mathbf{ON}.$$

Thus, Cor. 4.26(a) yields $\operatorname{rk}(y) \in \mathbf{ON}$ for each $y \in \mathbf{A}$. Then the second formula in (4.32) is also clear, since, again using Th. 3.36(b), sup and $\bigcup$ are the same on the ordinals. Finally, if $x, y \in \mathbf{A}$ with $x \,\mathbf{R}\, y$, then $x \in y_\downarrow$ and

$$\operatorname{rk}(x) < \mathbf{S}(\operatorname{rk}(x)) \leq \sup\big\{\mathbf{S}(\operatorname{rk}(t)) : t \in y_\downarrow\big\} = \operatorname{rk}(y)$$

proves rk to be strictly isotone. ■

Proposition 4.41. *For each* $\alpha \in \mathbf{ON}$*, one has* $\operatorname{rk}(\alpha) := \operatorname{rk}(\mathbf{ON}, \alpha, \in) = \alpha$*.*

Proof. We prove $\operatorname{rk}(\alpha) = \alpha$ for each $\alpha \in \mathbf{ON}$, using transfinite induction according to Cor. 4.26(a) (as we can not use Cor. 4.26(b)(ii) in the way it was stated). From (4.31), we know $\operatorname{rk}(0) = 0$. If $0 \neq \alpha \in \mathbf{ON}$ and $\operatorname{rk}(\beta) = \beta$ holds for each $\beta \in \alpha$, then

$$\operatorname{rk}(\alpha) = \sup\big\{\mathbf{S}(\beta) : \beta \in \alpha\big\} \stackrel{(*)}{=} \alpha,$$

where $(*)$ holds for limit ordinals α (since, then, $\mathbf{S}(\beta) < \alpha$ for each $\beta < \alpha$) as well as for successor ordinals $\alpha = \mathbf{S}(\gamma)$, $\gamma \in \alpha$ (since, then, $\alpha \in \big\{\mathbf{S}(\beta) : \beta \in \alpha\big\}$). Thus, Cor. 4.26(a) yields $\operatorname{rk}(\alpha) = \alpha$ for each $\alpha \in \mathbf{ON}$. ■

Remark 4.42. Theorem 4.29 has a converse in the sense that, if its conclusion, regarding the definability of functions $\mathbf{F}$ via recursion, holds for a set-like relation $\mathbf{R}$ on a class $\mathbf{A}$, then $\mathbf{R}$ must be well-founded: Indeed, if the conclusion of Th. 4.29 holds, then one can define a rank-like[14] function $\mathbf{F}: \mathbf{A} \longrightarrow \mathbf{V}$, using

$$\hat{\mathbf{S}}: \mathbf{V} \longrightarrow \mathbf{ON}, \quad \hat{\mathbf{S}}(x) := \begin{cases} \mathbf{S}(x) & \text{if } x \in \mathbf{ON}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{G}: \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}, \quad \mathbf{G}(x,s) := \begin{cases} \bigcup \big\{\hat{\mathbf{S}}(s(t)) : t \in \operatorname{dom}(s)\big\} & \text{for } s \text{ a function,} \\ 0 & \text{otherwise.} \end{cases}$$

Then Th. 4.29 yields $\mathbf{F}: \mathbf{V} \longrightarrow \mathbf{V}$ such that

$$\underset{a \in \mathbf{A}}{\forall} \mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}{\restriction}_{a_\downarrow}\big) = \bigcup \big\{\hat{\mathbf{S}}(\mathbf{F}(t)) : t \in a_\downarrow\big\} \in \mathbf{ON},$$

since $\hat{\mathbf{S}}$ maps into $\mathbf{ON}$ and the union of each set of ordinals is an ordinal by Th. 3.36(b). Since $\mathbf{F}(\mathbf{A}) \subseteq \mathbf{ON}$, we obtain, as in the proof of Prop. 4.40, for each $x, y \in \mathbf{A}$ with $x \,\mathbf{R}\, y$,

$$\mathbf{F}(x) < \mathbf{S}(\mathbf{F}(x)) \leq \sup \big\{\mathbf{S}(\mathbf{F}(t)) : t \in y_\downarrow\big\} = \mathbf{F}(y),$$

showing $\mathbf{F}$ to be strictly isotone. Thus, $\mathbf{R}$ must be well-founded by Prop. 4.21(f).

4.4 Ordinal Arithmetic

In the present section, we will study ordinal addition, multiplication, and exponentiation. We begin by continuing our study of ordinal addition that we had initiated in Ex. 4.34:

Theorem 4.43. **(a)** Recursion-Free Characterization of Ordinal Addition: *When we are comparing pairs of ordinals below, we use the lexicographic order* $<$ *on* $\mathbf{ON} \times \mathbf{ON}$*, as defined in Ex.* 4.23(c)*, which we already noted to constitute a strict well-order. It is then also a strict well-order when strongly restricted to any subclass of* $\mathbf{ON} \times \mathbf{ON}$*. In* (4.33) *below, proper classes of ordinals are, actually, not needed, as one could always consider* $<$ *on* $\gamma \times \gamma$ *for some sufficiently large ordinal* γ*. In* (4.33)*, we use the order type as defined in Th.* 3.44 *with respect to the lexicographic order* $<$*, restricted to the respective subset of* $\mathbf{ON} \times \mathbf{ON}$ *(or of* $\gamma \times \gamma$*). One has*

$$\underset{\alpha,\beta \in \mathbf{ON}}{\forall} \quad \alpha + \beta = \operatorname{type}\big(\{0\} \times \alpha \cup \{1\} \times \beta\big). \tag{4.33}$$

[14] In general, the rank function itself does not quite work, as pointed out in [Kun13, Ex. I.9.50].

(b) Associativity of Ordinal Addition:

$$\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma).$$

(c) Commutativity of Ordinal Addition on ω; Noncommutativity on **ON**:

$$\left(\forall_{m,n\in\omega} \quad m+n=n+m\right), \quad \textit{but} \quad 1+\omega=\omega<\omega+1.$$

(d) Left Cancellation of Ordinal Addition; No Right Cancellation:

$$\left(\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \alpha+\beta=\alpha+\gamma \quad\Rightarrow\quad \beta=\gamma\right), \quad \textit{but} \quad 1+\omega=\omega.$$

(e) Ordinal Subtraction:

$$\forall_{\alpha,\beta\in\mathbf{ON}} \quad \left(\alpha\le\beta \;\Rightarrow\; \exists!_{\gamma\in\mathbf{ON}} \;\; \alpha+\gamma=\beta\right).$$

Moreover, one has the representation $\gamma = \operatorname{type}(\beta\setminus\alpha)$.

(f) Left Addition is Isotone[15]:

$$\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \left(\beta\le\gamma \;\Rightarrow\; \beta+\alpha\le\gamma+\alpha\right).$$

(g) ω Is Closed under Ordinal Addition:

$$\forall_{m,n\in\omega} \quad m+n\in\omega.$$

Proof. (a): We need to provide an isomorphism $f:\{0\}\times\alpha\cup\{1\}\times\beta\longrightarrow\alpha+\beta$. We will show that an isomorphism is defined by

$$f:\{0\}\times\alpha\cup\{1\}\times\beta\longrightarrow\alpha+\beta, \quad f(i,x):=\begin{cases} x & \text{for } i=0,\\ \alpha+x & \text{for } i=1: \end{cases}$$

According to Lem. 3.40, it suffices to show f is strictly isotone and surjective. We first show f to be strictly isotone:

$$\begin{aligned}
x,y\in\alpha \;\wedge\; x<y \quad&\Rightarrow\quad f(0,x)=x<y=f(0,y),\\
x,y\in\beta \;\wedge\; x<y \quad&\Rightarrow\quad f(1,x)=\alpha+x\overset{\text{Ex. 4.34(c)}}{<}\alpha+y=f(1,y),\\
x\in\alpha \;\wedge\; y\in\beta \quad&\Rightarrow\quad f(0,x)=x<\alpha\overset{\text{Ex. 4.34(c)}}{\le}\alpha+y=f(1,y).
\end{aligned}$$

[15]Right addition is even strictly isotone, cf. Ex. 4.34(c).

We now show f to be surjective by transfinite induction on $\beta \in \mathbf{ON}$: If $\beta = 0$ and $x \in \alpha$, then $f(0,x) = x$. If $f : \{0\} \times \alpha \cup \{1\} \times \beta \longrightarrow \alpha + \beta$ is surjective, then so is $f : \{0\} \times \alpha \cup \{1\} \times \mathbf{S}(\beta) \longrightarrow \alpha + \mathbf{S}(\beta) = \mathbf{S}(\alpha + \beta)$, since $f(1,\beta) = \alpha + \beta$. If β is a limit ordinal and $f : \{0\} \times \alpha \cup \{1\} \times \gamma \longrightarrow \alpha + \gamma$ is surjective for each $\gamma \in \beta$, then $f : \{0\} \times \alpha \cup \{1\} \times \beta \longrightarrow \alpha + \beta = \bigcup\{\alpha + \gamma : \gamma \in \beta\}$ is surjective: If $\delta \in \alpha + \beta$, then there exists $\gamma \in \beta$ with $\delta \in \alpha + \gamma$ and there exists $(i,x) \in \{0\} \times \alpha \cup \{1\} \times \beta$ with $f(i,x) = \delta$, as $f : \{0\} \times \alpha \cup \{1\} \times \gamma \longrightarrow \alpha + \gamma$ is surjective.

(b): Let $\alpha, \beta, \gamma \in \mathbf{ON}$. For the proof, one can use (a) to show

$$(\alpha + \beta) + \gamma = \operatorname{type}\big(\{0\} \times \alpha \cup \{1\} \times \beta \cup \{2\} \times \gamma\big) = \alpha + (\beta + \gamma) \tag{4.34}$$

(we leave the details as an exercise).

(c): To prove commutativity on ω, we first show $m+1 = 1+m$ via induction on $m \in \omega$: The base case $0 + 1 = 1 + 0$ holds by Ex. 4.34(b); for the induction step, note that $m + 1 = 1 + m$ implies

$$\mathbf{S}(m) + 1 = \mathbf{S}(m+1) = \mathbf{S}(1+m) \overset{\text{Ex. 4.34(a)}}{=} (1+m) + 1 \overset{\text{(b)}}{=} 1 + (m+1) \overset{\text{Ex. 4.34(a)}}{=} 1 + \mathbf{S}(m).$$

From Ex. 4.34(b), we already know $m + 0 = 0 + m$ for each $m \in \omega$, and it remains to prove $m + n = n + m$ for each $n \in \mathbb{N}$, which we do via another induction: The base case $(n = 1)$ has already been done above. For the induction step, if $m + n = n + m$ holds for fixed $n \in \omega$ and each $m \in \omega$, then

$$\begin{aligned} m + \mathbf{S}(n) &= \mathbf{S}(m+n) = \mathbf{S}(n+m) = (n+m) + 1 = 1 + (n+m) \\ &\overset{\text{(b)}}{=} (1+n) + m = (n+1) + m = \mathbf{S}(n) + m, \end{aligned}$$

completing the induction on n and the prove of commutativity on ω. Finally, observe

$$\begin{aligned} 1 + \omega &= \bigcup\{1 + n : n \in \omega\} = \bigcup\{\mathbf{S}(n) : n \in \omega\} \\ &\overset{0=\emptyset}{=} \bigcup\{n : n \in \omega\} = \omega < \mathbf{S}(\omega) = \omega + 1. \end{aligned}$$

(d): $1 + \omega = \omega$ was already shown in (c). The law of left cancellation follows from Ex. 4.34(c) via contraposition: Let $\alpha, \beta, \gamma \in \mathbf{ON}$. If $\beta \neq \gamma$, then $\beta < \gamma$ or $\gamma < \beta$ and Ex. 4.34(c) implies $\alpha + \beta < \alpha + \gamma$ or $\alpha + \gamma < \alpha + \beta$.

(e): Let $\alpha, \beta \in \mathbf{ON}$ with $\alpha \leq \beta$. Uniqueness of $\gamma \in \mathbf{ON}$ with $\alpha + \gamma = \beta$ is clear from (d). Let $\gamma := \operatorname{type}(\beta \setminus \alpha)$ and $f : \gamma \longrightarrow \beta \setminus \alpha$ the corresponding isomorphism. We define an isomorphism

$$g : \{0\} \times \alpha \cup \{1\} \times \gamma \longrightarrow \beta, \quad g(i,x) := \begin{cases} x & \text{for } i = 0, \\ f(x) & \text{for } i = 1 : \end{cases}$$

Indeed, g is strictly isotone, since

$$\begin{aligned} x \in \alpha \ \wedge \ y \in \gamma \quad &\Rightarrow \quad g(0,x) = x < f(y) = g(1,y), \\ x, y \in \alpha \ \wedge \ x < y \quad &\Rightarrow \quad g(0,x) = x < y = g(0,y), \\ x, y \in \gamma \ \wedge \ x < y \quad &\Rightarrow \quad g(1,x) = f(x) < f(y) = g(1,y), \end{aligned}$$

and g is also surjective: If $\delta \in \alpha$, then $g(0,\delta) = \delta$; if $\delta \in \beta \setminus \alpha$, then $g(1, f^{-1}(\delta)) = f(f^{-1}(\delta)) = \delta$. In consequence, g is an isomorphism, proving (e).

(f): We fix $\beta, \gamma \in \mathbf{ON}$ with $\beta \leq \gamma$ and show $\beta + \alpha \leq \gamma + \alpha$ via transfinite induction on $\alpha \in \mathbf{ON}$: The base case ($\alpha = 0$) is true, since $\beta + 0 = \beta \leq \gamma = \gamma + 0$. Now assume $\beta \leq \gamma$ implies $\beta + \alpha \leq \gamma + \alpha$. Then

$$\begin{aligned} \beta + \mathbf{S}(\alpha) = \beta + (\alpha + 1) &\overset{\text{(b)}}{=} (\beta + \alpha) + 1 = \mathbf{S}(\beta + \alpha) \overset{\text{Prop. 3.38(c)}}{\leq} \mathbf{S}(\gamma + \alpha) \\ &= (\gamma + \alpha) + 1 \overset{\text{(b)}}{=} \gamma + (\alpha + 1) = \gamma + \mathbf{S}(\alpha). \end{aligned}$$

If α is a limit ordinal and $\beta + \xi \leq \gamma + \xi$ holds for each $\xi \in \alpha$, then

$$\beta + \alpha = \bigcup \{\beta + \xi : \xi \in \alpha\} \leq \bigcup \{\gamma + \xi : \xi \in \alpha\} = \gamma + \alpha,$$

thereby concluding the induction and the proof.

(g): Let $m \in \omega$. We conduct the proof via induction on $n \in \omega$: $m + 0 = m \in \omega$ provides the base case. If $m + n \in \omega$, then $m + \mathbf{S}(n) = \mathbf{S}(m + n) \in \omega$, thereby establishing the case. ■

Theorem 4.44. **(a)** Recursion-Free Characterization of Ordinal Multiplication: *As in Th.* 4.43(a)*, when comparing pairs of ordinals below, we use the lexicographic order* $<$ *on* $\mathbf{ON} \times \mathbf{ON}$*. As before, we can still restrict the lexicographic order to* $\gamma \times \gamma$ *for some sufficiently large ordinal* γ *(we do not need it on the full class* $\mathbf{ON} \times \mathbf{ON}$*). One has*

$$\forall_{\alpha, \beta \in \mathbf{ON}} \quad \alpha \cdot \beta = \operatorname{type}(\beta \times \alpha). \tag{4.35}$$

(b) Associativity of Ordinal Multiplication:

$$\forall_{\alpha, \beta, \gamma \in \mathbf{ON}} \quad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

(c) Left Distributivity of Ordinal Multiplication; No Right Distributivity on $\mathbf{ON}$:

$$\left(\forall_{\alpha, \beta, \gamma \in \mathbf{ON}} \quad \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \right), \quad \textit{but} \quad (1+1) \cdot \omega = \omega < \omega + \omega = 1 \cdot \omega + 1 \cdot \omega.$$

(d) Commutativity of Ordinal Multiplication on ω; Noncommutativity on $\mathbf{ON}$:

$$\Big(\forall_{m,n\in\omega} \quad m\cdot n = n\cdot m\Big), \quad \textit{but} \quad 2\cdot\omega = \omega < \omega\cdot 2.$$

(e) Left Cancellation of Ordinal Multiplication; No Right Cancellation:

$$\Big(\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \big(\alpha > 0 \;\wedge\; \alpha\cdot\beta = \alpha\cdot\gamma \quad\Rightarrow\quad \beta=\gamma\big)\Big), \quad \textit{but} \quad 1\cdot\omega = 2\cdot\omega = \omega.$$

(f) Left Multiplication is Isotone[16]:

$$\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \big(\beta \le \gamma \;\;\Rightarrow\;\; \beta\cdot\alpha \le \gamma\cdot\alpha\big).$$

(g) ω Is Closed under Ordinal Multiplication:

$$\forall_{m,n\in\omega} \quad m\cdot n \in \omega.$$

(h) Division with Remainder:

$$\forall_{\alpha,\beta\in\mathbf{ON}} \quad \Big(0<\beta \;\;\Rightarrow\;\; \exists!_{\gamma,\delta\in\mathbf{ON}} \;\big(\alpha = \beta\cdot\gamma + \delta \;\;\wedge\;\; 0\le\delta<\beta\big)\Big).$$

Proof. (a): Show that

$$f:\ \beta\times\alpha \longrightarrow \alpha\cdot\beta, \quad f(y,x) := \alpha\cdot y + x,$$

constitutes an isomorphism (exercise).

(b): Let $\alpha,\beta,\gamma\in\mathbf{ON}$. One shows

$$\gamma\times(\alpha\cdot\beta) \cong \gamma\times(\beta\times\alpha) \cong (\gamma\times\beta)\times\alpha \cong (\beta\cdot\gamma)\times\alpha, \tag{4.36}$$

where the outer two isomorphisms are immediate from (a), and providing the inner isomorphism is left as an exercise.

(c): By (a), it suffices to show that

$$f:\ \big(\{0\}\times\beta\cup\{1\}\times\gamma\big)\times\alpha \longrightarrow \{0\}\times(\beta\times\alpha)\cup\{1\}\times(\gamma\times\alpha),$$
$$f\big((i,y),x\big) := \big(i,(y,x)\big),$$

[16] Right multiplication is even strictly isotone, cf. Prop. 4.37(c).

constitutes an isomorphism. Indeed, surjectivity of f is immediate and strict isotonicity also holds, since, for each

$$(i_1, y_1, x_1), (i_2, y_2, x_2) \in \{0,1\} \times (\gamma \cup \beta) \times \alpha,$$

we obtain

$$\begin{aligned} i_1 < i_2 \quad &\Rightarrow \quad \big(i_1, (y_1, x_1)\big) < \big(i_2, (y_2, x_2)\big), \\ i_1 = i_2 \ \wedge\ y_1 < y_2 \quad \Rightarrow \quad (y_1, x_1) < (y_2, x_2) \quad &\Rightarrow \quad \big(i_1, (y_1, x_1)\big) < \big(i_2, (y_2, x_2)\big), \\ i_1 = i_2 \ \wedge\ y_1 = y_2 \ \wedge\ x_1 < x_2 \quad &\Rightarrow \quad \big(i_1, (y_1, x_1)\big) < \big(i_2, (y_2, x_2)\big), \end{aligned}$$

proving

$$\big((i_1, y_1), x_1\big) < \big((i_2, y_2), x_2\big) \quad \Rightarrow \quad f\big((i_1, y_1), x_1\big) < f\big((i_2, y_2), x_2\big).$$

We postpone the proof of $(1+1)\cdot\omega = \omega < \omega + \omega = 1\cdot\omega + 1\cdot\omega$ to the end of the proof of (d).

(d): To prove commutativity on ω, we, first, show

$$\forall_{m,n\in\omega} \quad n\cdot m + m = (n+1)\cdot m \tag{4.37}$$

via induction on m. The base case ($m = 0$) holds, since $0 = n\cdot 0 + 0 = (n+1)\cdot 0$. For the induction step, we compute

$$\begin{aligned} n\cdot \mathbf{S}(m) + \mathbf{S}(m) &\overset{(4.29b)}{=} (n\cdot m + n) + (m+1) \overset{\text{Th. 4.43(b)}}{=} (n\cdot m + (n+m)) + 1 \\ &\overset{\text{Th. 4.43(c)}}{=} (n\cdot m + (m+n)) + 1 \overset{\text{Th. 4.43(b)}}{=} ((n\cdot m + m) + n) + 1 \\ &\overset{\text{ind. hyp.}}{=} ((n+1)\cdot m + n) + 1 \overset{\text{Th. 4.43(c)}}{=} \mathbf{S}(n)\cdot m + \mathbf{S}(n) \\ &\overset{(4.29b)}{=} \mathbf{S}(n)\cdot\mathbf{S}(m) = (n+1)\cdot\mathbf{S}(m). \end{aligned}$$

We are now in a position to carry out the proof of $\forall_{m,n\in\omega}\ m\cdot n = n\cdot m$ via induction on n. We already know $m\cdot 0 = 0\cdot m = 0$ and $m\cdot 1 = 1\cdot m = m$ from Prop. 4.37(a),(b). For the induction step, we compute, for every $m, n \in \omega$,

$$m\cdot\mathbf{S}(n) \overset{(4.29b)}{=} m\cdot n + m \overset{\text{ind. hyp.}}{=} n\cdot m + m \overset{(4.37)}{=} (n+1)\cdot m = \mathbf{S}(n)\cdot m,$$

thereby completing the proof of commutativity on ω. Finally, for each $n \in \mathbb{N}$, we have $n + n \in \omega$ and $2\cdot n = n\cdot 2 = n + n > n$, implying

$$\begin{aligned} 2\cdot\omega \quad &= \quad \bigcup\{2\cdot n :\ n\in\omega\} = \sup\{n+n :\ n\in\omega\} = \sup\{n :\ n\in\omega\} = \omega \\ &\overset{\text{Ex. 4.34(c)}}{<} \quad \omega + \omega = \omega\cdot 1 + \omega\cdot 1 \overset{(c)}{=} \omega\cdot(1+1) = \omega\cdot 2, \end{aligned}$$

which also shows

$$(1+1)\cdot\omega = 2\cdot\omega = \omega < \omega+\omega = 1\cdot\omega + 1\cdot\omega,$$

completing the proof of the inequality in (c) as well.

(e),(f),(g): Exercise.

(h): We first show there exists a unique ordinal γ such that

$$\beta\cdot\gamma \leq \alpha < \beta\cdot\mathbf{S}(\gamma): \tag{4.38}$$

Consider the set $\sigma := \{\xi \in \mathbf{S}(\alpha) : \beta\cdot\xi \leq \alpha\}$. Then $0 \in \sigma$, i.e. $\sigma \neq \emptyset$. We claim σ to be a successor ordinal: If $\xi_0 \in \xi \in \sigma$, then, by Prop. 4.37(c) $\beta\cdot\xi_0 < \beta\cdot\xi \leq \alpha$, i.e. $\xi_0 \in \sigma$ and σ is transitive. Thus, $\sigma \in \mathbf{ON}$ by Cor. 3.35(b). Moreover,

$$\alpha = 1\cdot\alpha \overset{\text{(g)}}{\leq} \beta\cdot\alpha \overset{\text{Ex. 4.34(c)}}{<} \beta\cdot\alpha + \beta \overset{\text{(c)}}{=} \beta\cdot(\alpha+1) = \beta\cdot\mathbf{S}(\alpha). \tag{4.39}$$

If σ were a limit ordinal, then

$$\beta\cdot\sigma \overset{\text{(4.29c)}}{=} \sup\{\beta\cdot\xi : \xi\in\sigma\} \leq \alpha,$$

since α is an upper bound for the set $\{\beta\cdot\xi : \xi\in\sigma\}$. Thus, by (4.39), $\sigma < \mathbf{S}(\alpha)$ and $\sigma \in \sigma$ in contradiction to $\sigma \in \mathbf{ON}$. In consequence, σ must be a successor ordinal and there exists $\gamma \in \mathbf{ON}$ with $\sigma = \mathbf{S}(\gamma)$. Then $\gamma \in \sigma$ and, thus, $\beta\cdot\gamma \leq \alpha$. Also $\alpha < \beta\cdot\sigma = \beta\cdot\mathbf{S}(\gamma)$ (if $\beta\cdot\sigma \leq \alpha$, we, once again, had the contradiction $\sigma\in\sigma$). Thus, γ satisfies (4.38). To prove uniqueness, we assume $\gamma \in \mathbf{ON}$ to satisfy (4.38) and prove $\mathbf{S}(\gamma) = \sigma$: If $\xi \in \mathbf{S}(\gamma)$, then $\xi \leq \gamma$, implying $\beta\cdot\xi \leq \beta\cdot\gamma \leq \alpha$ by Prop. 4.37(c). Thus, by (4.39), $\xi \in \mathbf{S}(\alpha)$, showing $\xi\in\sigma$ and $\mathbf{S}(\gamma) \subseteq \sigma$. Conversely, if $\xi\in\sigma$, then $\beta\cdot\xi \leq \alpha$ and (4.38) yields $\xi < \mathbf{S}(\gamma)$, showing $\sigma \subseteq \mathbf{S}(\gamma)$, completing the prove of uniqueness of γ. If γ satisfies (4.38), then, by Th. 4.43(e), there exists a unique $\delta \in \mathbf{ON}$ such that $\alpha = \beta\cdot\gamma + \delta$. Moreover, $\delta < \beta$, as, otherwise,

$$\alpha < \beta\cdot\mathbf{S}(\gamma) = \beta\cdot(\gamma+1) = \beta\cdot\gamma + \beta \leq \beta\cdot\gamma + \delta.$$

Finally, if $\gamma_0, \delta_0 \in \mathbf{ON}$ with $\alpha = \beta\cdot\gamma_0 + \delta_0$ and $0 \leq \delta_0 < \beta$, then

$$\beta\cdot\gamma_0 \leq \alpha = \beta\cdot\gamma_0 + \delta_0 < \beta\cdot\gamma_0 + \beta = \beta\cdot(\gamma_0+1) = \beta\cdot\mathbf{S}(\gamma_0),$$

implying $\gamma_0 = \gamma$ and, thus, $\delta_0 = \delta$, thereby completing the proof of (h). ■

Definition 4.45. *Ordinal Exponentiation*[17]*:* For each $\alpha, \beta \in \mathbf{ON}$ and each limit ordinal λ, define[18]

$$\alpha_{\text{ord}}^{0} := 1, \tag{4.40a}$$
$$\alpha_{\text{ord}}^{\mathbf{S}(\beta)} := \alpha_{\text{ord}}^{\beta} \cdot \alpha, \tag{4.40b}$$
$$\alpha_{\text{ord}}^{\lambda} := \bigcup\{\alpha_{\text{ord}}^{\gamma} : 0 < \gamma < \lambda\} \overset{\text{Th. 3.36(b)}}{=} \sup\{\alpha_{\text{ord}}^{\gamma} : 0 < \gamma < \lambda\}. \tag{4.40c}$$

The above definition includes the definition of exponentiation on $\omega = \mathbb{N}_0$, where we also use the simpler notation

$$\underset{m,n\in\omega}{\forall} \quad m^n := m_{\text{ord}}^{n}. \tag{4.41}$$

To justify, using Cor. 4.30(a), that (4.40), for each $\alpha \in \mathbf{ON}$, defines a unique function $\hat{}: \mathbf{ON} \longrightarrow \mathbf{ON}$, $\xi \mapsto \alpha_{\text{ord}}^{\xi} := \hat{}(\xi)$, let $x_0 := 1$ and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} x(\beta) \cdot \alpha & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \mathbf{S}(\beta),\ \beta \in \mathbf{ON}, \\ & x(\beta) \in \mathbf{ON}, \\ \bigcup\{x(\gamma) : 0 < \gamma < \lambda\} & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \lambda,\ \lambda \text{ a limit ordinal}, \\ 0 & \text{otherwise}. \end{cases}$$

Then Cor. 4.30(a) provides a unique function $\hat{}: \mathbf{ON} \longrightarrow \mathbf{V}$ with $\alpha_{\text{ord}}^{0} = \hat{}(0) = x_0 = 0$ and $\underset{\xi\in\mathbf{ON}\setminus\{0\}}{\forall} \hat{}(\xi) = \mathbf{H}(\hat{}\restriction_{\xi})$. We use transfinite induction, using Cor. 4.26(b), to show that $\hat{}$ maps into $\mathbf{ON}$ and satisfies (4.40): $\hat{}(0) = 0 \in \mathbf{ON}$ provides the base case. If $\beta \in \mathbf{ON}$ and $\hat{}(\beta) = \alpha_{\text{ord}}^{\beta} \in \mathbf{ON}$, then

$$\alpha_{\text{ord}}^{\mathbf{S}(\beta)} = \hat{}(\mathbf{S}(\beta)) = \mathbf{H}(\hat{}\restriction_{\mathbf{S}(\beta)}) = \hat{}(\beta) \cdot \alpha = \alpha_{\text{ord}}^{\beta} \cdot \alpha \in \mathbf{ON},$$

yielding (4.40b); and, for each limit ordinal λ, assuming $\hat{}(\gamma) = \alpha_{\text{ord}}^{\gamma} \in \mathbf{ON}$ for each $\gamma \in \lambda$,

$$\alpha_{\text{ord}}^{\lambda} = \hat{}(\lambda) = \mathbf{H}(\hat{}\restriction_{\lambda}) = \bigcup\{\hat{}(\gamma) : 0 < \gamma < \lambda\} = \bigcup\{\alpha_{\text{ord}}^{\gamma} : 0 < \gamma < \lambda\} \overset{\text{Th. 3.36(b)}}{\in} \mathbf{ON},$$

yielding (4.40c), and completing the induction.

[17] For a recursion-free characterization of ordinal exponentiation, see, e.g., [Kun13, Ex. I.9.55].

[18] As it is necessary to distinguish between ordinal exponentiation and set exponentiation, we will use the slightly cumbersome subscript "ord" to indicate ordinal exponentiation, whereas no subscript means set exponentiation (or exponentiation of natural numbers): Once we have the power set axiom, we will define $A^B := \mathcal{F}(B, A)$ to mean the set of functions from B into A. Then $\#(m^n) = m_{\text{ord}}^{n}$ for $m, n \in \omega$ and one then often uses the same notation for both m^n and $\#(m^n)$. However, due to $2_{\text{ord}}^{\omega} = \omega < \#(2^\omega)$, the two notions must be distinguished if infinite sets are involved.

Theorem 4.46. (a) Ordinal Exponentiation Involving 0 and 1:

$$\forall_{\alpha\in\mathbf{ON}} \quad \Big(\alpha^0_{\text{ord}} = 1 \quad\wedge\quad \alpha^1_{\text{ord}} = \alpha \quad\wedge\quad 1^\alpha_{\text{ord}} = 1 \quad\wedge\quad (\alpha > 0 \;\Rightarrow\; 0^\alpha_{\text{ord}} = 0)\Big).$$

(b) Ordinal Exponential Functions Are Strictly Isotone for $\alpha > 1$:

$$\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \Big(\alpha > 1 \;\wedge\; \beta < \gamma \quad\Rightarrow\quad \alpha^\beta_{\text{ord}} < \alpha^\gamma_{\text{ord}}\Big).$$

(c) Ordinal Power Functions Are Isotone:

$$\left(\forall_{\alpha,\beta,\gamma\in\mathbf{ON}} \quad \Big(\beta \le \gamma \quad\Rightarrow\quad \beta^\alpha_{\text{ord}} \le \gamma^\alpha_{\text{ord}}\Big)\right), \quad \textit{but} \quad 2^\omega_{\text{ord}} = 3^\omega_{\text{ord}} = \omega.$$

Proof. (a): Exercise.

(b): We assume $\alpha > 1$ and $\beta < \gamma$, and conduct the proof via transfinite induction on γ: For the base case ($\gamma = 0$), there is nothing to show, since, in this case, $\beta < \gamma$ is false. Now assume $\beta < \gamma$ implies $\alpha^\beta_{\text{ord}} < \alpha^\gamma_{\text{ord}}$ and assume $\beta < \mathbf{S}(\gamma)$. Then $\beta \le \gamma$ by Prop. 3.38(c). Then, by induction (and as $\alpha > 1$),

$$\alpha^\beta_{\text{ord}} \le \alpha^\gamma_{\text{ord}} \overset{\text{Prop. 4.37(c)}}{<} \alpha^\gamma_{\text{ord}} \cdot \alpha = \alpha^{\mathbf{S}(\gamma)}_{\text{ord}},$$

as needed. If γ is a limit ordinal and $\alpha^\beta_{\text{ord}} < \alpha^\xi_{\text{ord}}$ for each $\xi \in \gamma$ with $\beta < \xi$, then, for $\beta < \gamma$ (as $\beta < \mathbf{S}(\beta) < \gamma$),

$$\alpha^\beta_{\text{ord}} < \alpha^{\mathbf{S}(\beta)}_{\text{ord}} \le \bigcup\{\alpha^\xi_{\text{ord}} : 0 < \xi < \gamma\} = \alpha^\gamma_{\text{ord}}.$$

(c): We fix $\beta, \gamma \in \mathbf{ON}$ with $\beta \le \gamma$ and show $\beta^\alpha_{\text{ord}} \le \gamma^\alpha_{\text{ord}}$ via transfinite induction on $\alpha \in \mathbf{ON}$: The base case ($\alpha = 0$) is true, since $\beta^0_{\text{ord}} = \gamma^0_{\text{ord}} = 1$. Now assume $\beta \le \gamma$ implies $\beta^\alpha_{\text{ord}} \le \gamma^\alpha_{\text{ord}}$. Then

$$\beta^{\mathbf{S}(\alpha)}_{\text{ord}} = \beta^\alpha_{\text{ord}} \cdot \beta \overset{\text{Prop. 4.37(c), Th. 4.44(f)}}{\le} \gamma^\alpha_{\text{ord}} \cdot \gamma = \gamma^{\mathbf{S}(\alpha)}_{\text{ord}}.$$

If α is a limit ordinal and $\beta^\xi_{\text{ord}} \le \gamma^\xi_{\text{ord}}$ holds for each $\xi \in \alpha$, then

$$\beta^\alpha_{\text{ord}} = \bigcup\{\beta^\xi_{\text{ord}} : 0 < \xi < \alpha\} \le \bigcup\{\gamma^\xi_{\text{ord}} : 0 < \xi < \alpha\} = \gamma^\alpha_{\text{ord}},$$

thereby concluding the induction and the proof of isotonicity. To prove $2^\omega_{\text{ord}} = 3^\omega_{\text{ord}} = \omega$, we first show

$$\forall_{k,n\in\omega\setminus\{0,1\}} \quad \Big(k^n \in \omega \quad\wedge\quad k^n > n\Big) \tag{4.42}$$

via induction on n: For the base case ($n = 2$), observe $k^2 = k \cdot k \in \omega$ by Th. 4.44(g) and $k^2 = k \cdot k > k \geq 2$ (using $k \geq 2$ and Prop. 4.37(c)). If the assertion of (4.42) holds for n, then $k^{\mathbf{S}(n)} = k^n \cdot k \in \omega$ by Th. 4.44(g) and $k^{\mathbf{S}(n)} = k^n \cdot k > n \cdot k \geq n \cdot 2 = n + n > n + 1 = \mathbf{S}(n)$, again using $k \geq 2$ and Prop. 4.37(c). Finally, using (4.42), we obtain for each $k \in \omega \setminus \{0, 1\}$,

$$k^{\omega}_{\text{ord}} = \bigcup \{k^n : n \in \omega\} = \sup\{k^n : n \in \omega\} \overset{(4.42)}{=} \sup\{n : n \in \omega\} = \omega,$$

thereby establishing the case. ∎

To prove the laws $\alpha^{\beta+\gamma}_{\text{ord}} = \alpha^{\beta}_{\text{ord}} \cdot \alpha^{\gamma}_{\text{ord}}$ and $\alpha^{\beta\cdot\gamma}_{\text{ord}} = (\alpha^{\beta}_{\text{ord}})^{\gamma}_{\text{ord}}$, it will be useful to, first, introduce the notions of continuity and normality for functions on the ordinals (here we will, roughly, follow [Gol96, Sec. 8]):

Definition 4.47. Let $\mathbf{F} : \mathbf{ON} \longrightarrow \mathbf{ON}$. We call $\mathbf{F}$ *normal* if, and only if, $\mathbf{F}$ is strictly isotone and *continuous*[19], satisfying the condition

$$\underset{\lambda \in \mathbf{ON}}{\forall} \Big(\lambda \text{ limit ordinal} \;\; \Rightarrow \;\; \mathbf{F}(\lambda) = \bigcup \{\mathbf{F}(\gamma) : \gamma < \lambda\} \Big). \tag{4.43}$$

Lemma 4.48. (a) *Let X be a nonemtpy set of ordinals, $\alpha := \bigcup X$. Then α is a limit ordinal if $\alpha \notin X$.*

(b) *For each ordinal α and each limit ordinal λ, $\alpha + \lambda$ is a limit ordinal.*

Proof. (a): According to Th. 3.36(b), $\alpha = \bigcup X = \sup X$. Suppose $\alpha \notin X$ and $\beta < \alpha$. As $\alpha = \sup X$, there exists $\gamma \in X$ with $\beta < \gamma < \alpha$, where the last equality is due to the fact that $\alpha \notin X$ and $\alpha = \sup X$. Since $\mathbf{S}(\beta) \leq \gamma$, $\mathbf{S}(\beta) < \alpha$, showing α is a limit ordinal.

(b): If $\alpha \in \mathbf{ON}$ and λ is a limit ordinal, then $\alpha + \lambda = \bigcup\{\alpha + \gamma : \gamma < \lambda\}$ by (4.27c). Thus, applying (a) with $X := \{\alpha + \gamma : \gamma < \lambda\}$ and α replaced by $\alpha + \lambda$, we obtain $\alpha + \lambda$ to be a limit ordinal (since $\alpha + \lambda \notin \alpha + \lambda$ by Lem. 3.29). ∎

[19] The continuity is with respect to the so-called *order topology* arising from the order on $\mathbf{ON}$ (cf. [Phi16b, Ex. 1.52(ii)] and Appendix A), where there exists the subtlety that we do not have the order topology on the proper class $\mathbf{ON}$, i.e. it is, actually, the continuity with respect to the order topology on every ordinal sufficiently large to contain any concrete α under consideration. One can then show that, for strictly isotone functions $F : \alpha \longrightarrow \tilde{\alpha}$ with $\alpha, \tilde{\alpha} \in \mathbf{ON}$, continuity is equivalent to a condition of the form (4.43), cf. Prop. A.7(d),(e), where one does not obtain additional conditions at $\lambda := 0$ and successor ordinals λ, as these constitute isolated points in the order topology, cf. Lem. A.5 and Prop. A.7(d),(a). However, as a caveat, it is pointed out that, for nonmonotone functions, (4.43) is neither necessary nor sufficient for continuity, see Ex. A.8(a),(b),(c).

Proposition 4.49. *Let* $\mathbf{F} : \mathbf{ON} \longrightarrow \mathbf{ON}$ *be normal. If* X *is a nonempty subset of* $\mathbf{ON}$*, then*

$$\mathbf{F}\left(\bigcup X\right) = \bigcup \big\{\mathbf{F}(\gamma) : \gamma \in X\big\}. \tag{4.44}$$

Proof. According to Th. 3.36(b), $\alpha := \bigcup X = \sup X \in \mathbf{ON}$. First, consider the case that $\alpha \in X$ (i.e. $\alpha = \max X$). Then, since $\mathbf{F}$ is isotone,

$$\forall_{\gamma \in X} \quad \Big(\gamma \leq \alpha \quad \Rightarrow \quad \mathbf{F}(\gamma) \leq \mathbf{F}(\alpha)\Big),$$

showing

$$\mathbf{F}(\alpha) = \max \big\{\mathbf{F}(\gamma) : \gamma \in X\big\} = \bigcup \big\{\mathbf{F}(\gamma) : \gamma \in X\big\},$$

as claimed. It remains to consider the case, where $\alpha \notin X$. Then, according to Lem. 4.48(a), α must be a limit ordinal and the assumed continuity of $\mathbf{F}$ yields

$$\mathbf{F}(\alpha) = \bigcup \big\{\mathbf{F}(\gamma) : \gamma \in \alpha\big\}.$$

Since $\alpha = \sup X$, we have that $\gamma \in X$ implies $\gamma < \alpha$ (as $\alpha \notin X$), i.e. $X \subseteq \alpha$ and, thus

$$\begin{aligned} &\big\{\mathbf{F}(\gamma) : \gamma \in X\big\} \subseteq \big\{\mathbf{F}(\gamma) : \gamma \in \alpha\big\} \\ \Rightarrow \quad &\bigcup \big\{\mathbf{F}(\gamma) : \gamma \in X\big\} \subseteq \bigcup \big\{\mathbf{F}(\gamma) : \gamma \in \alpha\big\} = \mathbf{F}(\alpha). \end{aligned}$$

Since $\subseteq$ is $\leq$ on $\mathbf{ON}$, we have shown $\sup \big\{\mathbf{F}(\gamma) : \gamma \in X\big\} \leq \mathbf{F}(\alpha)$. To prove (4.44), we need to show equality, i.e. we still need to verify that no $\beta \in \mathbf{ON}$ with $\beta < \mathbf{F}(\alpha)$ can be an upper bound of $\big\{\mathbf{F}(\gamma) : \gamma \in X\big\}$. If $\beta < \mathbf{F}(\alpha) = \sup \big\{\mathbf{F}(\gamma) : \gamma \in \alpha\big\}$, then β is not an upper bound of $\big\{\mathbf{F}(\gamma) : \gamma \in \alpha\big\}$ and there exists $\gamma \in \alpha$ such that $\beta < \mathbf{F}(\gamma)$. Since $\gamma < \alpha$ and $\alpha = \sup X$, γ is not an upper bound of X and there exists $\xi \in X$ with $\gamma < \xi$. As $\mathbf{F}$ is strictly isotone, we obtain $\beta < \mathbf{F}(\gamma) < \mathbf{F}(\xi)$, proving β is not an upper bound of $\big\{\mathbf{F}(\gamma) : \gamma \in X\big\}$, i.e. $\mathbf{F}(\alpha) = \sup \big\{\mathbf{F}(\gamma) : \gamma \in X\big\}$, as desired. ■

Corollary 4.50. **(a)** *With respect to Def.* 4.47*, the following holds:*

$$\forall_{\alpha \in \mathbf{ON}} \quad \mathbf{f} : \mathbf{ON} \longrightarrow \mathbf{ON}, \quad \mathbf{f}(\beta) := \alpha + \beta \;\; \textit{is normal},$$

$$\forall_{\alpha \in \mathbf{ON} \setminus \{0\}} \quad \mathbf{g} : \mathbf{ON} \longrightarrow \mathbf{ON}, \quad \mathbf{g}(\beta) := \alpha \cdot \beta \;\; \textit{is normal},$$

$$\forall_{\alpha \in \mathbf{ON} \setminus \{0,1\}} \quad \mathbf{h} : \mathbf{ON} \longrightarrow \mathbf{ON}, \quad \mathbf{h}(\beta) := \alpha^{\beta}_{\mathrm{ord}} \;\; \textit{is normal.}$$

(b) *If* X *is a nonempty subset of* $\mathbf{ON}$*, then*

$$\forall_{\alpha \in \mathbf{ON}} \quad \alpha + \bigcup X = \bigcup \{\alpha + \gamma : \gamma \in X\},$$

$$\forall_{\alpha \in \mathbf{ON}} \quad \alpha \cdot \bigcup X = \bigcup \{\alpha \cdot \gamma : \gamma \in X\},$$

$$\forall_{\alpha \in \mathbf{ON} \setminus \{0\}} \quad \alpha^{\bigcup X}_{\mathrm{ord}} = \bigcup \{\alpha^{\gamma}_{\mathrm{ord}} : \gamma \in X\}.$$

Proof. (a): We need to verify $\mathbf{f}, \mathbf{g}, \mathbf{h}$ to be continuous and strictly isotone. However, $\mathbf{f}$ and $\mathbf{g}$ are continuous directly from their respective definitions (cf. (4.27c), (4.29c)), and, for $\alpha > 0$, $\mathbf{h}$ is also continuous, since then $\alpha^0 = 1 \leq \alpha^\gamma$ for each $\gamma \in \mathbf{ON}$ and, thus,

$$\mathbf{h}(\lambda) = \alpha_{\mathrm{ord}}^{\lambda} \overset{(4.40c)}{=} \bigcup \{\alpha_{\mathrm{ord}}^{\gamma} : 0 < \gamma < \lambda\} = \bigcup \{\alpha_{\mathrm{ord}}^{\gamma} : \gamma < \lambda\} = \bigcup \{\mathbf{h}(\gamma) : \gamma < \lambda\}$$

for each limit ordinal λ. Moreover, $\mathbf{f}$ is strictly isotone by Ex. 4.34(c), $\mathbf{g}$ is strictly isotone by Prop. 4.37(c) for $\alpha > 0$, and $\mathbf{h}$ is strictly isotone by Prop. 4.46(b) for $\alpha > 1$.

(b): For the values of α allowed in (a), the equalities are an immediate consequence of (a) and Prop. 4.49. For $\alpha = 0$, the equation for multiplication still holds, as both side are then equal to 0. For $\alpha = 1$, the equation for exponentiation still holds, as both side are then equal to 1. ■

Theorem 4.51. Ordinal Exponentiation Laws:

$$\underset{\alpha,\beta,\gamma \in \mathbf{ON}}{\forall} \left(\alpha_{\mathrm{ord}}^{\beta+\gamma} = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\gamma} \quad \wedge \quad \alpha_{\mathrm{ord}}^{\beta\cdot\gamma} = (\alpha_{\mathrm{ord}}^{\beta})_{\mathrm{ord}}^{\gamma} \right).$$

Proof. Both laws can be proved similarly via transfinite induction on γ. We carry out the prove for the first law and leave the second one as an exercise. For

$$\underset{\alpha,\beta,\gamma \in \mathbf{ON}}{\forall} \quad \alpha_{\mathrm{ord}}^{\beta+\gamma} = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\gamma},$$

the base case holds due to

$$\alpha_{\mathrm{ord}}^{\beta+0} = \alpha_{\mathrm{ord}}^{\beta} = \alpha_{\mathrm{ord}}^{\beta} \cdot 1 = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{0}.$$

Next, if $\alpha_{\mathrm{ord}}^{\beta+\gamma} = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\gamma}$ holds, then

$$\alpha_{\mathrm{ord}}^{\beta+\mathbf{S}(\gamma)} = \alpha_{\mathrm{ord}}^{\mathbf{S}(\beta+\gamma)} = \alpha_{\mathrm{ord}}^{\beta+\gamma} \cdot \alpha = (\alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\gamma}) \cdot \alpha = \alpha_{\mathrm{ord}}^{\beta} \cdot (\alpha_{\mathrm{ord}}^{\gamma} \cdot \alpha) = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\mathbf{S}(\gamma)}$$

holds as well. If γ is a limit ordinal and $\alpha_{\mathrm{ord}}^{\beta+\xi} = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\xi}$ holds for each $\xi \in \gamma$, then

$$\begin{aligned} \alpha_{\mathrm{ord}}^{\beta+\gamma} &= \alpha_{\mathrm{ord}}^{\bigcup\{\beta+\xi:\, 0<\xi<\gamma\}} \overset{\text{Cor. 4.50(b)}}{=} \bigcup \{\alpha_{\mathrm{ord}}^{\beta+\xi} : 0 < \xi < \gamma\} \\ &= \bigcup \{\alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\xi} : 0 < \xi < \gamma\} \\ &\overset{\text{Cor. 4.50(b)}}{=} \alpha_{\mathrm{ord}}^{\beta} \cdot \bigcup \{\alpha_{\mathrm{ord}}^{\xi} : 0 < \xi < \gamma\} = \alpha_{\mathrm{ord}}^{\beta} \cdot \alpha_{\mathrm{ord}}^{\gamma}, \end{aligned}$$

completing the induction. ■

Normal functions have the somewhat surprising property of having arbitrarily large fixed points:

Theorem 4.52. *If* $f : \mathbf{ON} \longrightarrow \mathbf{ON}$ *is normal, then*

$$\forall_{\alpha\in\mathbf{ON}} \ \exists_{\beta\in\mathbf{ON}} \ \Big(\alpha \leq \beta \ \wedge \ f(\beta) = \beta\Big). \tag{4.45}$$

Proof. Let $X := \{f^n(\alpha) : n \in \omega\}$, where the ordinals $f^n(\alpha)$ are defined recursively by Ex. 4.31(a). We show that $\beta := \bigcup X$ satisfies (4.45): Due to $\alpha = f^0(\alpha) \in X$, we have $X \neq \emptyset$ and $\alpha \leq \beta$. Moreover, as f is normal, we compute

$$f(\beta) = f\left(\bigcup\{f^n(\alpha) : n \in \omega\}\right) \stackrel{(4.44)}{=} \bigcup\{f(f^n(\alpha)) : n \in \omega\} \stackrel{(*)}{=} \bigcup\{f^n(\alpha) : n \in \omega\} = \beta,$$

where $(*)$ holds, as $f^0(\alpha) = \alpha \leq f(\alpha) = f^1(\alpha)$ (i.e. $f^0(\alpha) \subseteq f^1(\alpha)$) due to f being strictly isotone and (4.28). ■

Example 4.53. Applying Th. 4.52 to the normal functions of Cor. 4.50(a), given $\alpha, \xi \in \mathbf{ON}$, there exist $\beta_1, \beta_2, \beta_3 \in \mathbf{ON}$ with $\beta_1, \beta_2, \beta_3 \geq \xi$ such that

$$\begin{aligned} \alpha + \beta_1 &= \beta_1, \\ \alpha \cdot \beta_2 &= \beta_2 \quad (\text{for } \alpha \neq 0), \\ \alpha_{\text{ord}}^{\beta_3} &= \beta_3 \quad (\text{for } \alpha \neq 0, 1). \end{aligned}$$

5 Power Set Axiom and Cardinality

It is consistent with Axioms 0 – 6 that no uncountable sets exist. There is one more basic construction principle for sets that is not covered by Axioms 0 – 6 and that will provide sets of arbitrarily large cardinality, namely the formation of power sets. This needs another axiom:

Axiom 7 *Power Set*:

$$\forall_X \ \exists_{\mathcal{M}} \ \forall_Y \ \Big(Y \subseteq X \ \Rightarrow \ Y \in \mathcal{M}\Big). \tag{5.1}$$

Thus, the power set axiom states that, for each set X, there exists a set $\mathcal{M}$ that contains all subsets Y of X as elements.

Definition 5.1. If X is a set and $\mathcal{M}$ is given by the power set axiom, then we call

$$\mathcal{P}(X) := \{Y \in \mathcal{M} : Y \subseteq X\}$$

the *power set*[20] of X.

[20] In the literature, another common notation for $\mathcal{P}(X)$ is 2^X. As 2^X is, actually, typically defined to be the set of functions from X into $2 = \{0, 1\}$ (cf. Not. 5.3), writing 2^X for $\mathcal{P}(X)$ means one identifies each $B \subseteq X$ with its *characteristic function* $\chi_B : X \longrightarrow \{0, 1\}$, $\chi_B(x) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B \end{cases}$ (cf. Th. 5.7(b) below).

Example 5.2. Recalling the toy models $M_1, \dots, M_{10}$ of Def. 2.1 plus model M_{11} of Ex. 4.32, check as an exercise that Axiom 7 holds in models M_2, M_3, M_{10}, but is violated in all models M_i with $i \in \{1, 4, \dots, 9, 11\}$.

As before, we summarize the models' properties we found so far in a table:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T	T
$\neg$(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F	T
Axiom 3 (Pairing)	F	T	F	F	F	F	F	F	F	T	T
Axiom 4 (Union)	T	T	T	T	T	T	T	T	F	T	F
Axiom 5 (Replacement)	T	T	T	F	F	F	F	F	F	T	T
Axiom 6 (Infinity)	F	F	F	F	F	F	F	F	F	T	F
Axiom 7 (Power)	F	T	T	F	F	F	F	F	F	T	F

—

If A and B are sets, then, from the power set axiom, we now know the set of relations over A and B, which is precisely $\mathcal{P}(A \times B)$, to exist as well. In consequence, the subset, consisting of the functions $f : A \longrightarrow B$ is a set as well, giving rise to the following notation:

Notation 5.3. The set of all functions with domain A and codomain B is denoted by $\mathcal{F}(A, B)$ or B^A, i.e.

$$\mathcal{F}(A, B) := B^A := \big\{(f : A \longrightarrow B) : A = \operatorname{dom}(f) \wedge B = \operatorname{codom}(f)\big\}. \tag{5.2}$$

—

In Def. 4.11, we already introduced notions regarding a set's size and defined a well-ordered set's cardinality in (4.5). We will now continue the subject of comparing sets by size and, in particular, we will see that a set's power set is always strictly larger than the original set, which will allow us to construct "arbitrarily large" sets. We begin by introducing some additional related notation:

Definition and Remark 5.4. Let A, B be sets.

(a) Write $A \approx B$ if, and only if, there exists a bijective function $\varphi : A \longrightarrow B$ (was already defined in Def. 4.11(a)); $A \not\approx B$ means $\neg(A \approx B)$.

(b) Write $A \preccurlyeq B$ if, and only if, there exists an injective function $\varphi : A \longrightarrow B$; $A \not\preccurlyeq B$ means $\neg(A \preccurlyeq B)$.

(c) Write $A \prec B$ if, and only if, $A \preccurlyeq B$ and $B \not\preccurlyeq A$.

Then, clearly, $A \subseteq B$ implies $A \preccurlyeq B$; $\preccurlyeq$ is transitive and reflexive; $\prec$ is transitive and irreflexive ($\neg(A \prec A)$).

—

To prove $A \approx B$, it is often easier to show $A \preccurlyeq B$ and $B \preccurlyeq A$, i.e. it is often easier to construct two injective functions rather than to, directly, construct a bijective function. Fortunately, the Schröder-Bernstein Th. 5.6 below shows that $A \preccurlyeq B$ and $B \preccurlyeq A$ implies $A \approx B$. Following [Kun12, p. 50], we will base the proof on the following lemma (which can be considered the core of the entire result):

Lemma 5.5. *Let A, B be sets. Then*

$$B \subseteq A \;\wedge\; A \preccurlyeq B \quad \Rightarrow \quad A \approx B.$$

Proof. Assume $B \subseteq A$ and $A \preccurlyeq B$, and let $f : A \longrightarrow B$ be injective. For $n \in \omega$, we use the standard notation f^n, where the $f^n : A \longrightarrow A$ are recursively defined by

$$f^0 := \mathrm{Id}_A, \quad \underset{n\in\omega}{\forall} \; f^{n+1} := f \circ f^n.$$

We then observe

$$f^0(A) = A \supseteq f^0(B) = B \supseteq f^1(A) \supseteq f^1(B) \supseteq f^2(A) \supseteq f^2(B) \supseteq \dots$$

and, indeed, we prove

$$\underset{n\in\omega}{\forall} \; \Big(f^n(B) \subseteq f^n(A) \;\wedge\; f^{n+1}(A) \subseteq f^n(B) \Big) \tag{5.3}$$

via induction on n: $f^0(B) = B \subseteq A = f^0(A)$ and $f^1(A) \subseteq B = f^0(B)$ hold due to the assumptions. If $f^n(B) \subseteq f^n(A)$ and $f^{n+1}(A) \subseteq f^n(B)$ hold, then, applying f to both sides of the inclusions, immediately yields $f^{n+1}(B) \subseteq f^{n+1}(A)$ and $f^{n+2}(A) \subseteq f^{n+1}(B)$ as well. We now define

$$\underset{n\in\omega}{\forall} \; \Big(H_n := f^n(A) \setminus f^n(B), \quad K_n := f^n(B) \setminus f^{n+1}(A) \Big)$$

and

$$P := \bigcap \big\{ f^n(A) : n \in \omega \big\} = \bigcap \big\{ f^n(B) : n \in \omega \big\}.$$

Letting

$$\mathcal{A} := \{P\} \cup \{H_n : n \in \omega\} \cup \{K_n : n \in \omega\}, \quad \mathcal{B} := \mathcal{A} \setminus \{H_0\} = \mathcal{A} \setminus \{A \setminus B\},$$

we obtain disjoint partitions of A and B:

$$A = \bigcup \mathcal{A} \quad \wedge \quad B = \bigcup \mathcal{B} \quad \wedge \quad \underset{C,D \in \mathcal{A}}{\forall} \Big(C \neq D \;\Rightarrow\; C \cap D = \emptyset \Big) : \tag{5.4}$$

Indeed, if $x \in B$ and $x \notin P$, then

$$M := \big\{ n \in \omega : \, x \notin f^n(A) \big\} \neq \emptyset,$$

and we let $m := \min M$. Since $f(A) \subseteq B$, $m \geq 2$, and we have $x \in f^{m-1}(A) \setminus f^m(A)$. Thus, if $x \in f^{m-1}(B)$, then $x \in K_{m-1} = f^{m-1}(B) \setminus f^m(A)$; if $x \notin f^{m-1}(B)$, then $x \in H_{m-1} = f^{m-1}(A) \setminus f^{m-1}(B)$, proving $B \subseteq \bigcup \mathcal{B}$ and, thus, $B = \bigcup \mathcal{B}$ ($\bigcup \mathcal{B} \subseteq B$ is clear, since $f(A) \subseteq B$). Since $\mathcal{A} = \mathcal{B} \cup \{H_0\}$, $A = \bigcup \mathcal{A}$ is now also proved. It remains to show the third statement in (5.4): If $x \in P$, then x is in each $f^n(A)$ and each $f^n(B)$, $n \in \omega$, showing $P \cap H_n = \emptyset$ and $P \cap K_n = \emptyset$ for each $n \in \omega$. Let $n \in \omega$ and $m \in \mathbb{N}$. Then $H_n \cap f^n(B) = \emptyset$ implies $H_n \cap K_n = \emptyset$ and, since $f^{n+m}(A) \subseteq f^n(B)$ and $f^{n+m}(B) \subseteq f^n(B)$ by (5.3), $H_n \cap H_{n+m} = \emptyset$ and $H_n \cap K_{n+m} = \emptyset$. Analogously, $K_n \cap K_{n+m} = \emptyset$ and $K_n \cap H_{n+m} = \emptyset$, completing the proof of (5.4).

Next, we note that, for each $n \in \omega$, $f : H_n \longrightarrow H_{n+1}$ is bijective: Since f is injective on A, we just need to show $f(H_n) = H_{n+1} = f^{n+1}(A) \setminus f^{n+1}(B)$: If $x \in H_n$, then $f(x) \in f^{n+1}(A)$ is clear. If $f(x) \in f^{n+1}(B)$, then $x \in f^n(B)$, proving $f(H_n) \subseteq H_{n+1}$. If $y \in H_{n+1}$, then there exists $a \in A \setminus B$ with $y = f^{n+1}(a)$. Then $x := f^n(a) \in H_n$ with $f(x) = y$ (were $x \in f^n(B)$, then there existed $a \neq b \in B$ with $f^n(b) = x = f^n(a)$ in contradiction to f being injective)[21].

We can now define the bijection

$$g : A \longrightarrow B, \quad g(x) := \begin{cases} f(x) & \text{for } x \in \bigcup\{H_n : \, n \in \omega\}, \\ x & \text{for } x \in P \cup \bigcup\{K_n : \, n \in \omega\} : \end{cases}$$

While g is surjective onto B according to (5.4), g restricted to $P \cup \bigcup\{K_n : \, n \in \omega\}$ is the identity and, thus, bijective, whereas g restricted to $\bigcup\{H_n : \, n \in \omega\}$ is bijective, as we have seen each $f : H_n \longrightarrow H_{n+1}$ to be bijective. ■

Theorem 5.6 (Schröder-Bernstein)**.** *Let A, B be sets. Then*

$$A \approx B \quad \Leftrightarrow \quad A \preccurlyeq B \;\wedge\; B \preccurlyeq A, \tag{5.5a}$$

$$A \prec B \quad \Leftrightarrow \quad A \preccurlyeq B \;\wedge\; A \not\approx B. \tag{5.5b}$$

[21] Analogously, $f : K_n \longrightarrow K_{n+1}$ is bijective for each $n \in \omega$, where one now just has to switch the roles of A and B for the proof; however, we do not need to use the bijectivity $f : K_n \longrightarrow K_{n+1}$

Proof. The nontrivial part of the proof is to show that the existence of an injective $f : A \longrightarrow B$ and an injective $g : B \longrightarrow A$ implies the existence of a bijective function $h : A \longrightarrow B$, which is now easy, since we have already done the main work in Lem. 5.5 above[22]: We have $C := g(B) \subseteq A$ and $(g \circ f) : A \longrightarrow C$ is injective. Then Lem. 5.5 yields $A \approx C$. Since also $C \approx B$ via g, we have $A \approx B$ as desired. It is clear that $A \approx B$ implies $A \preccurlyeq B$ and $B \preccurlyeq A$. Now (5.5b) also follows, since

$$A \prec B \quad \stackrel{\text{Def. and Rem. 5.4(c)}}{\Leftrightarrow} \quad A \preccurlyeq B \;\wedge\; B \not\preccurlyeq A \quad \stackrel{(5.5a)}{\Leftrightarrow} \quad A \preccurlyeq B \;\wedge\; A \not\approx B,$$

thereby concluding the proof. ■

Theorem 5.7. *Let A be a set.*

(a) $A \prec \mathcal{P}(A)$.

(b) $\mathcal{P}(A) \approx 2^A$.

Proof. (a): The function

$$f : A \longrightarrow \mathcal{P}(A), \quad f(a) := \{a\},$$

is, clearly, injective, proving $A \preccurlyeq \mathcal{P}(A)$. To prove $A \prec \mathcal{P}(A)$, in view of (5.5b), it suffices to show there does exist a surjective function $f : A \longrightarrow \mathcal{P}(A)$. If $A = \emptyset$, then $A \not\approx \mathcal{P}(A) = \{\emptyset\}$ is clear. Seeking a contradiction, assume $A \neq \emptyset$ and that there does exist a surjective function $f : A \longrightarrow \mathcal{P}(A)$. Define

$$B := \{x \in A : x \notin f(x)\}.$$

Now B is a subset of A, i.e. $B \in \mathcal{P}(A)$ and the assumption that f is surjective implies the existence of $a \in A$ such that $f(a) = B$. If $a \in B$, then $a \notin f(a) = B$, i.e. $a \in B$ implies $a \in B \wedge \neg(a \in B)$, so $a \notin B$ must hold to avoid this contradiction. However, $a \notin B$ implies $a \in f(a) = B$, i.e., again a contradiction. In conclusion, we have shown our original assumption that there exists a surjective function $f : A \longrightarrow \mathcal{P}(A)$ implies the contradiction $a \in B \wedge \neg(a \in B)$, proving no surjective function from A onto $\mathcal{P}(A)$ can exist.

(b): We show

$$\chi : \mathcal{P}(A) \longrightarrow \{0,1\}^A, \quad \chi(B) := \chi_B,$$

[22] There is a shorter proof that seems more elegant, but, in contrast to the proof via Lem. 5.5, it makes use of the power set axiom, cf. [Phi19a, Th. 3.12]. Another proof that does not make use of the power set axiom can be found in [Phi16a, Th. A.56], but it is even lengthier than the proof presented here.

where, for each $B \subseteq A$,

$$\chi_B : A \longrightarrow \{0,1\}, \quad \chi_B(x) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases}$$

is the so-called *characteristic function* of the set B, to be bijective: χ is injective: Let $B, C \in \mathcal{P}(A)$ with $B \neq C$. By possibly switching the names of B and C, we may assume there exists $x \in B$ such that $x \notin C$. Then $\chi_B(x) = 1$, whereas $\chi_C(x) = 0$, showing $\chi(B) \neq \chi(C)$, proving χ is injective. χ is surjective: Let $f : A \longrightarrow \{0,1\}$ be an arbitrary function and define $B := \{x \in A : f(x) = 1\}$. Then $\chi(B) = \chi_B = f$, proving χ is surjective. ■

Proposition 5.8. *Let A, B, C, D be sets.*

(a) *If $A \preccurlyeq B$ and $C \preccurlyeq D$, then $C^A \preccurlyeq D^B$.*

(b) *If $2 \preccurlyeq C$, then $A \prec \mathcal{P}(A) \approx 2^A \preccurlyeq C^A$.*

(c) $(A^B)^C \approx A^{B \times C}$.

(d) *If $B \cap C = \emptyset$, then $A^{B \cup C} \approx A^B \times A^C$.*

Proof. (a): If $D = \emptyset$, then $C^A = D^B = \emptyset$. Thus, let $d_0 \in D$. Moreover, let $a : A \longrightarrow B$ and $c : C \longrightarrow D$ be injective, $B_0 := a(A) \subseteq B$. Define

$$f : C^A \longrightarrow D^B, \quad f(g)(x) := \begin{cases} (c \circ g \circ a^{-1})(x) & \text{for } x \in B_0, \\ d_0 & \text{otherwise.} \end{cases}$$

Then $f(g) \in D^B$. If $g_1, g_2 \in C^A$ with $g_1 \neq g_2$, then there exists $y \in A$ with $g_1(y) \neq g_2(y)$. Letting $x := a(y) \in B_0 \subseteq B$, injectivity of c yields

$$f(g_1)(x) = c(g_1(y)) \neq c(g_2(y)) = f(g_2)(x),$$

showing $f(g_1) \neq f(g_2)$, i.e. f is injective.

(b): $A \prec \mathcal{P}(A) \approx 2^A$ was shown in Th. 5.7 and $2 \preccurlyeq C$ implies $2^A \preccurlyeq C^A$ due to (a).

(c),(d): Exercise. ■

Proposition 5.9. **(a)** *If $\alpha \in \mathbf{ON}$ and $X \subseteq \alpha$, then $\beta := \operatorname{type}(X, \in) \leq \alpha$.*

(b) *If A is a set and $\alpha \in \mathbf{ON}$ is such that $A \preccurlyeq \alpha$, then there exists a strict well-order $<$ on A and there exists $\beta \in \mathbf{ON}$ with $\beta \leq \alpha$ and $A \approx \beta$.*

(c) *If* $\alpha, \beta, \gamma \in \mathbf{ON}$ *with* $\alpha \leq \beta \leq \gamma$ *and* $\alpha \approx \gamma$*, then* $\alpha \approx \beta \approx \gamma$*.*

(d) *If* $\omega \leq \alpha$*, then* $\alpha \approx \mathbf{S}(\alpha)$*.*

Proof. (a): If $X = \emptyset$, then $\beta = 0 \in \alpha$. Thus, let $X \neq \emptyset$ and let $f : \beta \longrightarrow X$ be an isomorphism. Then, in particular, f is strictly isotone and we have

$$\forall_{\gamma \in \beta} \quad \gamma \leq f(\gamma) \tag{5.6}$$

by Prop. 4.35(b). In consequence,

$$\sup \beta \leq \sup\{f(\gamma) : \gamma \in \beta\} = \sup X \leq \sup \alpha,$$

proving $\beta \leq \alpha$ (otherwise, we had $\beta = \mathbf{S}(\alpha)$ and α a limit ordinal, implying $f(\alpha) = \max X < \alpha$, in contradiction to (5.6)).

(b),(c),(d): Exercise. ■

While Prop. 5.9(d) shows that many ordinals have the same size, recalling (4.5), it makes sense to try to use suitable ordinals to measure the size of sets. Also bearing in mind Prop. 5.9(c), which says that the class **ON** consists of segments of ordinals of the same size, we will now define so-called *cardinals*:

Definition 5.10. $\kappa \in \mathbf{ON}$ is called a *cardinal number* or *cardinal* if, and only if, for each $\alpha \in \kappa$, one has $\alpha \prec \kappa$. We denote the class of all cardinals by **Card**.

Theorem 5.11. (a) *For* $\kappa \in \mathbf{ON}$*, the following statements are equivalent:*

(i) $\kappa \in \mathbf{Card}$.

(ii) *There does not exist* $\alpha \in \kappa$ *with* $\alpha \approx \kappa$*.*

(iii) $\kappa = \min\big\{\alpha \in \mathbf{ON} : \alpha \approx \kappa\big\}$.

(b) *If* $\kappa \in \mathbf{Card}$ *and* $\omega \leq \kappa$*, then* κ *is a limit ordinal.*

(c) $\omega \subseteq \mathbf{Card}$.

(d) *If* X *is a set of cardinals, then* $\sup X \in \mathbf{Card}$*.*

(e) $\omega \in \mathbf{Card}$.

Proof. (a): Since $\alpha \in \kappa$ implies $\alpha \preccurlyeq \kappa$, the equivalence between (i) and (ii) is clear from (5.5b). Since $\kappa \in \{\alpha \in \mathbf{ON} : \alpha \approx \kappa\}$, the equivalence between (ii) and (iii) is also immediate.

(b): If $\kappa \in \mathbf{Card}$ and $\omega \leq \kappa$, then, by Prop. 5.9(d), κ is not a successor ordinal.

(c) is an immediate consequence of Th. 4.14.

(d): Proceeding by contraposition, assume $\sup X = \bigcup X \notin \mathbf{Card}$. Then there exists $\alpha < \sup X$ with $\alpha \approx \sup X$. Since α is not an upper bound of X, there exists $\beta \in X$ with $\alpha < \beta$ and, then, Prop. 5.9(c) shows $\alpha \approx \beta$, i.e. $\beta \notin \mathbf{Card}$ and X is not a set of cardinals.

(e) is now an immediate consequence of (c) and (d). ∎

In (4.5), we assigned a cardinality $\#A \in \mathbf{ON}$ (and we now even know $\#A \in \mathbf{Card}$) to each set A with a strict well-order $<$ and we remarked that one would need the axiom of choice (AC), Axiom 9 of Sec. 7 below, to assign a cardinality to every set. However, for the time being, we would still like to proceed without AC, which leads us to the following definition:

Definition 5.12. We call a set A *well-orderable* if, and only if, there exists a strict well-order $<$ on A. Let $\mathbf{WO}$ denote the class of all well-orderable sets.

Corollary 5.13. **(a)** *If A is a set, then the following statements are equivalent:*

(i) $A \in \mathbf{WO}$.

(ii) $A \preccurlyeq \alpha$ *for some* $\alpha \in \mathbf{ON}$.

(iii) $A \approx \alpha$ *for some* $\alpha \in \mathbf{ON}$.

(b) *If $A \in \mathbf{WO}$, then $\#A \in \mathbf{Card}$ and $\#A = \min\{\alpha \in \mathbf{ON} : \alpha \approx A\}$.*

(c) *If A, B are sets, $A \in \mathbf{WO}$, and $f : A \longrightarrow B$ is surjective, then $B \in \mathbf{WO}$ and $\#B \leq \#A$.*

(d) *If B is a set and $\kappa \in \mathbf{Card}$, then $B \preccurlyeq \kappa$ if, and only if, there exists a surjective $f : \kappa \longrightarrow B$.*

(e) *If $A, B \in \mathbf{WO}$, then*

(i) $A \preccurlyeq B$ *if, and only if,* $\#A \leq \#B$.

(ii) $A \approx B$ *if, and only if,* $\#A = \#B$.

(iii) $A \prec B$ *if, and only if,* $\#A < \#B$.

Proof. (a): (ii) implies (i) and (iii) by Prop. 5.9(b). That (iii) implies (ii) is clear, and (i) implies (ii) by Th. 3.44.

(b): By (4.5),

$$\#A = \min\big\{\alpha \in \mathbf{ON} :\ \alpha \approx \operatorname{type}(A,<)\big\}.$$

Thus, $A \approx \operatorname{type}(A,<)$ implies $\#A = \min\big\{\alpha \in \mathbf{ON} :\ \alpha \approx A\big\}$ and Th. 5.11(a)(iii) implies $\#A \in \mathbf{Card}$.

(c): As $A \in \mathbf{WO}$, let $<$ denote a strict well-order on A. Let $f : A \longrightarrow B$ be surjective and define

$$g : B \longrightarrow A, \quad g(x) := \min f^{-1}(\{x\}) :$$

Then g is well-defined (since the surjectivity of f implies $f^{-1}(\{x\}) \neq \emptyset$ for each $x \in B$) and injective (since $x \neq y$ implies $f^{-1}(\{x\}) \cap f^{-1}(\{y\}) = \emptyset$). Let $\kappa := \#A$ and let $h : A \longrightarrow \kappa$ be bijective. Then $(h \circ g) : B \longrightarrow \kappa$ is injective, showing $B \preccurlyeq \kappa$. By Prop. 5.9(b), $B \in \mathbf{WO}$ and $B \approx \beta$ for some $\beta \in \mathbf{ON}$ with $\beta \leq \kappa$. According to (b), $\#B \leq \beta \leq \kappa = \#A$.

(d): If $f : \kappa \longrightarrow B$ is surjective, then, by (c), $B \in \mathbf{WO}$ and $B \approx \#B \leq \kappa$, i.e. $B \preccurlyeq \kappa$. If $B \preccurlyeq \kappa$, then, by Prop. 5.9(b), $B \in \mathbf{WO}$ and $B \approx \beta$ for some $\beta \in \mathbf{ON}$ with $\beta \leq \kappa$. Since, clearly, one can map κ surjectively onto β, one can also map κ surjectively onto B.

(e): Since $A \approx \#A$ and $B \approx \#B$, it suffices to prove the statements for cardinals. Thus, let $A, B \in \mathbf{Card}$. Then $A = \#A$ and $B = \#B$, i.e. (ii) is trivially true. (i): If $A \leq B$, then $A \subseteq B$ and $A \preccurlyeq B$ is clear. If $A \preccurlyeq B$ and $A \leq B$ does not hold, then $B < A$ and (5.5a) implies $A \approx B$ in contradiction to $B < A$. For (iii), note

$$A \prec B \quad\Leftrightarrow\quad A \preccurlyeq B \ \wedge\ A \not\approx B \quad\overset{\text{(i),(ii)}}{\Leftrightarrow}\quad A \leq B \ \wedge\ A \neq B \quad\Leftrightarrow\quad A < B,$$

thereby completing the proof. ■

Without AC (i.e. in ZF), it is not possible to proof that $\mathcal{P}(\omega) \in \mathbf{WO}$ (or that $\mathbb{R} \in \mathbf{WO}$), see [Kun12, p. 54] for references. It is, thus, perhaps, somewhat surprising that one does not need AC to produce arbitrarily large cardinals, which is a consequence of the following theorem:

Theorem 5.14 (Hartogs)**.** *For every set A, there exists $\kappa \in \mathbf{Card}$ such that $\kappa \not\preccurlyeq A$.*

Proof. Let W be the set of all $(X,R) \in \mathcal{P}(A) \times \mathcal{P}(A \times A)$ such that $R \subseteq X \times X$ is a relation on X that strictly well-orders X (i.e. W consists of all strict well-orders on subsets of A). Set

$$\kappa := \sup\big\{\operatorname{type}(X,R) + 1 :\ (X,R) \in W\big\} \in \mathbf{ON}.$$

We show that, in fact, $\kappa \in \mathbf{Card}$ and $\kappa \not\preccurlyeq A$: If $\alpha \in \mathbf{ON}$, then $\alpha \preccurlyeq A$ holds if, and only if, $\alpha = \operatorname{type}(X,R) \approx X$ for some $(X,R) \in W$: Indeed, if $f : \alpha \longrightarrow A$ is injective, then $f : \alpha \longrightarrow X := f(\alpha) \subseteq A$ is bijective, and

$$\underset{x,y\in X}{\forall} \quad x < y \;:\Leftrightarrow\; f^{-1}(x) < f^{-1}(y)$$

defines a relation on X such that $f : (\alpha, \in) \longrightarrow (X,<)$ is, clearly, an isomorphism. Thus, $(X,<) \in W$ and $\alpha = \operatorname{type}(X,<)$. Since $\kappa > \alpha$ for each $\alpha = \operatorname{type}(X,R)$ with $(X,R) \in W$, we obtain $\kappa \not\preccurlyeq A$ as desired. If $\beta < \kappa$, then there exists $(X,R) \in W$ such that $\beta < \operatorname{type}(X,R)+1$ (as κ is defined as the sup). If $\beta \approx \kappa$, then Prop. 5.9(c) implies $\beta \approx \operatorname{type}(X,R) \approx \kappa$ and $\kappa \preccurlyeq X \preccurlyeq A$, in contradiction to $\kappa \not\preccurlyeq A$. ■

Definition 5.15. If A is a set and $\alpha \in \mathbf{ON}$, then define

$$\operatorname{al}(A) := \aleph(A) := \min\{\kappa \in \mathbf{Card} : \kappa \not\preccurlyeq A\}, \quad \alpha^+ := \aleph(\alpha)$$

(clearly, $\aleph(A)$ is the cardinal constructed in the proof of Th. 5.14). The symbol is called *aleph* and $\aleph : \mathbf{V} \longrightarrow \mathbf{Card}$ is known as *Hartogs aleph function*. Via transfinite recursion over $\mathbf{ON}$, we now also define cardinals $\aleph_\alpha = \omega_\alpha$: For each $\alpha \in \mathbf{ON}$ and each limit ordinal λ, let

$$\aleph_0 := \omega_0 := \omega, \tag{5.7a}$$

$$\aleph_{\mathbf{S}(\alpha)} := \omega_{\mathbf{S}(\alpha)} := (\omega_\alpha)^+, \tag{5.7b}$$

$$\aleph_\lambda := \omega_\lambda := \sup\{\omega_\gamma : \gamma < \lambda\}. \tag{5.7c}$$

To justify, using Cor. 4.30(a), that (5.7) defines a unique function $\mathbf{F} : \mathbf{ON} \longrightarrow \mathbf{Card}$, $\mathbf{F}(\alpha) = \omega_\alpha$, let $x_0 := \omega$ and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} (x(\alpha))^+ & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \mathbf{S}(\alpha),\ \alpha \in \mathbf{ON}, \\ & x(\alpha) \in \mathbf{Card}, \\ \bigcup\{x(\gamma) : \gamma < \lambda\} & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \lambda,\ \lambda \text{ a limit ordinal}, \\ 0 & \text{otherwise}. \end{cases}$$

Then Cor. 4.30(a) provides a unique function $\mathbf{F} : \mathbf{ON} \longrightarrow \mathbf{V}$ with $\mathbf{F}(0) = x_0 = \omega$ and $\underset{\xi\in\mathbf{ON}\setminus\{0\}}{\forall} \mathbf{F}(\xi) = \mathbf{H}\big(\mathbf{F}\restriction_\xi\big)$. We use transfinite induction, using Cor. 4.26(b), to show that $\mathbf{F}$ maps into $\mathbf{Card}$ and satisfies (5.7): $\mathbf{F}(0) = \omega \in \mathbf{Card}$ provides the base case. If $\alpha \in \mathbf{ON}$ and $\mathbf{F}(\alpha) = \omega_\alpha \in \mathbf{Card}$, then

$$\omega_{\mathbf{S}(\alpha)} = \mathbf{F}(\mathbf{S}(\alpha)) = \mathbf{H}\big(\mathbf{F}\restriction_{\mathbf{S}(\alpha)}\big) = (\mathbf{F}(\alpha))^+ = (\omega_\alpha)^+ \in \mathbf{Card},$$

yielding (5.7b); and, for each limit ordinal λ, assuming $\mathbf{F}(\gamma) = \omega_\gamma \in \mathbf{Card}$ for each $\gamma \in \lambda$,

$$\omega_\lambda = \mathbf{F}(\lambda) = \mathbf{H}\big(\mathbf{F}\restriction_\lambda\big) = \bigcup\{\mathbf{F}(\gamma) : \gamma < \lambda\} = \bigcup\{\omega_\gamma : \gamma < \lambda\} \in \mathbf{Card},$$

yielding (5.7c), and completing the induction.

Proposition 5.16. **(a)** *If $\alpha \in \mathbf{ON}$, then $\alpha^+ = \min\{\kappa \in \mathbf{Card} : \alpha < \kappa\}$.*

(b) *For each $\alpha, \beta \in \mathbf{ON}$, $\alpha < \beta$ implies $\omega_\alpha < \omega_\beta$.*

(c) *The function*

$$\mathbf{f} : \mathbf{ON} \longrightarrow \mathbf{Card}, \quad \mathbf{f}(\alpha) := \omega_\alpha,$$

is a normal function and, for each nonempty set X of ordinals,

$$\omega_{\sup X} = \sup\{\omega_\alpha : \alpha \in X\}. \tag{5.8}$$

Moreover,

$$\underset{\alpha \in \mathbf{ON}}{\forall} \quad \underset{\beta \in \mathbf{ON}}{\exists} \quad \Big(\alpha \leq \beta \quad \wedge \quad \omega_\beta = \beta\Big). \tag{5.9}$$

(d) *If A is a set, then A is an infinite cardinal if, and only if, $A = \omega_\alpha$ for some $\alpha \in \mathbf{ON}$:*

$$A \in \mathbf{Card} \setminus \omega \quad \Leftrightarrow \quad \underset{\alpha \in \mathbf{ON}}{\exists} \; A = \omega_\alpha.$$

Proof. (a) holds, since, for each $\alpha \in \mathbf{ON}$ and each $\kappa \in \mathbf{Card}$, we have

$$\kappa \not\leq \alpha \quad \Leftrightarrow \quad \neg(\kappa \leq \alpha) \quad \Leftrightarrow \quad \alpha < \kappa.$$

(b): We conduct the proof via transfinite induction on β: For the base case ($\beta = 0$), there is nothing to show, since, in this case, $\alpha < \beta$ is false. Now assume $\alpha < \beta$ implies $\omega_\alpha < \omega_\beta$ and assume $\alpha < \mathbf{S}(\beta)$. Then $\alpha \leq \beta$ by Prop. 3.38(c). Then, by induction, $\omega_\alpha \leq \omega_\beta < (\omega_\beta)^+ = \omega_{\mathbf{S}(\beta)}$, as needed. If β is a limit ordinal and $\alpha < \gamma$ implies $\omega_\alpha < \omega_\gamma$ for each $\gamma < \beta$, then, since $\alpha < \mathbf{S}(\alpha) < \beta$,

$$\omega_\alpha < \omega_{\mathbf{S}(\alpha)} \leq \sup\{\omega_\gamma : \gamma < \beta\} = \omega_\beta.$$

(c): For each $\alpha \in \mathbf{ON}$, $\omega_\alpha \in \mathbf{Card}$, as was already shown in Def. 5.15. Moreover, $\mathbf{f}$ is normal, since $\mathbf{f}$ is continuous by (5.7c) and $\mathbf{f}$ is strictly isotone by (b). Then (5.8) holds due to Prop. 4.49 and (5.9) is immediate from Th. 4.52.

(d): If $\alpha \in \mathbf{ON}$, then (b) shows $\omega \leq \omega_\alpha$, i.e. ω_α is infinite, whereas $\omega_\alpha \in \mathbf{Card}$ was already shown in (c). Conversely, seeking a contradiction, assume there exists $\kappa \in \mathbf{Card} \setminus \omega$ such that $\kappa \neq \omega_\alpha$ for each $\alpha \in \mathbf{ON}$ (i.e. $\mathbf{f}$, as defined in (c), is not surjective onto $\mathbf{Card} \setminus \omega$). Then we may assume $\kappa = \min\{\alpha \in \mathbf{Card} \setminus \omega : \alpha \notin \mathbf{f}(\mathbf{ON})\}$. Define

$$X := \{\alpha \in \mathbf{ON} : \mathbf{f}(\alpha) < \kappa\}.$$

Then $\omega \in X$, i.e. $X \neq \emptyset$. Let $\sigma := \sup X$. Then, by (c),

$$\lambda := \omega_\sigma = \mathbf{f}(\sigma) = \sup\{\omega_\alpha : \alpha \in X\}.$$

Then $\lambda \in \mathbf{Card} \setminus \omega$ by (b) and Th. 5.11(d). We have $\lambda \leq \kappa$, since κ is an upper bound for $K := \{\omega_\alpha : \alpha \in X\}$. On the other hand, no cardinal $\kappa_0 < \kappa$ can be an upper bound for K: By the definition of κ, there exists $\alpha \in X$ such that $\kappa_0 = \omega_\alpha$. Then $\kappa_0 < (\kappa_0)^+ = \omega_{\alpha+1} \in K$. Thus, $\lambda = \kappa$, which is a contradiction, since $\lambda = \omega_\sigma \in \mathbf{f}(\mathbf{ON})$. ∎

Theorem 5.17. *For each infinite ordinal $\alpha \geq \omega$, $\#(\alpha \times \alpha) = \#\alpha$ (in particular, if $\kappa \in \mathbf{Card} \setminus \omega$, then $\#(\kappa \times \kappa) = \kappa$).*

Proof. Once we have shown $\#(\kappa \times \kappa) = \kappa$ for each $\kappa \in \mathbf{Card} \setminus \omega$, if $\alpha \in \mathbf{ON} \setminus \omega$ and $\kappa := \#\alpha$, then $\alpha \times \alpha \approx \kappa \times \kappa \approx \kappa \approx \alpha$. Thus, without loss of generality, let $\kappa \in \mathbf{Card} \setminus \omega$. While $\kappa \preccurlyeq \kappa \times \kappa$ is clear from the injective function $f : \kappa \longrightarrow \kappa \times \kappa$, $f(\alpha) := (\alpha, 0)$, we need to prove $\kappa \approx \kappa \times \kappa$. To this end, we let $<$ denote lexicographic order on $\mathbf{ON} \times \mathbf{ON}$ (as before) and define another strict well-order $<_0$ on $\mathbf{ON} \times \mathbf{ON}$, which will yield $\kappa = \operatorname{type}(\kappa \times \kappa, <_0)$ for each $\kappa \in \mathbf{Card} \setminus \omega$ (and, in particular, $\kappa \approx \kappa \times \kappa$). Thus, define

$$\forall_{\alpha_1,\beta_1,\alpha_2,\beta_2 \in \mathbf{ON}} \quad (\alpha_1,\beta_1) <_0 (\alpha_2,\beta_2) \quad :\Leftrightarrow \quad \begin{aligned} & \max\{\alpha_1,\beta_1\} < \max\{\alpha_2,\beta_2\} \\ & \vee \ \big(\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} \\ & \qquad \wedge \ (\alpha_1,\beta_1) < (\alpha_2,\beta_2)\big). \end{aligned}$$

We verify that $<_0$ constitutes a strict well-order on $\mathbf{ON} \times \mathbf{ON}$:

$<_0$ is asymmetric:

$$\begin{aligned} & (\alpha_1,\beta_1) <_0 (\alpha_2,\beta_2) \\ \Rightarrow \quad & \max\{\alpha_1,\beta_1\} < \max\{\alpha_2,\beta_2\} \\ & \vee \ \big(\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} \ \wedge \ (\alpha_1,\beta_1) < (\alpha_2,\beta_2)\big) \\ \Rightarrow \quad & \neg(\max\{\alpha_2,\beta_2\} < \max\{\alpha_1,\beta_1\}) \\ & \vee \ \big(\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} \ \wedge \ \neg((\alpha_2,\beta_2) < (\alpha_1,\beta_1))\big) \\ \Rightarrow \quad & \neg\big((\alpha_2,\beta_2) <_0 (\alpha_1,\beta_1)\big), \end{aligned}$$

showing $<_0$ to be asymmetric.

$<_0$ is transitive:

Assume $(\alpha_1,\beta_1) <_0 (\alpha_2,\beta_2)$ and $(\alpha_2,\beta_2) <_0 (\alpha_3,\beta_3)$. There are four cases to consider: (i) If $\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} = \max\{\alpha_3,\beta_3\}$, then $(\alpha_1,\beta_1) < (\alpha_2,\beta_2)$ and $(\alpha_2,\beta_2) < (\alpha_3,\beta_3)$, implying $(\alpha_1,\beta_1) < (\alpha_3,\beta_3)$ and $(\alpha_1,\beta_1) <_0 (\alpha_3,\beta_3)$. (ii) If $\max\{\alpha_1,\beta_1\} < \max\{\alpha_2,\beta_2\} < \max\{\alpha_3,\beta_3\}$, then $\max\{\alpha_1,\beta_1\} < \max\{\alpha_3,\beta_3\}$ and $(\alpha_1,\beta_1) <_0 (\alpha_3,\beta_3)$. (iii) If $\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} < \max\{\alpha_3,\beta_3\}$, then $\max\{\alpha_1,\beta_1\} < \max\{\alpha_3,\beta_3\}$ and $(\alpha_1,\beta_1) <_0 (\alpha_3,\beta_3)$. (iv) If $\max\{\alpha_1,\beta_1\} < \max\{\alpha_2,\beta_2\} = \max\{\alpha_3,\beta_3\}$, then $\max\{\alpha_1,\beta_1\} < \max\{\alpha_3,\beta_3\}$ and $(\alpha_1,\beta_1) <_0 (\alpha_3,\beta_3)$.

$<_0$ satisfies trichotomy:

$$\neg\big((\alpha_1,\beta_1) <_0 (\alpha_2,\beta_2)\big) \wedge \neg\big((\alpha_2,\beta_2) <_0 (\alpha_1,\beta_1)\big)$$
$$\Rightarrow \quad \max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} \wedge (\alpha_1,\beta_1) = (\alpha_2,\beta_2) \quad \Rightarrow \quad (\alpha_1,\beta_1) = (\alpha_2,\beta_2),$$

proving $<_0$ to satisfy trichotomy.

Every nonempty subset X of $\mathbf{ON} \times \mathbf{ON}$ has a min with respect to $<_0$: If $\emptyset \neq X \subseteq \mathbf{ON} \times \mathbf{ON}$, then, letting

$$M := \big\{ \max\{\alpha,\beta\} : (\alpha,\beta) \in X \big\},$$

we have $\emptyset \neq M \subseteq \mathbf{ON}$ and, thus, may set $m := \min M$. Now, letting

$$X_0 := \big\{(\alpha,\beta) \in X : m = \max\{\alpha,\beta\}\big\},$$

we have $\emptyset \neq X_0 \subseteq \mathbf{ON} \times \mathbf{ON}$ and, thus, may set $(\mu,\nu) := \min X_0$, where the min is taken with respect to the lexicographic $<$ on $\mathbf{ON}\times\mathbf{ON}$, which we know to form a strict well-order (cf. Ex. 4.23(c)). We show $(\mu,\nu) = \min X$ with respect to $<_0$: Let $(\alpha,\beta) \in X$. If $\max\{\mu,\nu\} < \max\{\alpha,\beta\}$, then $(\mu,\nu) <_0 (\alpha,\beta)$. If $\max\{\mu,\nu\} = \max\{\alpha,\beta\} = m$, then $(\alpha,\beta) \in X_0$ and $(\mu,\nu) \leq (\alpha,\beta)$ by the definition of (μ,ν), implying $(\mu,\nu) = (\alpha,\beta)$ or $(\mu,\nu) <_0 (\alpha,\beta)$, as desired. This completes the proof that $<_0$ constitutes a strict well-order on $\mathbf{ON} \times \mathbf{ON}$.

Next, we prove

$$\forall_{\kappa \in \mathbf{Card}\setminus\omega} \quad \kappa = \operatorname{type}(\kappa \times \kappa, <_0) \tag{5.10}$$

via transfinite induction according to Th. 4.25: Seeking a contradiction, assume (5.10) does not hold and let $\mathbf{X} := \{\kappa \in \mathbf{Card} \setminus \omega : \kappa \neq \operatorname{type}(\kappa \times \kappa, <_0)\}$. As $\mathbf{X} \neq \emptyset$, by Th. 4.25, we may set $\mu := \min \mathbf{X}$ and $\delta := \operatorname{type}(\mu \times \mu, <_0)$. Note that

$$\forall_{\alpha \in \mathbf{ON}} \quad \big(\alpha < \mu \Rightarrow \kappa := \#(\alpha \times \alpha) < \mu\big): \tag{5.11}$$

If $\alpha \in \omega$, then $\kappa = \alpha^2 \in \omega$ by Th. 4.44(a),(g) and Th. 5.11(c), i.e. $\kappa < \mu$. If $\alpha \geq \omega$, then, from the definition of μ, $\operatorname{type}(\#\alpha \times \#\alpha) = \#\alpha$, i.e. $\#\alpha \times \#\alpha \approx \#\alpha$. Since $\alpha \approx \#\alpha$, we obtain $\alpha \times \alpha \approx \alpha < \mu$, thereby proving (5.11).

Now let $f : \delta \longrightarrow \mu \times \mu$ denote the isomorphism $f : (\delta, <) \cong (\mu \times \mu, <_0)$. If $\delta < \mu$, then one obtains the contradiction $\mu \preccurlyeq \mu \times \mu \approx \delta \prec \mu$ (where $\delta \not\approx \mu$, since $\mu \in \mathbf{Card}$). If $\delta > \mu$, then there exist $\alpha, \beta \in \mu$ such that $f(\mu) = (\alpha,\beta) \in \mu \times \mu$. As the infinite cardinal μ is a limit ordinal by Th. 5.11(b), $\gamma := \max\{\alpha,\beta\} + 1 = \mathbf{S}(\max\{\alpha,\beta\}) < \mu$. Thus, $f(\mu) <_0 (\gamma,\gamma)$ and, since f is strictly isotone, $f{\restriction_\mu}\colon \mu \longrightarrow \gamma \times \gamma$, implying

$$\mu \preccurlyeq \gamma \times \gamma \overset{(5.11)}{\prec} \mu,$$

once again, a contradiction. Thus, $\delta = \mu$, which is also a contradiction (to the definition of μ), proving $\mathbf{X} = \emptyset$, (5.10) and the theorem. ∎

Theorem 5.18. *Ordinal arithmetic does not raise cardinality in the sense that*

$$\underset{\alpha,\beta\in\mathbf{ON}}{\forall} \begin{pmatrix} 2 \leq \min\{\alpha,\beta\} \;\wedge\; \omega \leq \max\{\alpha,\beta\} \\ \Rightarrow \;\; \#(\alpha+\beta) = \#(\alpha\cdot\beta) = \#(\alpha_{\mathrm{ord}}^{\beta}) = \max\{\#\alpha,\#\beta\} \end{pmatrix}. \tag{5.12}$$

Proof. Let $\gamma := \max\{\alpha,\beta\}$. Then $\omega \leq \gamma$. We have

$$\gamma \leq \alpha+\beta \leq \gamma+\gamma = \gamma\cdot 2 \leq \gamma\cdot\gamma$$

by Ex. 4.34(c), Th. 4.43(f), and Prop. 4.37(c). Thus,

$$\gamma\times\gamma \overset{\text{Th. 5.17}}{\approx} \gamma \preccurlyeq \alpha+\beta \preccurlyeq \gamma\cdot\gamma \overset{\text{Th. 4.44(a)}}{\approx} \gamma\times\gamma,$$

proving $\#\gamma = \#(\alpha+\beta)$. Analogously, we have $\gamma \leq \alpha\cdot\beta \leq \gamma\cdot\gamma$, implying $\gamma\times\gamma \preccurlyeq \alpha\cdot\beta \preccurlyeq \gamma\times\gamma$, and, thus, $\#\gamma = \#(\alpha\cdot\beta)$. Next, note that $\gamma \leq 2_{\mathrm{ord}}^{\gamma}$ (for example, due to $f : \delta \longrightarrow f(\delta)$, $f(\xi) := 2_{\mathrm{ord}}^{\xi}$, being an isomorphism on each ordinal δ, plus Prop. 4.35(b)). Using Th. 4.46(b),(c), we obtain $\gamma \leq \alpha_{\mathrm{ord}}^{\beta}$, showing $\gamma \preccurlyeq \alpha_{\mathrm{ord}}^{\beta}$. Showing the existence of an injection $f : \alpha_{\mathrm{ord}}^{\beta} \longrightarrow \gamma$ is more complicated: Let $\delta := \mathbf{S}(\gamma)$, $\kappa := \#\gamma$. Then, by Prop. 5.9(d), $\delta \approx \gamma \approx \kappa$. Thus, there exists a bijective $f : \delta \longrightarrow \kappa$ and, by Th. 5.17, there exists a bijective $g : \kappa\times\kappa \longrightarrow \kappa$. We now define, for each $\xi \in \delta$, an injective function $h_\xi : \alpha_{\mathrm{ord}}^{\xi} \longrightarrow \kappa$, via transfinite recursion on $\xi \in \delta$ (we will, first, describe the definition in the way commonly done in the literature, followed by a justification via Cor. 4.30(b)): For $\xi = 0$, let

$$h_0 : \alpha_{\mathrm{ord}}^{0} = 1 = \{0\} \longrightarrow \kappa, \quad h_0(0) := 0. \tag{5.13a}$$

Given $\xi, \mathbf{S}(\xi) \in \delta$ and $h_\xi : \alpha_{\mathrm{ord}}^{\xi} \longrightarrow \kappa$, define $h_{\mathbf{S}(\xi)} : \alpha_{\mathrm{ord}}^{\mathbf{S}(\xi)} \longrightarrow \kappa$ as follows: Recall $\alpha_{\mathrm{ord}}^{\mathbf{S}(\xi)} = \alpha_{\mathrm{ord}}^{\xi}\cdot\alpha$, i.e., according to Th. 4.44(a), there exists a bijective function $F : \alpha_{\mathrm{ord}}^{\mathbf{S}(\xi)} \longrightarrow \alpha\times\alpha_{\mathrm{ord}}^{\xi}$, $F(a) = (F_1(a), F_2(a))$. Let

$$h_{\mathbf{S}(\xi)} : \alpha_{\mathrm{ord}}^{\mathbf{S}(\xi)} \longrightarrow \kappa, \quad h_{\mathbf{S}(\xi)}(a) := g\big(f(F_1(a)), h_\xi(F_2(a))\big). \tag{5.13b}$$

If $\xi \in \delta$ is a limit ordinal, given $h_\mu : \alpha_{\mathrm{ord}}^{\mu} \longrightarrow \kappa$ for each $\mu \in \xi$, f and g be the same bijections as above and define

$$G : \alpha_{\mathrm{ord}}^{\xi} \longrightarrow \xi, \quad G(a) := \min\{\mu\in\xi : a \in \alpha_{\mathrm{ord}}^{\mu}\}$$

(which is well-defined, since $\alpha_{\mathrm{ord}}^{\xi} = \{\alpha_{\mathrm{ord}}^{\mu} : 0 < \mu < \xi\}$). Let

$$h_\xi : \alpha_{\mathrm{ord}}^{\xi} \longrightarrow \kappa, \quad h_\xi(a) := g\big(f(G(a)), h_{G(a)}(a)\big). \tag{5.13c}$$

To justify, using Cor. 4.30(b), that (5.13) defines a unique function $\eta: \delta \longrightarrow \mathcal{P}(\alpha_{\text{ord}}^{\delta} \times \kappa)$, $\xi \mapsto h_\xi := \eta(\xi)$ (such that h_ξ is, actually, an injective function from $\alpha_{\text{ord}}^{\delta}$ into κ), let $x_0 := \{(0,0)\}$ and $\mathbf{H}: \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x): \alpha_{\text{ord}}^{\mathbf{S}(\xi)} \longrightarrow \kappa, \quad \mathbf{H}(x)(a) := g\big(f(F_1(a)), x(\xi)(F_2(a))\big)$$

if x is a function with $\operatorname{dom}(x) = \mathbf{S}(\xi)$, where $\xi, \mathbf{S}(\xi) \in \delta$, and $x(\xi) \in \mathcal{F}(\alpha_{\text{ord}}^{\xi}, \kappa) = \kappa^{\alpha_{\text{ord}}^{\xi}}$ (with g, f, F_1, F_2 as defined above),

$$\mathbf{H}(x): \alpha_{\text{ord}}^{\xi} \longrightarrow \kappa, \quad \mathbf{H}(x)(a) := g\big(f(G(a)), x(G(a))(a)\big)$$

if x is a function with $\operatorname{dom}(x) = \xi$, $\xi \in \delta$ a limit ordinal, and $x(\mu) \in \mathcal{F}(\alpha_{\text{ord}}^{\mu}, \kappa) = \kappa^{\alpha_{\text{ord}}^{\mu}}$ for each $\mu \in \xi$ (with g, f, G as defined above),

$$\mathbf{H}(x) := 0 \quad \text{otherwise.}$$

Then Cor. 4.30(b) provides a unique function $\eta: \delta \longrightarrow \mathbf{V}$ with $h_0 = \eta(0) = x_0$, satisfying (5.13a), and $\forall_{\xi \in \delta \setminus \{0\}} \ \eta(\xi) = \mathbf{H}(\eta\restriction_\xi)$. We use transfinite induction, using Cor. 4.26(c), to show that, for each $\xi \in \delta$, $h_\xi := \eta(\xi): \alpha_{\text{ord}}^{\xi} \longrightarrow \kappa$ is an injectve function, satisfying (5.13): $h_0 = \eta(0) = x_0$, satisfying (5.13a), provides the base case. If $\xi, \mathbf{S}(\xi) \in \delta$ and $\eta(\xi) = h_\xi: \alpha_{\text{ord}}^{\xi} \longrightarrow \kappa$ is injective, then

$$h_{\mathbf{S}(\xi)} = \eta(\mathbf{S}(\xi)) = \mathbf{H}(\eta\restriction_{\mathbf{S}(\xi)}): \alpha_{\text{ord}}^{\mathbf{S}(\xi)} \longrightarrow \kappa,$$
$$h_{\mathbf{S}(\xi)}(a) = \eta(\mathbf{S}(\xi))(a) = \mathbf{H}(\eta\restriction_{\mathbf{S}(\xi)})(a) = g\big(f(F_1(a)), h_\xi(F_2(a))\big),$$

yielding (5.13b). Moreover, $h_{\mathbf{S}(\xi)}$ is injective, since $F = (F_1, F_2)$, f, h_ξ, and g are injective. If $\xi \in \delta$ is a limit ordinal and $\eta(\mu) = h_\mu: \alpha_{\text{ord}}^{\mu} \longrightarrow \kappa$ is injective for each $\mu \in \xi$, then

$$h_\xi = \eta(\xi) = \mathbf{H}(\eta\restriction_\xi): \alpha_{\text{ord}}^{\xi} \longrightarrow \kappa,$$
$$h_\xi(a) = \eta(\xi)(a) = \mathbf{H}(\eta\restriction_\xi)(a) = g\big(f(G(a)), h_{G(a)}(a)\big),$$

yielding (5.13c). Moreover, h_ξ is injective: If $a, b \in \alpha_{\text{ord}}^{\xi}$ and $G(a) \neq G(b)$, then $f(G(a)) \neq f(G(b))$ and $h_\xi(a) \neq h_\xi(b)$; if $G(a) = G(b)$ and $a \neq b$, then $h_{G(a)}(a) \neq h_{G(a)}(b)$, again implying $h_\xi(a) \neq h_\xi(b)$. This completes the induction. Since $\beta \leq \gamma < \delta$, we have, in particular, proved the existence of an injective map $h_\beta: \alpha_{\text{ord}}^{\beta} \longrightarrow \kappa$. Since $\kappa \approx \gamma$, the proof of $\alpha_{\text{ord}}^{\beta} \preccurlyeq \gamma$ is also complete. ∎

6 Foundation

Foundation is, perhaps, the least important of the axioms in ZF. It basically cleanses the mathematical universe of unnecessary "clutter", i.e. of certain pathological sets that are of no importance to standard mathematics anyway.

Axiom 8 *Foundation:*

$$\underset{X}{\forall}\left(\left(\underset{x}{\exists}\, x \in X\right) \Rightarrow \underset{x\in X}{\exists}\, \neg\underset{z}{\exists}\left(z \in x \;\wedge\; z \in X\right)\right). \tag{6.1}$$

Thus, the foundation axiom states that every nonempty set X contains an element x that is disjoint to X.

In the following, if we make use of Axiom 8 to prove some result, the axiom will be explicitly mentioned.

Proposition 6.1. **(a)** *Axiom* 8 *is equivalent to the statement that* $\in$ *is well-founded on* $\mathbf{V}$.

(b) *Axiom* 8 *implies* $\in$ *to be acyclic on* $\mathbf{V}$ *(i.e. the transitive closure* $\in^*$ *is irreflexive on* $\mathbf{V}$*, cf. Def.* 4.19(d)*). Thus, there do not exist sets* $x_1, x_2, \dots, x_n$*,* $n \in \mathbb{N}$*, such that*

$$x_1 \in x_2 \in \cdots \in x_n \in x_1. \tag{6.2a}$$

In particular, sets can not be members of themselves:

$$\neg\underset{x}{\exists}\, x \in x. \tag{6.2b}$$

Proof. (a): The relation $\in$ is well-founded on $\mathbf{V}$ if, and only if, every nonempty set X contains an element x that is $\in$-minimal in X. Since x is $\in$-minimal in X if, and only if, there does not exist $z \in X$ such that $z \in x$, we see that (6.1) is precisely the statement that $\in$ is well-founded on $\mathbf{V}$.

(b): That $\in$ is acyclic is immediate from (a) and Prop. 4.21(c). Then (6.2) is also clear, since (6.2a) is the same as saying there exists an $\in$-path from x_1 to x_1 (cf. Def. 4.19(c)). ■

Remark 6.2. To obtain (6.2b), one, actually, needs only two axioms, namely foundation and comprehension (Axioms 8 and 2): If there exists a set z with $z \in z$, then we may apply Axiom 2, i.e.

$$\underset{Y}{\exists}\;\underset{x}{\forall}\;\left(x \in Y \;\Leftrightarrow\; (x \in X \wedge \phi)\right),$$

with $X := z$ and $\phi : x = X$ to obtain a set Y containing precisely $x = X = z$ (we may think of Y as $Y = \{z\}$, but in absence of extensionality, there could be sets $U \neq Y$ that also contain precisely z). However, Y now violates (6.1), since Y contains only z and both Y and z contain z.

Indeed, comprehension is needed in the above argument: Modell M_{10} of Def. 2.1 satisfies all axioms of ZF (Axioms 0 – 8), except comprehension, and (6.2b) fails in M_{10} (cf. Ex. 6.3 below). On the other hand, model M_2 satisfies (6.2b), but not (6.1).

Example 6.3. As before, $M_1, \dots, M_{10}$ denote the toy models of Def. 2.1, M_{11} denotes the model of Ex. 4.32.

One finds that Axiom 8 holds in all the models, except in M_2 and M_4 (we consider M_1, M_2, M_{10}, and leave the others as exercises – for M_{11} you may use Prop. 6.14(a) and Prop. 6.13(b) below):

M_1 satisfies (6.1), since D_1 does not contain any nonempty sets.

M_2 violates (6.1), since $a \in D_2$ is nonempty, but a contains only the element a, which is not disjoint to a (as $a\,E_2\,a$).

M_{10} satisfies (6.1), as b is the only nonemtpy set and b contains the empty set a.

As before, we summarize the models' properties we found so far in a table:

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}
Axiom 0 (Existence)	T	T	T	T	T	T	T	T	T	T	T
Axiom 1 (Extensionality)	T	T	T	T	F	T	T	F	T	T	T
$\neg$(2.1) (has empty set)	T	F	F	F	T	T	T	T	T	T	T
Axiom 2 (Comprehension)	T	F	F	F	T	F	T	T	T	F	T
Axiom 3 (Pairing)	F	T	F	F	F	F	F	F	F	T	T
Axiom 4 (Union)	T	T	T	T	T	T	T	T	F	T	F
Axiom 5 (Replacement)	T	T	T	F	F	F	F	F	F	T	T
Axiom 6 (Infinity)	F	F	F	F	F	F	F	F	F	T	F
Axiom 7 (Power)	F	T	T	F	F	F	F	F	F	T	F
Axiom 8 (Foundation)	T	F	T	F	T	T	T	T	T	T	T

—

According to Def. and Rem. 4.38, if Axiom 8 holds, then the rank $\mathrm{rk}(\mathbf{V}, x, \in)$ is defined for each set x. Thus, not surprisingly, the rank function is a useful tool in connexion with the axiom of foundation so that we pursue our related studies, begun at the end of Sec. 4.3, a little further.

Lemma 6.4. *Let* $\mathbf{A}, \mathbf{B}$ *be classes,* $\mathbf{A} \subseteq \mathbf{B}$*, and let* $\mathbf{R}$ *be a well-founded and set-like relation on* $\mathbf{B}$ *with transitive closure* $\mathbf{R}^*$*.*

(a) *For each* $a \in \mathbf{A}$*, we have* $\operatorname{rk}(\mathbf{A}, a, \mathbf{R}) \leq \operatorname{rk}(\mathbf{B}, a, \mathbf{R})$.

(b) *For each* $a \in \mathbf{A}$ *such that* $\operatorname{pred}(\mathbf{B}, a, \mathbf{R}^*) \subseteq \mathbf{A}$*, we have* $\operatorname{rk}(\mathbf{A}, a, \mathbf{R}) = \operatorname{rk}(\mathbf{B}, a, \mathbf{R})$.

Proof. (a): We show the inequality via transfinite induction, using Th. 4.25: Assuming $\mathbf{X} := \big\{a \in \mathbf{A} : \operatorname{rk}(\mathbf{A}, a, \mathbf{R}) > \operatorname{rk}(\mathbf{B}, a, \mathbf{R})\big\} \neq \emptyset$, Th. 4.25 provides an $\mathbf{R}$-minimal element $y \in \mathbf{X}$. Then, by (4.32),

$$\begin{aligned}\operatorname{rk}(\mathbf{A}, y, \mathbf{R}) &= \sup\big\{\mathbf{S}\big(\operatorname{rk}(\mathbf{A}, x, \mathbf{R})\big) : x \in \mathbf{A} \wedge x\,\mathbf{R}\,y\big\}\\ &\leq \sup\big\{\mathbf{S}\big(\operatorname{rk}(\mathbf{B}, x, \mathbf{R})\big) : x \in \mathbf{B} \wedge x\,\mathbf{R}\,y\big\} = \operatorname{rk}(\mathbf{B}, y, \mathbf{R}),\end{aligned}$$

which is in contradiction to $y \in \mathbf{X}$, proving $\mathbf{X} = \emptyset$ and (a).

(b): The proof is, once again, via transfinite induction, using Th. 4.25: Assuming $\mathbf{X} := \big\{a \in \mathbf{A} : \operatorname{rk}(\mathbf{A}, a, \mathbf{R}) \neq \operatorname{rk}(\mathbf{B}, a, \mathbf{R}) \wedge \operatorname{pred}(\mathbf{B}, a, \mathbf{R}^*) \subseteq \mathbf{A}\big\} \neq \emptyset$, Th. 4.25 provides an $\mathbf{R}$-minimal element $y \in \mathbf{X}$. Then, by (4.32),

$$\begin{aligned}\operatorname{rk}(\mathbf{A}, y, \mathbf{R}) &= \sup\big\{\mathbf{S}\big(\operatorname{rk}(\mathbf{A}, x, \mathbf{R})\big) : x \in \mathbf{A} \wedge x\,\mathbf{R}\,y\big\}\\ &\overset{(*)}{=} \sup\big\{\mathbf{S}\big(\operatorname{rk}(\mathbf{B}, x, \mathbf{R})\big) : x \in \mathbf{B} \wedge x\,\mathbf{R}\,y\big\} = \operatorname{rk}(\mathbf{B}, y, \mathbf{R}),\end{aligned}$$

which is in contradiction to $y \in \mathbf{X}$, proving $\mathbf{X} = \emptyset$ and (a). At $(*)$, we used that y is minimal in $\mathbf{X}$ and $y \in \mathbf{X}$ implies $\operatorname{pred}(\mathbf{B}, y, \mathbf{R}^*) \subseteq \mathbf{A}$, which, together with $x \in \mathbf{B}$ and $x\,\mathbf{R}\,y$ implies both $x \in \mathbf{A}$ and $\operatorname{pred}(\mathbf{B}, x, \mathbf{R}^*) \subseteq \operatorname{pred}(\mathbf{B}, y, \mathbf{R}^*) \subseteq \mathbf{A}$. Thus, the second sup is, actually, taken over the same set as the first sup. ■

Definition 6.5. If y is a set, then we call

$$\operatorname{tcl}(y) := \operatorname{pred}(\mathbf{V}, y, \in^*) = \{x : x \in^* y\} \tag{6.3}$$

the *transitive closure* of the set y (caveat: while this definition is quite common in the literature, one should take note that, if the set y is a relation on some class, than two distinct notions of transitive closure are defined for y, namely y^* and $\operatorname{tcl}(y)$).

Lemma 6.6. *Assume* $\mathbf{T}$ *to be a transitive class (i.e.* $x \in X \in \mathbf{T}$ *implies* $x \in \mathbf{T}$*) and let* y *be a set.*

(a) *If* $y \subseteq \mathbf{T}$*, then* $\operatorname{tcl}(y) \subseteq \mathbf{T}$.

(b) *If* $y \in \mathbf{T}$*, then* $\operatorname{tcl}(y) \subseteq \mathbf{T}$.

Proof. (a): Let $y \subseteq \mathbf{T}$. If $x \in \operatorname{tcl}(y)$, then there exists an $\in$-path $\pi : D \longrightarrow \pi(D)$ such that $D = \mathbf{S}(n)$, $n \in \mathbb{N}$, $\pi(0) = x$, $\pi(n) = y$. Inductively, one sees that each

$i \in \{0, \dots, n-1\}$, $\pi(i) \in \mathbf{T}$: $\pi(n-1) \in \pi(n) = y$, i.e. $\pi(n-1) \in \mathbf{T}$, as $y \subseteq \mathbf{T}$ by assumption. If $\pi(i) \in \mathbf{T}$ and $i-1 \geq 0$, then $\pi(i-1) \in \pi(i)$, implying $\pi(i-1) \in \mathbf{T}$, since $\mathbf{T}$ is transitive[23].

(b) follows from (a), since $y \in \mathbf{T}$ implies $y \subseteq \mathbf{T}$, as $\mathbf{T}$ is transitive. ■

Proposition 6.7. *For each set y, the following holds:*

(a) $\mathrm{tcl}(y)$ *is a transitive set.*

(b) *We have the representations*

$$\mathrm{tcl}(y) = \bigcap\{T : y \subseteq T \wedge T \text{ is transitive}\} = \bigcup\left\{\bigcup^n y : n \in \omega\right\}, \tag{6.4}$$

where, using the notation introduced in Prop. 4.24 with $\mathbf{A} := \mathbf{V}$ and $\mathbf{R} :=\in$,

$$\underset{n\in\omega}{\forall} \quad \left(\bigcup^n y := D_{\mathbf{S}(n)}(y), \quad \bigcup^{\mathbf{S}(n)} y = \bigcup\bigcup^n y\right); \tag{6.5}$$

also note

$$\bigcup^0 y = D_1(y) = y.$$

In particular, if T is a transitive set with $y \subseteq T$, then $\mathrm{tcl}(y) \subseteq T$, i.e., in combination with (a), $\mathrm{tcl}(y)$ *is the smallest transitive superset of y.*

Proof. (a): Since $\in$ is set-like by Ex. 4.23(b), $\mathrm{tcl}(y)$ is a set by Prop. 4.24(c). If $x \in X \in \mathrm{tcl}(y)$, then $x \in^* X$ and $X \in^* y$, implying $x \in^* y$, since $\in^*$ is transitive. Thus, $x \in \mathrm{tcl}(y)$, showing $\mathrm{tcl}(y)$ to be transitive.

(b): Let $\mathbf{A} := \{T : y \subseteq T \wedge T \text{ is transitive}\}$. If $T \in \mathbf{A}$, then Lem. 6.6(a) (applied with $\mathbf{T} := T$) implies $\mathrm{tcl}(y) \subseteq T$, showing $\mathrm{tcl}(y) \subseteq \bigcap \mathbf{A}$. On the other hand, since $y \subseteq \mathrm{tcl}(y)$ and $\mathrm{tcl}(y)$ is transitive by (a), we have $\mathrm{tcl}(y) \in \mathbf{A}$ and, thus, $\bigcap \mathbf{A} \subseteq \mathrm{tcl}(y)$, proving $\mathrm{tcl}(y) = \bigcap \mathbf{A}$, i.e. the first equality in (6.4). To prove the second equality in (6.4), we compute

$$\begin{aligned}\mathrm{tcl}(y) = \mathrm{pred}(\mathbf{V}, y, \in^*) &\overset{\text{Prop. 4.24(c)}}{=} \bigcup\{D_n(y) : n \in \mathbb{N}\} \\ &= \bigcup\{D_{\mathbf{S}(n)}(y) : n \in \omega\} = \bigcup\left\{\bigcup^n y : n \in \omega\right\},\end{aligned}$$

thereby establishing the case. Moreover, $\bigcup^0 y = D_1(y) = \mathrm{pred}(\mathbf{V}, y, \in) = \{x : x \in y\} = y$. To prove the second equality in (6.5), we show

$$\underset{n\in\mathbb{N}}{\forall} \quad D_{\mathbf{S}(n)}(y) = \bigcup D_n(y) : \tag{6.6}$$

[23] Alternatively, if one prefers the induction to go forward from 1 to n, one can conduct an analogous proof using Prop. 4.24(b),(c).

Indeed, for each $n \in \mathbb{N}$,

$$x \in D_{\mathbf{S}(n)}(y) \quad \Leftrightarrow \quad \underset{\pi: D \longrightarrow \pi(D)}{\exists} \begin{pmatrix} \pi \text{ is } \in\text{-path } \wedge \ D = n+2 \\ \wedge \ \pi(0) = x \ \wedge \ \pi(n+1) = y \end{pmatrix} \quad \Leftrightarrow \quad x \in \bigcup D_n(y),$$

thereby proving (6.6). ■

Example 6.8. If $y = \big\{\{\{1\}\}\big\}$, then

$$\operatorname{tcl}(y) = \big\{0, 1, \{1\}, \{\{1\}\}\big\}.$$

Lemma 6.9. *If y is a set, then $\in$ is well-founded on $\operatorname{tcl}(y)$ if, and only if, $\in$ is well-founded on $\{y\} \cup \operatorname{tcl}(y)$.*

Proof. For $y \in \operatorname{tcl}(y)$, there is nothing to prove. Thus, assume $y \notin \operatorname{tcl}(y)$, i.e. $\neg(y \in^* y)$. It is also clear that, if $\in$ is well-founded on the larger set $\{y\} \cup \operatorname{tcl}(y)$, then it is also well-founded on the smaller set $\operatorname{tcl}(y)$. For the remaining implication, assume $\in$ to be well-founded on $\operatorname{tcl}(y)$ and let $\emptyset \neq X \subseteq \{y\} \cup \operatorname{tcl}(y)$. If $X \subseteq \operatorname{tcl}(y)$, then X contains a minimal element, since $\in$ is well-founded on $\operatorname{tcl}(y)$. If $X = \{y\}$, then y is minimal in X. On the other hand, if $Y := X \setminus \{y\} \neq \emptyset$, then $Y \subseteq \operatorname{tcl}(y)$ and, thus, there exists a minimal element $m \in Y$. Then m is also minimal in X, since $y \in m$ implied $y \in^* m \in^* y$, i.e. $y \in^* y$, in contradiction to the assumption $\neg(y \in^* y)$. Thus, $\in$ is well-founded $\{y\} \cup \operatorname{tcl}(y)$. ■

Definition 6.10. We call a set y *well-founded* if, and only if, $\in$ is well-founded on $\operatorname{tcl}(y)$. If y is well-founded, then we define

$$\operatorname{rk}(y) := \operatorname{rk}\big(\{y\} \cup \operatorname{tcl}(y), y, \in\big). \tag{6.7}$$

Moreover, we let $\mathbf{WF}$ denote the class of all well-founded sets.

Proposition 6.11. *Assume $\mathbf{T}$ to be a transitive class. If $\in$ is well-founded on $\mathbf{T}$, then $\mathbf{T} \subseteq \mathbf{WF}$ and*

$$\underset{y \in \mathbf{T}}{\forall} \quad \operatorname{rk}(y) \overset{(6.7)}{=} \operatorname{rk}\big(\{y\} \cup \operatorname{tcl}(y), y, \in\big) = \operatorname{rk}(\mathbf{T}, y, \in). \tag{6.8}$$

Proof. If $y \in \mathbf{T}$, then Lem. 6.6(b) implies $\operatorname{tcl}(y) \subseteq \mathbf{T}$, i.e. $\in$ being well-founded on $\mathbf{T}$ implies $\in$ to be well-founded on $\operatorname{tcl}(y)$, yielding $y \in \mathbf{WF}$ and $\mathbf{T} \subseteq \mathbf{WF}$. Since also

$$\operatorname{pred}\big(\{y\} \cup \operatorname{tcl}(y), y, \in^*\big) = \operatorname{tcl}(y) \subseteq \mathbf{T},$$

Lem. 6.4(b) implies (6.8). ■

Theorem 6.12. **(a)** $\mathbf{ON} \subseteq \mathbf{WF}$ *and, thus,* $\mathbf{WF}$ *is a proper class. Moreover*

$$\forall_{\alpha \in \mathbf{ON}} \quad \mathrm{rk}(\alpha) = \mathrm{rk}(\mathbf{ON}, \alpha, \in) = \alpha.$$

(b) *Assuming the axiom of foundation (Axiom 8), one has*

$$\forall_{\alpha \in \mathbf{ON}} \quad \mathrm{rk}(\alpha) = \mathrm{rk}(\mathbf{V}, \alpha, \in) = \mathrm{rk}(\mathbf{ON}, \alpha, \in) = \alpha.$$

(c) *The following statements are equivalent:*

(i) *The axiom of foundation (Axiom 8) holds.*

(ii) $\in$ *is well-founded on* $\mathbf{V}$*.*

(iii) $\mathbf{V} = \mathbf{WF}$*.*

Proof. (a) is a direct consequence of Prop. 6.11 applied with $\mathbf{T} := \mathbf{ON}$, since $\in$ is well-founded on $\mathbf{ON}$ and we know $\mathbf{ON}$ to be a transitive class by Prop. 3.30, where $\mathrm{rk}(\mathbf{ON}, \alpha, \in) = \alpha$ was already shown in Prop. 4.41.

(b): Since Axiom 8 means $\in$ is well-founded on $\mathbf{V}$ and $\mathbf{V}$ is trivially transitive, we obtain (b) by applying Prop. 6.11 with $\mathbf{T} := \mathbf{V}$.

(c): For the equivalence between (i) and (ii), see Prop. 6.1(a). If Axiom 8 holds, then, applying Prop. 6.11 with $\mathbf{T} := \mathbf{V}$ yields $\mathbf{V} \subseteq \mathbf{WF}$, whereas $\mathbf{WF} \subseteq \mathbf{V}$ is trivially true, proving (i) implies (iii). For the converse, if y is a nonempty set without an $\in$-minimal element, then, since $y \subseteq \mathrm{tcl}(y)$, $\in$ is not well-founded on $\mathrm{tcl}(y)$, showing $y \notin \mathbf{WF}$, showing (iii) implies (ii) (via contraposition). ■

Proposition 6.13. **(a)** *If* $x \in y \in \mathbf{WF}$*, then* $x \in \mathbf{WF}$*, i.e.* $\mathbf{WF}$ *is a transitive class. Moreover,* $\mathrm{rk}(x) < \mathrm{rk}(y)$*.*

(b) $\in$ *is well-founded on* $\mathbf{WF}$*.*

(c) *If* x *is a set, then* $x \in \mathbf{WF}$ *if, and only if,* $x \subseteq \mathbf{WF}$*.*

(d) *For each* $y \in \mathbf{WF}$*, we have*

$$\mathrm{rk}(y) = \mathrm{rk}(\mathbf{WF}, y, \in) = \sup\left\{\mathbf{S}(\mathrm{rk}(x)) : x \in y\right\}. \tag{6.9}$$

(e) *If* $x \subseteq y \in \mathbf{WF}$*, then* $x \in \mathbf{WF}$ *and* $\mathrm{rk}(x) \leq \mathrm{rk}(y)$*.*

Proof. (a): If $a \in \operatorname{tcl}(x)$ and $x \in y$, then $a \in \operatorname{tcl}(y)$ (cf. Prop. 4.24(b)), showing $\operatorname{tcl}(x) \subseteq \operatorname{tcl}(y)$. Thus, if $\in$ is well-founded on $\operatorname{tcl}(y)$, then $\in$ is well-founded on $\operatorname{tcl}(x)$, showing $x \in y \in \mathbf{WF}$ implies $x \in \mathbf{WF}$. Since $y \in \mathbf{WF}$, $\in$ is well-founded on $T := \{y\} \cup \operatorname{tcl}(y)$ and, applying Prop. 6.11 with $\mathbf{T} := T$ yields

$$\operatorname{rk}(x) = \operatorname{rk}(T, x, \in) < \operatorname{rk}(T, y, \in) = \operatorname{rk}(y),$$

where the strict inequality holds, since, according to Prop. 4.40, $a \mapsto \operatorname{rk}(T, a, \in)$ is strictly isotone.

(b): Since $\operatorname{rk}(\mathbf{WF}) \subseteq \mathbf{ON}$ by Prop. 4.40, and $\operatorname{rk} : \mathbf{WF} \longrightarrow \mathbf{ON}$ is strictly isotone by (a), $\in$ is well-founded on $\mathbf{WF}$ by Prop. 4.21(f).

(c): If $x \in \mathbf{WF}$, then $x \subseteq \mathbf{WF}$ by (a). Conversely, if $x \subseteq \mathbf{WF}$, then Lem. 6.6(b) yields $\operatorname{tcl}(x) \subseteq \mathbf{WF}$ and, since $\in$ is well-founded on $\mathbf{WF}$ by (b), $\in$ is well-founded on $\operatorname{tcl}(x)$, showing $x \in \mathbf{WF}$.

(d): Since we know from (a) and (b) that $\mathbf{WF}$ is a transitive class with $\in$ well-founded on $\mathbf{WF}$, we can apply Prop. 6.11 with $\mathbf{T} := \mathbf{WF}$ to obtain

$$\operatorname{rk}(y) = \operatorname{rk}(\mathbf{WF}, y, \in) = \sup\left\{\mathbf{S}\big(\operatorname{rk}(\mathbf{WF}, x, \in)\big) : x \in y_\downarrow\right\} \overset{y_\downarrow = y}{=} \sup\left\{\mathbf{S}(\operatorname{rk}(x)) : x \in y\right\}.$$

(e): If $x \subseteq y \in \mathbf{WF}$, then $x \in \mathbf{WF}$ due to (c). If $a \in x$, then $a \in y$ and (a) yields $\mathbf{S}(\operatorname{rk}(a)) \leq \operatorname{rk}(y)$ for each $a \in x$. Thus, by (d),

$$\operatorname{rk}(x) = \sup\left\{\mathbf{S}(\operatorname{rk}(a)) : a \in x\right\} \leq \operatorname{rk}(y),$$

as desired. ■

Proposition 6.14. *If $x, y \in \mathbf{WF}$, then the following assertions hold true:*

(a) $\{x, y\} \in \mathbf{WF}$ *and* $\operatorname{rk}(\{x, y\}) = \max\left\{\operatorname{rk}(x), \operatorname{rk}(y)\right\} + 1$.

(b) $\left\{\{x\}, \{x, y\}\right\} \in \mathbf{WF}$ *and* $\operatorname{rk}\left(\left\{\{x\}, \{x, y\}\right\}\right) = \max\left\{\operatorname{rk}(x), \operatorname{rk}(y)\right\} + 2$.

(c) $\mathcal{P}(x) \in \mathbf{WF}$ *and* $\operatorname{rk}(\mathcal{P}(x)) = \operatorname{rk}(x) + 1$.

(d) $\bigcup x \in \mathbf{WF}$ *and* $\operatorname{rk}\left(\bigcup x\right) \leq \operatorname{rk}(x)$.

(e) $x \cup y \in \mathbf{WF}$ *and* $\operatorname{rk}(x \cup y) = \max\left\{\operatorname{rk}(x), \operatorname{rk}(y)\right\}$.

(f) $\operatorname{tcl}(x) \in \mathbf{WF}$ *and* $\operatorname{rk}(\operatorname{tcl}(x)) = \operatorname{rk}(x)$.

Proof. (a): If $x, y \in \mathbf{WF}$, then $\{x, y\} \subseteq \mathbf{WF}$ and, thus, $\{x, y\} \in \mathbf{WF}$ by Prop. 6.13(c). The equality is then immediate from (6.9).

(b): $\big\{\{x\}, \{x, y\}\big\} \in \mathbf{WF}$ by (a) and Prop. 6.13(c). The equality is then immediate from (a) and (6.9).

(c): If $a \in \mathcal{P}(x)$, then $a \subseteq x \in \mathbf{WF}$ and $a \in \mathbf{WF}$ by Prop. 6.13(e). In consequence, $\mathcal{P}(x) \subseteq \mathbf{WF}$ and Prop. 6.13(c) implies $\mathcal{P}(x) \in \mathbf{WF}$. From Prop. 6.13(e), we also know $\mathrm{rk}(a) \leq \mathrm{rk}(x)$ for each $a \in \mathcal{P}(x)$. Since $x \in \mathcal{P}(x)$, (6.9) yields $\mathrm{rk}(\mathcal{P}(x)) = \mathrm{rk}(x) + 1$.

(d): If $a \in \bigcup x$, then there exits $b \in x$ such that $a \in b \in x$. Since $\mathbf{WF}$ is transitive, one obtains, first, $b \in \mathbf{WF}$ and, then, $a \in \mathbf{WF}$. Thus, $\bigcup x \subseteq \mathbf{WF}$ and Prop. 6.13(c) yields $\bigcup x \in \mathbf{WF}$. From Prop. 6.13(a) and $a \in b \in x$, we obtain $\mathbf{S}(\mathrm{rk}(a)) < \mathrm{rk}(x)$, implying $\mathrm{rk}(\bigcup x) \leq \sup\big\{\mathbf{S}(\mathrm{rk}(a)) : a \in \bigcup x\big\} \leq \mathrm{rk}(x)$.

(e): Since $x \cup y = \bigcup\{x, y\}$, $x \cup y \in \mathbf{WF}$ by (a) and (d). If

$$\mathrm{rk}(x) = M := \max\{\mathrm{rk}(x), \mathrm{rk}(y)\},$$

then

$$\begin{aligned} \mathrm{rk}(x) &= \sup\big\{\mathbf{S}(\mathrm{rk}(a)) : a \in x\big\} \leq \sup\big\{\mathbf{S}(\mathrm{rk}(a)) : a \in x \cup y\big\} = \mathrm{rk}(x \cup y) \\ &\overset{\mathrm{rk}(x)=M}{=} \sup\big\{\mathbf{S}(\mathrm{rk}(a)) : a \in x\big\} = \mathrm{rk}(x), \end{aligned}$$

showing $\mathrm{rk}(x \cup y) = \max\big\{\mathrm{rk}(x), \mathrm{rk}(y)\big\}$.

(f): If $x \in \mathbf{WF}$, then Lem. 6.6(b) yields $\mathrm{tcl}(x) \subseteq \mathbf{WF}$ and, applying Prop. 6.13(c), we obtain $\mathrm{tcl}(x) \in \mathbf{WF}$. Since $x \subseteq \mathrm{tcl}(x)$, $\mathrm{rk}(x) \leq \mathrm{rk}(\mathrm{tcl}(x))$ by Prop. 6.13(e). If $x = \emptyset$, then $\mathrm{rk}(x) = 0 = \mathrm{rk}(\mathrm{tcl}(x))$ is clear. On the other hand, for each $a \in \mathrm{tcl}(x)$, $\mathrm{rk}(a) < \mathrm{rk}(x)$ follows using induction along an $\in$-path: If $a \in \mathrm{tcl}(x)$, then there exists an $\in$-path $\pi : D \longrightarrow \pi(D)$ such that $D = \mathbf{S}(n)$, $n \in \mathbb{N}$, $\pi(0) = a$, $\pi(n) = x$. Inductively, one sees that, for each $i \in \{1, \dots, n-1\}$, $\mathrm{rk}(\pi(i-1)) < \mathrm{rk}(\pi(i)) < \mathrm{rk}(x)$, since $\pi(i-1) \in \pi(i)$ implies $\mathrm{rk}(\pi(i-1)) < \mathrm{rk}(\pi(i))$ by Prop. 6.13(a). Thus,

$$\mathrm{rk}(\mathrm{tcl}(x)) = \sup\big\{\mathbf{S}(\mathrm{rk}(a)) : a \in \mathrm{tcl}(x)\big\} \leq \mathrm{rk}(x),$$

proving $\mathrm{rk}(\mathrm{tcl}(x)) = \mathrm{rk}(x)$. ■

Definition 6.15. For each $\alpha \in \mathbf{ON}$, define

$$R(\alpha) := V_\alpha := \{x \in \mathbf{WF} : \mathrm{rk}(x) < \alpha\}. \tag{6.10}$$

The $R(\alpha)$ are called the *von Neumann stages*.

Lemma 6.16. *For each set x, one has*

$$\underset{\alpha \in \mathbf{ON}}{\forall} \quad \big(x \in R(\mathbf{S}(\alpha)) \quad \Leftrightarrow \quad x \subseteq R(\alpha)\big).$$

Proof. Let $x \in R(\mathbf{S}(\alpha))$ and $a \in x$. Then Prop. 6.13(a) implies $a \in \mathbf{WF}$ and $\mathrm{rk}(a) < \mathrm{rk}(x) < \alpha + 1$, showing $\mathrm{rk}(a) < \alpha$ and $a \in R(\alpha)$. Thus, $x \subseteq R(\alpha)$. Conversely, let $x \subseteq R(\alpha)$. Then Prop. 6.13(c) yields $x \in \mathbf{WF}$ and Prop. 6.13(a),(d) yield $\mathrm{rk}(x) = \sup\big\{\mathbf{S}(\mathrm{rk}(a)) : a \in x\big\} \leq \alpha$, showing $x \in R(\mathbf{S}(\alpha))$. ■

Theorem 6.17. **(a)** *For each $\alpha \in \mathbf{ON}$, $R(\alpha) = V_\alpha$ is a set and, moreover*

$$R(0) = V_0 = \emptyset, \tag{6.11a}$$

$$R(\mathbf{S}(\alpha)) = V_{\mathbf{S}(\alpha)} = \mathcal{P}(R(\alpha)), \tag{6.11b}$$

$$R(\alpha) = V_\alpha = \bigcup\{R(\gamma) : \gamma < \alpha\} \text{ if } \alpha \text{ is a limit ordinal.} \tag{6.11c}$$

(b) $\mathbf{WF} = \bigcup\{R(\alpha) : \alpha \in \mathbf{ON}\} = \bigcup\{V_\alpha : \alpha \in \mathbf{ON}\}$[24].

Proof. (a): We prove each $R(\alpha)$ to be a set, using transfinite induction on $\alpha \in \mathbf{ON}$, also proving (6.11) along the way: Since there exists no set x with $\mathrm{rk}(x) < 0$, (6.11a) is clear from (6.10). Since $\emptyset$ is a set, this also provides the base case for the induction. Next, we see that (6.11b) is precisely the assertion of Lem. 6.16. If $R(\alpha)$ is a set by induction hypothesis, then $R(\mathbf{S}(\alpha)) = \mathcal{P}(R(\alpha))$ is a set by the power set axiom (Axiom 7). Now let α be a limit ordinal. While $\bigcup\{R(\gamma) : \gamma < \alpha\} \subseteq R(\alpha)$ is clear from (6.10), if $x \in R(\alpha)$, then $\mathrm{rk}(x) < \alpha$ and $x \in R(\mathrm{rk}(x) + 1) \subseteq \bigcup\{R(\gamma) : \gamma < \alpha\}$, since α is a limit ordinal. Thus $R(\alpha) \subseteq \bigcup\{R(\gamma) : \gamma < \alpha\}$, proving (6.11c). If each $R(\gamma)$, $\gamma \in \alpha$, is a set by induction hypothesis, the $\{R(\gamma) : \gamma < \alpha\}$ is a set by replacement (Axiom 5) and $R(\alpha)$ is a set by union (Axiom 4).

(b) holds, since $\mathrm{rk}(\mathbf{WF}) \subseteq \mathbf{ON}$. ■

7 The Axiom of Choice

7.1 Statement of the Axiom of Choice

In addition to the axioms of ZF discussed in the previous section, there is one more axiom, namely the axiom of choice (AC) that, together with ZF, makes up ZFC, the axiom system at the basis of current standard mathematics. Even though AC is used and accepted by most mathematicians, it does have the reputation of being somewhat less "natural". Thus, many mathematicians try to avoid the use of AC, where possible, and it is often pointed out explicitly, if a result depends on the use of AC (but this practice is

[24]Due to this representation, **WF** is also known as the *von Neumann universe* or the *von Neumann hierarchy of sets.*

by no means consistent and one might sometimes be surprised, which seemingly harmless result does actually depend on AC in some subtle nonobvious way). We will now state the axiom:

Axiom 9 *Axiom of Choice (AC):*

$$\underset{\mathcal{M}}{\forall}\quad \left(\emptyset \notin \mathcal{M} \;\Rightarrow\; \underset{f:\,\mathcal{M}\longrightarrow\bigcup\mathcal{M}}{\exists} \left(\underset{M\in\mathcal{M}}{\forall} f(M) \in M\right)\right).$$

Thus, the axiom of choice postulates, for each nonempty set $\mathcal{M}$, whose elements are all nonempty sets, the existence of a *choice function*, that means a function that assigns, to each $M \in \mathcal{M}$, an element $m \in M$.

Example 7.1. For example, the axiom of choice postulates, for each nonempty set A, the existence of a choice function on $\mathcal{P}(A) \setminus \{\emptyset\}$ that assigns each nonempty subset of A one of its elements.

—

The axiom of choice is remarkable since, at first glance, it might seem so natural that one can hardly believe it is not provable from the axioms in ZF. However, one can actually show that it is neither provable nor disprovable from ZF (see, e.g., [Jec73, Th. 3.5, Th. 5.16]). If you want to convince yourself that the existence of choice functions is, indeed, a tricky matter, try to define a choice function on $\mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ without AC (but do not spend too much time on it – one can show this is actually impossible to accomplish).

7.2 Equivalences of the Axiom of Choice

7.2.1 Equivalence Proofs Without Using Foundation

Theorem 7.5 below provides several important equivalences of AC. We start with some preparatory definitions, where Def. 7.2(a) will be used in Th. 7.5(iv) and Def. 7.2(b) will be used in Th. 7.5(v),(vi).

Definition 7.2. (a) If A is a set and $\mathcal{F} \subseteq \mathcal{P}(A)$, then we say that $\mathcal{F}$ is *of finite character* if, and only if, a subset X of A is an element of $\mathcal{F}$ if, and only if, every finite subset of X is an element of $\mathcal{F}$, i.e. if, and only if,

$$\underset{X\subseteq A}{\forall}\quad \left(X \in \mathcal{F} \quad\Leftrightarrow\quad \underset{B\subseteq X}{\forall} \left(\#B \in \omega \;\Rightarrow\; B \in \mathcal{F}\right)\right).$$

(b) Suppose $<$ is a strict partial order on the set X. Then a nonempty subset C of X is called a *chain* if, and only if, $<$ strongly restricted to C constitutes a strict total order on C. Moreover, a chain $C \subseteq X$ is then called *maximal* if, and only if, no strict superset Y of C (i.e. no $Y \subseteq X$ such that $C \subsetneq Y$) is a chain.

—

The following Lem. 7.3 will be used in the proof of (ii) $\Rightarrow$ (iv) of Th. 7.5 below.

Lemma 7.3. *If A is a set, $\mathcal{F} \subseteq \mathcal{P}(A)$ is of finite character, $X \in \mathcal{F}$, and $Y \subseteq X$, then $Y \in \mathcal{F}$.*

Proof. If $B \subseteq Y$ is finite, then B is also a finite subset of X. Thus, $B \in \mathcal{F}$, as $X \in \mathcal{F}$ and $\mathcal{F}$ is of finite character. In consequence, $Y \in \mathcal{F}$. ■

The following Prop. 7.4 is the key ingredient to the proof of AC $\Leftrightarrow$ (ii) in Th. 7.5 below.

Proposition 7.4. *If A is a set, then $A \in$ **WO** if, and only if, there exists a choice function (as defined in Axiom 9) on $\mathcal{P}(A) \setminus \{\emptyset\}$.*

Proof. If $A \in$ **WO**, then we can define a choice function on $\mathcal{P}(A) \setminus \{\emptyset\}$ by

$$f : \mathcal{P}(A) \setminus \{\emptyset\} \longrightarrow A, \quad f(B) := \min B,$$

where the min is taken with respect to some fixed well-order on A. For the converse, let $f : \mathcal{P}(A) \setminus \{\emptyset\} \longrightarrow A$ be a choice function and let $\kappa := \operatorname{al}(A)$ (cf. Def. 5.15). Moreover, let x be some set that is not in A (for definiteness, one could take x to be the smallest ordinal that is not in A). Via transfinite recursion on $\alpha \in \kappa$, define

$$g : \kappa \longrightarrow A \cup \{x\},$$
$$g(\alpha) := \begin{cases} f\big(A \setminus \{g(\beta) : \beta < \alpha\}\big) & \text{if } A \setminus \{g(\beta) : \beta < \alpha\} \neq \emptyset, \\ x & \text{otherwise :} \end{cases} \tag{7.1}$$

To justify, via Cor. 4.30(b), that (7.1), indeed, defines a unique function $g : \kappa \longrightarrow A \cup \{x\}$, let $x_0 := f(A)$, $\gamma := \kappa$, and $\mathbf{H} : \mathbf{V} \longrightarrow \mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} f\big(A \setminus \{x(\beta) : \beta < \alpha\}\big) & \text{if } x \text{ is a function with } \operatorname{dom}(x) = \alpha, \\ & \alpha \in \kappa,\ A \setminus \{x(\beta) : \beta < \alpha\} \neq \emptyset, \\ x & \text{otherwise.} \end{cases}$$

Then Cor. 4.30(b) provides a unique function $g : \kappa \longrightarrow g(\kappa)$ with $g(0) = x_0 = f(A) = f\big(A \setminus \{g(\beta) : \beta < 0\}\big)$ and, for each $\alpha \in \kappa \setminus \{0\}$,

$$g(\alpha) = \mathbf{H}\big(g{\restriction_\alpha}\big) = \begin{cases} f\big(A \setminus \{g(\beta) : \beta < \alpha\}\big) & \text{if } A \setminus \{g(\beta) : \beta < \alpha\} \neq \emptyset, \\ x & \text{otherwise,} \end{cases}$$

thereby proving $g(\kappa) \subseteq A \cup \{x\}$ and (7.1).

Then, for each $\alpha \in \kappa$ with $g(\alpha) \neq x$, $\beta \in \alpha$ implies $g(\beta) \neq g(\alpha)$. In consequence, we must have $x \in g(\kappa)$, since, otherwise, $g : \kappa \longrightarrow A$ were injective, in contradiction to $\kappa \not\preccurlyeq A$. Letting $\mu := \min\{\alpha \in \kappa : g(\mu) = x\}$, we have $\{g(\beta) : \beta < \mu\} = A$, i.e. $g{\restriction_\mu}\colon \mu \longrightarrow A$ is bijective. Thus, according to Cor. 5.13(a), $A \in \mathbf{WO}$. ■

Theorem 7.5 (Equivalences of AC, Not Using Foundation (Axiom 8))**.** *The following statements* (i) – (vi) *are equivalent to AC (Axiom* 9 *above).*

(i) *Every Cartesian product*

$$\prod_{i \in I} A_i := \left\{ \left(f : I \longrightarrow \bigcup_{j \in I} A_j \right) : \underset{i \in I}{\forall}\ f(i) \in A_i \right\} \tag{7.2}$$

of nonempty sets A_i, where I is a nonempty index set, is nonempty.

(ii) Zermelo's Well-Ordering Theorem: $\mathbf{V} = \mathbf{WO}$, *i.e. every set can be strictly well-ordered.*

(iii) *If x, y are sets, then $x \preccurlyeq y$ or $y \preccurlyeq x$ (cf. Def. and Rem.* 5.4(b)*).*

(iv) Tukey's Lemma: *If A is a set, $\mathcal{F} \subseteq \mathcal{P}(A)$ is of finite character, and $X \in \mathcal{F}$, then there exists a $\subsetneq$-maximal element $Y \in \mathcal{F}$ such that $X \subseteq Y$.*

(v) Hausdorff's Maximality Principle: *Every nonempty set X with a strict partial order $<$ contains a maximal chain (cf. Def.* 7.2(b)*).*

(vi) Zorn's Lemma: *Let X be a nonempty set with a strict partial order $<$. If every chain $C \subseteq X$ (as defined in Def.* 7.2(b)*) has an upper bound in X (such chains with upper bounds are sometimes called* inductive*), then X contains a maximal element.*

Proof. "(i) ⇔ AC": Assume (i). Given a nonempty set of nonempty sets $\mathcal{M}$, let $I := \mathcal{M}$ and, for each $M \in \mathcal{M}$, let $A_M := M$. If $f \in \prod_{M \in I} A_M$, then, according to (7.2), for each $M \in I = \mathcal{M}$, one has $f(M) \in A_M = M$, proving AC holds. Conversely, assume AC. Consider a family $(A_i)_{i \in I}$ such that $I \neq \emptyset$ and each $A_i \neq \emptyset$. Let $\mathcal{M} := \{A_i : i \in I\}$.

Then, by AC, there is a function $g:\mathcal{M}\longrightarrow\bigcup\mathcal{M}=\bigcup_{j\in I}A_j$ such that $g(M)\in M$ for each $M\in\mathcal{M}$. Then we can define

$$f:I\longrightarrow\bigcup_{j\in I}A_j,\quad f(i):=g(A_i)\in A_i,$$

to prove (i).

For the remaining proof, we follow [Kun12, Sec. I.12].

Next, we will show AC $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii):

"AC $\Rightarrow$ (ii)": If A is a set, then, by AC, there exists a choice function on $\mathcal{P}(A)\setminus\{\emptyset\}$ and, thus, $A\in\mathbf{WO}$ according to Prop. 7.4.

"(ii) $\Rightarrow$ AC": Given a nonempty set of nonempty sets $\mathcal{M}$, let $X:=\bigcup\mathcal{M}$. Since, by (ii), $X\in\mathbf{WO}$, there exists a strict well-order $<$ on X. Then every nonempty $Y\subseteq X$ has a min with respect to $<$. As every $M\in\mathcal{M}$ is a nonempty subset of X, we can define a choice function $f:\mathcal{M}\longrightarrow X$, $f(M):=\min M\in M$, thereby proving AC.

"(ii) $\Rightarrow$ (iii)": If $x,y\in\mathbf{WO}$, then $\#x,\#y\in\mathbf{Card}\subseteq\mathbf{ON}$, i.e. $\#x\leq\#y$ (and $x\preccurlyeq y$) or $\#y\leq\#x$ (and $y\preccurlyeq x$).

"(iii) $\Rightarrow$ (ii)": If A is a set and $\kappa:=\operatorname{al}(A)$ (cf. Def. 5.15), then $\kappa\not\preccurlyeq A$. Thus, by (iii), $A\preccurlyeq\kappa\in\mathbf{ON}$, implying $A\in\mathbf{WO}$ according to Cor. 5.13(a).

We will now show (ii) $\Rightarrow$ (iv) $\Rightarrow$ AC:

"(ii) $\Rightarrow$ (iv)": Let A be a set, let $\mathcal{F}\subseteq\mathcal{P}(A)$ be of finite character, and $X\in\mathcal{F}$. We have to construct a $\subsetneq$-maximal element $Y\in\mathcal{F}$ such that $X\subseteq Y$. Since $A\in\mathbf{WO}$, we can let $\kappa:=\#A$ and there exists a bijection $f:\kappa\longrightarrow A$. If we let, for each $\alpha\in\kappa$, $x_\alpha:=f(\alpha)$, then $A=\{x_\alpha:\alpha\in\kappa\}$ and we define a family $(Y_\alpha)_{\alpha\in\mathbf{S}(\kappa)}$ of sets in $\mathcal{F}$ via transfinite recursion on $\alpha\in\mathbf{S}(\kappa)$: Let

$$Y_0:=X, \tag{7.3a}$$

$$Y_{\mathbf{S}(\alpha)}:=\begin{cases}Y_\alpha\cup\{x_\alpha\} & \text{if } Y_\alpha\cup\{x_\alpha\}\in\mathcal{F},\\ Y_\alpha & \text{otherwise,}\end{cases} \tag{7.3b}$$

$$Y_\alpha:=\bigcup\{Y_\gamma:\gamma<\alpha\}\quad\text{if }\alpha\text{ is a limit ordinal.} \tag{7.3c}$$

Via transfinite induction on $\alpha\in\mathbf{S}(\kappa)$, we now show

$$\underset{\alpha\in\mathbf{S}(\kappa)}{\forall}\Big(X\subseteq Y_\alpha\ \wedge\ \big(\beta\in\alpha\ \Rightarrow\ Y_\beta\subseteq Y_\alpha\big)\ \wedge\ Y_\alpha\in\mathcal{F}\Big): \tag{7.4}$$

The base case ($\alpha=0$) holds, since $Y_0=X\in\mathcal{F}$ and $\beta\in\alpha$ is false. If $\alpha\in\kappa$, $Y_\beta\subseteq Y_\alpha$ for each $\beta\in\alpha$, and $X\subseteq Y_\alpha\in\mathcal{F}$, then $Y_\beta\subseteq Y_{\mathbf{S}(\alpha)}$ for each $\beta\in\alpha$ and $X\subseteq Y_{\mathbf{S}(\alpha)}\in\mathcal{F}$

are immediate from (7.3b). Now let $\alpha \in \mathbf{S}(\kappa)$ be a limit ordinal and assume $Y_\beta \subseteq Y_\gamma$ for each $\beta \in \gamma$ and $X \subseteq Y_\gamma \in \mathcal{F}$ for each $\beta \in \gamma \in \alpha$. Then $X \subseteq Y_\alpha$ and $Y_\beta \subseteq Y_\alpha$ for each $\beta \in \alpha$ are clear from (7.3c). If $B \subseteq Y_\alpha$ is finite, then, for each $b \in B$, there is $\gamma_b \in \alpha$ such that $b \in Y_{\gamma_b}$. let $\mu := \max\{\gamma_b : b \in B\}$. Then $B \subseteq Y_\mu \in \mathcal{F}$, since, for each $b \in B$, $b \in Y_{\gamma_b} \subseteq Y_\mu$. Thus, $B \in \mathcal{F}$ and, as $\mathcal{F}$ is of finite character, $Y_\alpha \in \mathcal{F}$. This concludes the induction proof of (7.4). Letting $Y := Y_\kappa$, we already know $X \subseteq Y \in \mathcal{F}$ and it merely remains to show that Y is $\subsetneq$-maximal in $\mathcal{F}$. To this end, assume $Y \subsetneq Z \subseteq A$ and let $\alpha \in \kappa$ be such that $x_\alpha \in Z \setminus Y$. Then $Y_\alpha \cup \{x_\alpha\} \notin \mathcal{F}$, since, otherwise, $x_\alpha \in Y_{\mathbf{S}(\alpha)} \subseteq Y$. In consequence, by Lem. 7.3, $Z \supseteq Y_\alpha \cup \{x_\alpha\} \notin \mathcal{F}$ implies $Z \notin \mathcal{F}$.

"(iv) ⇒ AC": Let $\mathcal{M}_0$ be a nonempty set of nonemtpy sets, where we first assume the elements of $\mathcal{M}_0$ to be disjoint, i.e. $M \cap N = \emptyset$ for $M \neq N$, $M, N \in \mathcal{M}_0$. We will use (iv) to obtain $Y \subseteq A := \bigcup \mathcal{M}_0$ that intersects each $M \in \mathcal{M}_0$ in a singleton, i.e.

$$\underset{M \in \mathcal{M}_0}{\forall} \quad \#(Y \cap M) = 1. \tag{7.5}$$

To apply (iv), define

$$\mathcal{F} := \left\{ X \subseteq A : \underset{M \in \mathcal{M}_0}{\forall} \; \Big(X \cap M = \emptyset \; \vee \; \#(X \cap M) = 1 \Big) \right\}$$

(one can think of $\mathcal{F}$ as the set of all approximations to the desired Y). Next, note $\mathcal{F}$ to be of finite character: Indeed, if $X \subseteq A$ and $X \notin \mathcal{F}$, then there exist $M \in \mathcal{M}_0$ and $x, y \in X \cap M$ with $x \neq y$. Then $\{x, y\} \cap M = \{x, y\}$, showing $\{x, y\} \notin \mathcal{F}$. Thus, $\{x, y\}$ is a finite subset of X that is not in $\mathcal{F}$. If $X \subseteq A$ has a finite subset B such that $\#(B \cap M) > 1$ for some $M \in \mathcal{M}_0$, then $B \cap M \subseteq X \cap M$ implies $X \notin \mathcal{F}$, proving $\mathcal{F}$ to be of finite character. Clearly, $\emptyset \in \mathcal{F}$ and, by (iv), $\mathcal{F}$ contains a $\subsetneq$-maximal element Y. Then Y satisfies (7.5): Otherwise, there exists $M \in \mathcal{M}_0$ such that $Y \cap M = \emptyset$ (since $Y \in \mathcal{F}$). If $m \in M$, then, since the elements of $\mathcal{M}_0$ are disjoint, $Y \cup \{m\} \in \mathcal{F}$, in contradiction to the maximality of Y. We now consider a nonempty set $\mathcal{M}$ of nonemtpy sets. We need to construct a choice function $f : \mathcal{M} \longrightarrow \bigcup \mathcal{M}$. To apply the first part of the proof, let $\mathcal{M}_0 := \big\{ \{M\} \times M : M \in \mathcal{M} \big\}$. Then the elements of $\mathcal{M}_0$ are, clearly, disjoint, and we have proved the existence of a set $Y \subseteq \bigcup \mathcal{M}_0$ that intersects each $\{M\} \times M \in \mathcal{M}_0$ in a unique element, $Y \cap (\{M\} \times M) = \{(M, Y(M))\}$, $Y(M) \in M$. Then, actually, $Y \subseteq \mathcal{M} \times \bigcup \mathcal{M}$ can be taken as the desired choice function, or we write $f : \mathcal{M} \longrightarrow \bigcup \mathcal{M}$, $f(M) := Y(M)$.

To finish the proof of the theorem, we still show (iv) ⇒ (v) ⇒ (vi) ⇒ (iv):

"(iv) ⇒ (v)": Let X be a nonempty set with a strict partial order $<$. Let $\mathcal{F} := \{C \subseteq X : C = \emptyset \vee C \text{ is a chain}\}$. Then $\mathcal{F}$ is of finite character: If $A \subseteq X$ has a finite subset that is not totally ordered, then A is not a chain; if $\emptyset \neq A \subseteq X$ is not a chain, then there exist $x, y \in A$ with $x \neq y$ and $\neg(x < y)$ and $\neg(y < x)$, showing $\{x, y\} \notin \mathcal{F}$. Since

$\mathcal{F}$ is of finite character, (iv) yields a $\subsetneq$-maximal element $Y \in \mathcal{F}$, which is a maximal chain, proving (v).

"(v) $\Rightarrow$ (vi)": To prove Zorn's lemma from (v), let X be a nonempty set with a strict partial order $<$ such that every chain $C \subseteq X$ has an upper bound. Due to (v), we can assume $C \subseteq X$ to be a *maximal* chain. Let $m \in X$ be an upper bound for the maximal chain C. We claim that m is a maximal element with respect to $<$: Indeed, if there were $x \in X$ such that $m < x$, then $x \notin C$ (since m is an upper bound for C) and $C \cup \{x\}$ constituted a strict superset of C that were also a chain, contradicting the maximality of C.

"(vi) $\Rightarrow$ (iv)": Let A be a set, let $\mathcal{F} \subseteq \mathcal{P}(A)$ be of finite character, and $X \in \mathcal{F}$. We have to construct a $\subsetneq$-maximal element $Y \in \mathcal{F}$ such that $X \subseteq Y$. Let $\mathcal{F}_X := \{B \in \mathcal{F} : X \subseteq B\}$. Then $\mathcal{F}_X \neq \emptyset$ (since $X \in \mathcal{F}_X$) and $\mathcal{F}_X$ is endowed with the strict partial order $< := \subsetneq$. If $\emptyset \neq \mathcal{C} \subseteq \mathcal{F}_X$ is a chain, then let $M := \bigcup \mathcal{C}$. Then $M \in \mathcal{F}_X$: Indeed, if $C \in \mathcal{C}$, then $X \subseteq C \subseteq M$. If $\emptyset \neq B \subseteq M$ and B is finite, then, for each $b \in B$, there ist $C_b \in \mathcal{C}$ such that $b \in C_b$. Let $D := \max\{C_b : b \in B\}$. Then $B \subseteq D \in \mathcal{C}$, showing $B \in \mathcal{F}$, since $D \in \mathcal{F}$. Thus, $M \in \mathcal{F}$, since $\mathcal{F}$ is of finite character. Thus, $M \in \mathcal{F}_X$ is an upper bound for $\mathcal{C}$ (since $C \in \mathcal{C}$ implies $C \subseteq M$) and $(\mathcal{F}_X, <)$ satisfies the hypothesis of Zorn's lemma. Thus, (vi) yields a maximal element $Y \in \mathcal{F}_X$, thereby proving (iv). ■

7.2.2 Equivalence Proofs Making Use of Foundation

Theorem 7.7 below is similar to Th. 7.5 of the previous section as it provides several statements that are equivalent to AC. However, in contrast to the proof of Th. 7.5, the proof of Th. 7.7 makes use of the axiom of foundation (Axiom 8). More precisely, there is exactly one implication, where foundation is used, namely in the proof of (iv) $\Rightarrow$ AC. To provide more background, let ZFA denote the variant of ZF, where the axioms are modified to allow for so-called *atoms* (also known as *urelements*), which are nonset entities without elements (see, e.g. [Jec73, Ch. 4.1]) and let ZF^- denote ZF without Axiom 8. It is shown in [Jec73, Ch. 9], using the technique of so-called *permutation models*, that, for each of the implications AC $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) of Th. 7.7, there exists a model of ZFA, where the reverse implication fails. In [Hal17, Ch. 6], it is claimed without reference that ZF^- were known to be consistent with ($\neg$AC + Th. 7.7(ii) (the antichain principle)). In [Kun13, Ch. II.9], it is described how one can adapt the technique of permutation models to the setting of ZF^-, which should allow one to replicate the results of [Jec73, Ch. 9] with ZFA replaced by ZF^-, but I am not aware of any reference, where this has been carried out in detail. It was shown in [Bla84] that Th. 7.7(v) (the statement that every vector space has a basis) implies (without using foundation) Th. 7.7(i) (the axiom of multiple choice). While this means AC and Th. 7.7(v) are equivalent in ZF, to my knowlegde, it is still an open problem, whether AC and Th. 7.7(v) are equivalent in ZF^-.

We proceed with a proposition, that will be used to show (iii) $\Rightarrow$ (iv) in Th. 7.7:

Proposition 7.6. *Let X be a set. If there exists a strict well-order on X, then there exists a strict total order on $\mathcal{P}(X)$.*

Proof. If $X = \emptyset$, then the statement is trivially true. Thus, let $<$ be a strict well-order on $X \neq \emptyset$ and define a relation $<$ on $\mathcal{P}(X)$ by letting

$$\forall_{A,B\in\mathcal{P}(X)} \quad A < B \quad :\Leftrightarrow \quad \Big(A \neq B \ \wedge \ \min(A \,\Delta\, B) \in B \setminus A\Big), \tag{7.6}$$

where Δ denotes the *symmetric difference*, i.e.

$$A \,\Delta\, B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

This is sometimes called the *lexicographic order* on $\mathcal{P}(X)$, induced by $<$, and we will show that $<$ is, indeed, a strict total order on $\mathcal{P}(X)$: First, note $<$ to be well-defined on $\mathcal{P}(X)$, since $A \neq B$ implies $A \,\Delta\, B \neq \emptyset$ such that $\min(A \,\Delta\, B)$ exists (since $<$ is a strict well-order on X). Next, we verify $<$ to be asymmetric on $\mathcal{P}(X)$: If $A < B$, then $A \neq B$ and $m := \min(A \,\Delta\, B) \in B \setminus A$, implying $m \notin A$ (and $m \notin A \setminus B$), showing $\neg(B < A)$. To see that $<$ satisfies trichotomy on $\mathcal{P}(X)$ note that, if $A \neq B$ and $m := \min(A \,\Delta\, B) \in B \setminus A$, then $A < B$, whereas $m \in A \setminus B$ means $B < A$. Verifying transitivity of $<$ on $\mathcal{P}(X)$ is more complicated: Assume $A < B < C$. We need to show $A < C$. We know $A \neq C$, since $A < B < A$ were in contradiction to the asymmetry shown above. Thus, letting $p := \min(A \,\Delta\, C)$, it remains to show $p \in C \setminus A$, where we may use $m := \min(A \,\Delta\, B) \in B \setminus A$ and $n := \min(B \,\Delta\, C) \in C \setminus B$. We show $p \in C \setminus A$ by distinguishing cases, comparing p with m: Case $p = m$: Then $p \notin A$ and $p \in A \,\Delta\, C = (A \setminus C) \cup (C \setminus A)$ implies $p \in C \setminus A$. Case $p < m$: Seeking a contradiction, assume $p \in A$. Then, from the definitions of m, p and $p < m$, we obtain $p \in (A \cap B) \setminus C$, implying

$$p \in B \,\Delta\, C \quad \Rightarrow \quad n < p. \tag{7.7}$$

Thus, we have $n < p < m$ and the transitivity of $<$ on X yields $n < m$, implying

$$n \notin A \,\Delta\, B \quad \Rightarrow \quad n \notin A \setminus B \quad \stackrel{n\notin B}{\Rightarrow} \quad n \notin A \quad \stackrel{n\in C}{\Rightarrow} \quad n \in C \setminus A \Rightarrow \quad p \leq n,$$

in contradiction to (7.7). Thus, $p \notin A$ and $p \in C \setminus A$ as desired. Case $m < p$: We will show that this case is not possible, as it leads to another contradiction. Indeed, $m < p$ implies

$$m \notin A \,\Delta\, C \quad \stackrel{m\notin A}{\Rightarrow} \quad m \notin C \quad \stackrel{m\in B}{\Rightarrow} \quad m \in B \,\Delta\, C \quad \Rightarrow \quad n < m. \tag{7.8}$$

Thus, $n < m < p$ and $n < p$, again using transitivity of $<$ on X. Moreover,

$$\begin{aligned} n < p \quad &\Rightarrow \quad n \notin A \,\Delta\, C \quad \overset{n \in C}{\Rightarrow} \quad n \in A \quad \Rightarrow \quad n \in (A \cap C) \setminus B \\ &\Rightarrow \quad n \in A \,\Delta\, B \quad \Rightarrow \quad m < n, \end{aligned}$$

in contradiction to (7.8). This concludes the proof of $A < C$ and of the transitivity of $<$ on $\mathcal{P}(X)$. As this was the last missing property to show $<$ constitutes a strict total order on $\mathcal{P}(X)$, the proof is complete. ■

Theorem 7.7 (Equivalences of AC, Using Foundation (Axiom 8))**.** *The following statements* (i) – (v) *are equivalent to AC (Axiom* 9 *above), provided one assumes the axiom of foundation (Axiom* 8*) – where foundation is used to prove an implication is indicated precisely, once we have listed the equivalent statements:*

(i) Axiom of Multiple Choice (AMC):

$$\underset{\mathcal{M}}{\forall} \left(\emptyset \notin \mathcal{M} \Rightarrow \underset{f:\,\mathcal{M} \longrightarrow \mathcal{P}(\bigcup \mathcal{M})}{\exists} \left(\underset{M \in \mathcal{M}}{\forall} \big(f(M) \subseteq M \ \wedge\ 0 < \# f(M) < \infty \big) \right) \right),$$

i.e., for each nonempty set $\mathcal{M}$, whose elements are all nonempty sets, there exists a function that assigns, to each $M \in \mathcal{M}$, a finite nonempty subset of M.

(ii) Antichain Principle: *Every nonempty set X with a strict partial order $<$ contains a maximal antichain, where $A \subseteq X$ is called an* antichain *if, and only if,*

$$\underset{x,y \in A}{\forall} \Big(\neg(x < y) \ \wedge\ \neg(y < x) \Big), \tag{7.9}$$

and an antichain A is called maximal *if, and only if, no strict superset Y of A (i.e. no $Y \subseteq X$ such that $A \subsetneq Y$) is an antichain.*

(iii) *If X is a set with a strict total order $<$, then $X \in$ **WO** (i.e. there exists a strict well order on X).*

(iv) *If $X \in$ **WO**, then $\mathcal{P}(X) \in$ **WO**.*

(v) *Every vector space V over a field F has a basis $B \subseteq V$.*

(a) *The implications AC $\Rightarrow$* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) *hold without assuming Axiom* 8*.*

(b) *Assuming Axiom* 8*,* (iv) *implies AC.*

(c) *Without assuming Axiom* 8*, AC $\Rightarrow$* (v) *and* (v) $\Rightarrow$ (i)*.*

Proof. For the proof of (a) and (b), we follow [Jec73, Th. 9.1], whereas (c) is proved as Th. 7.8 below.

"AC $\Rightarrow$ (i)": This is immediate, since each singleton is a nonempty finite set.

"(i) $\Rightarrow$ (ii)": Let $<$ be a strict partial order on $X \neq \emptyset$ and let $\mathcal{F}$ denote the set of all finite nonempty subsets of X. According to AMC, there exists a function $f : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow \mathcal{F}$ such that

$$\forall_{A \in \mathcal{P}(X) \setminus \{\emptyset\}} \quad f(A) \subseteq A.$$

Moreover, for each $F \in \mathcal{F}$, let $F_{\min} := \{a \in F : a \text{ is minimal in } F\}$. Then, by Prop. 4.21(e), $F_{\min} \neq \emptyset$, and we can define

$$g : \mathcal{P}(X) \longrightarrow \mathcal{F} \cup \{\emptyset\}, \quad g(A) := \begin{cases} (f(A))_{\min} & \text{for } A \neq \emptyset, \\ \emptyset & \text{for } A = \emptyset \end{cases}$$

(then each $g(A)$ is empty or a nonempty finite antichain). To obtain a maximal antichain, we define, for each $\alpha \in \mathbf{ON}$, a set $A_\alpha \subseteq X$ via transfinite recursion: Let $A_0 := g(X)$ and, for each $\alpha > 0$, let $Y_\alpha := \bigcup\{A_\beta : \beta \in \alpha\}$ and

$$X_\alpha := \left\{ x \in X \setminus Y_\alpha : \forall_{y \in Y_\alpha} \Big(\neg(x < y) \wedge \neg(y < x) \Big) \right\}, \quad A_\alpha := g(X_\alpha).$$

Then, for each $\alpha \in \mathbf{ON}$ and $\beta \in \alpha$, $X_\alpha \cap A_\beta = \emptyset$ and, thus, (since $A_\alpha \subseteq X_\alpha$) $A_\alpha \cap A_\beta = \emptyset$. In particular, if $A_\alpha \neq \emptyset$, then $A_\alpha \neq A_\beta$. Letting $\kappa := \operatorname{al}(\mathcal{F})$, we know $\kappa \not\preceq \mathcal{F}$ by Def. 5.15. Thus, there must exist $\beta \in \kappa$ such that $A_\beta = \emptyset$ (otherwise, $\iota : \kappa \longrightarrow \mathcal{F}$, $\iota(\alpha) := A_\alpha$ were injective, in contradiction to $\kappa \not\preceq \mathcal{F}$). Letting $\mu := \min\{\beta \in \kappa : A_\beta = \emptyset\}$, we claim $A := \bigcup\{A_\alpha : \alpha \in \mu\}$ to be a maximal antichain in X: Indeed, if $x, y \in A$ with $x \neq y$, then there exist $\alpha, \beta \in \kappa$ such that $x \in A_\alpha$ and $y \in A_\beta$. If $\alpha = \beta$, then $x, y \in A_\alpha = g(X_\alpha) = (f(X_\alpha))_{\min}$, i.e. x and y are both minimal in $f(X_\alpha)$ and, thus, neither $x < y$ nor $y < x$. If $\alpha \neq \beta$, we may, without loss of generality, assume $\beta < \alpha$ (otherwise, switch the names of x and y). Then $x \in X_\alpha$ and $y \in Y_\alpha$, i.e. neither $x < y$ nor $y < x$ by the definitionof X_α, proving A to be an antichain. Moreover, A is a maximal antichain, since $A = Y_\mu$ and

$$A_\mu = g(X_\mu) = \emptyset \quad \Rightarrow \quad X_\mu = \emptyset,$$

i.e. there does not exist $x \in X \setminus A$ such that $A \cup \{x\}$ is an antichain.

"(ii) $\Rightarrow$ (iii)": Let X be a set with a strict total order $<$. According to Prop. 7.4, it suffices to show that there exists a choice function on $\mathcal{P}(X) \setminus \{\emptyset\}$. To this end, let

$$P := \big\{(A, a) : A \subseteq X \wedge a \in A\big\}$$

and define a strict partial order $<_P$ on P by letting

$$\forall_{(A,a),(B,b)\in P} \quad (A,a) <_P (B,b) \;:\Leftrightarrow\; \Big(A = B \,\wedge\, a < b\Big).$$

Indeed, $<_P$ is asymmetric and transitive: If $(A,a) <_P (B,b)$, then $A = B$ and $a < b$, implying $\neg(b < a)$ (since $<$ is asymmetric) and $\neg((B,b) <_P (A,a))$; if $(A,a) <_P (B,b) <_P (C,c)$, then $A = B = C$ and $a < b < c$, implying $A = C$ and $a < c$ (since $<$ is transitive), implying $(A,a) <_P (C,c)$. According to (ii), there exists a maximal antichain $Q \subseteq P$. We claim Q to be (the graph of) a choice function $f : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow X$: If $\emptyset \neq A \subseteq X$, then there exists $a \in A$ with $(A,a) \in Q$ (otherwise, $Q \cup \{(A,a)\}$ were an antichain, in contradiction to the maximality of Q). On the other hand, since $<$ satisfies trichotomy on X, if $(A,b) \in Q$, then $b = a$. Thus, we see that, for each $\emptyset \neq A \subseteq X$, there exists a *unique* $a = f(A) \in A$ with $(A,a) \in Q$, i.e. is (the graph of) a choice function, as desired.

"(iii) $\Rightarrow$ (iv)": If $X \in \mathbf{WO}$, then, by Prop. 7.6, there exists a strict total order on $\mathcal{P}(X)$ and, then, (iii) implies $\mathcal{P}(X) \in \mathbf{WO}$.

"Axiom 8 $\wedge$ (iv) $\Rightarrow$ AC": Assuming Axiom 8, we know $\mathbf{V} = \mathbf{WF}$ from Th. 6.12(c). Moreover, by Th. 6.17(b), $\mathbf{WF} = \bigcup\{R(\alpha) : \alpha \in \mathbf{ON}\}$, the $R(\alpha)$ denoting the sets of all sets with rank less than α (cf. (6.10)). Thus, if X is a nonempty set of nonempty sets, then there exists $\alpha_0 \in \mathbf{ON}$ such that $X \in R(\alpha_0)$. If $x \in X$, then, by Prop. 6.13(a), $\mathrm{rk}(x) < \mathrm{rk}(X) < \alpha_0$, showing $X \subseteq R(\alpha_0)$. In consequence, if we can show that there exists a strict well-order $<_{\alpha_0}$ on $R(\alpha_0)$, then we can define a choice function $f : X \longrightarrow \bigcup X$, $f(x) := \min x \in x$, taking the min with respect to $<_{\alpha_0}$. To obtain the strict well-order on $R(\alpha_0)$, we will define, via transfinite recursion on $\alpha \in \mathbf{S}(\alpha_0)$, a strict well-order $<_\alpha$ on each $R(\alpha)$ with $\alpha \in \mathbf{S}(\alpha_0)$, making use of the representations of the $R(\alpha)$ from (6.11)[25]: For $\alpha = 0$, we have $R(\alpha) = \emptyset$ and let $<_0 := \emptyset$. For the successor step of the recursion, let $\kappa := \mathrm{al}(R(\alpha_0))$ (then $\kappa \not\preceq R(\alpha_0)$ by Def. 5.15). As an ordinal, $\kappa \in \mathbf{WO}$, and we apply (iv) to obtain a strict well-order $<^\kappa$ on $\mathcal{P}(\kappa)$. If $\alpha \in \alpha_0$, then, given the strict well-order $<_\alpha$ on $R(\alpha)$, we have a corresponding isomorphism $f : R(\alpha) \longrightarrow \beta := \mathrm{type}(R(\alpha), <_\alpha)$, where $\kappa \not\preceq R(\alpha_0)$ implies $\beta < \kappa$. We now define the strict well-order $<_{\mathbf{S}(\alpha)}$ on $R(\mathbf{S}(\alpha)) = \mathcal{P}(R(\alpha))$ by letting

$$\forall_{x,y\in R(\mathbf{S}(\alpha))} \quad \Big(x <_{\mathbf{S}(\alpha)} y \quad :\Leftrightarrow \quad f(x) <^\kappa f(y)\Big) : \tag{7.10}$$

Note that the isomorphism $f : R(\alpha) \longrightarrow \beta$ induces a bijective map $f : \mathcal{P}(R(\alpha)) \longrightarrow$

[25] At first glance, one might think one could use a similar, but easier, argument to obtain strict well-orders even on every $R(\alpha)$ with $\alpha \in \mathbf{ON}$. However, if one applied (iv) infinitely often (to obtain $<_{\mathbf{S}(\alpha)}$ from $<_\alpha$), then one would already use (some version of) AC to choose a well-order from each nonempty set of well-orders. In contrast, the provided argument does not use AC, as it applies (iv) only once.

$\mathcal{P}(\beta) \subseteq \mathcal{P}(\kappa)$ and (7.10) guarantees this map to constitute an isomorphism

$$f: (\mathcal{P}(R(\alpha)), <_{\mathbf{S}(\alpha)}) \cong (\mathcal{P}(\beta), <^\kappa),$$

i.e. $<_{\mathbf{S}(\alpha)}$ is a strict well-order on $R(\mathbf{S}(\alpha)) = \mathcal{P}(R(\alpha))$, as $<^\kappa$ is a strict well-order on $\mathcal{P}(\beta)$. If α is a limit ordinal and we already have strict well-orders $<_\gamma$ on each $R(\gamma)$ with $\gamma \in \alpha$, then we define a strict well-order $<_\alpha$ on $R(\alpha) = \bigcup\{R(\gamma) : \gamma < \alpha\}$ by letting

$$\underset{x,y \in R(\alpha)}{\forall} \Big(x <_\alpha y \quad :\Leftrightarrow \quad \big(\operatorname{rk}(x) < \operatorname{rk}(y) \ \vee \ \rho := \operatorname{rk}(x) = \operatorname{rk}(y) \wedge x <_{\mathbf{S}(\rho)} y\big)\Big):$$

Indeed, $<_\alpha$ is then a strict well-order on $R(\alpha)$: Asymmetry is due to the asymmetry of $<$ on $\mathbf{ON}$ and the asymmetry of $<_{\mathbf{S}(\rho)}$ on $R(\mathbf{S}(\rho))$; likewise, transitivity of $<_\alpha$ is due to the transitivity of $<$ on $\mathbf{ON}$ and the transitivity of $<_{\mathbf{S}(\rho)}$ on $R(\mathbf{S}(\rho))$; if $\emptyset \neq A \subseteq R(\alpha)$ and $\rho := \min\{\operatorname{rk}(x) : x \in A\}$, then, clearly, the $<_{\mathbf{S}(\rho)}$-minimum of $\{x \in A : \operatorname{rk}(x) = \rho\}$ is the $<_\alpha$-minimum of A. This completes the transfinite recursion and, in particular, yields the desired strict well-order $<_{\alpha_0}$ on $R(\alpha_0)$.

Thus, we have proved (a) and (b). As mentioned above, the proof of (c) is postponed to Th. 7.8 below. ∎

Theorem 7.8. *The following holds without assuming the axiom of foundation (Axiom* 8*):*

(a) *AC (Axiom* 9*) implies that every vector space V over a field F has a basis $B \subseteq V$.*

(b) *The statement that every vector space V over a field F has a basis $B \subseteq V$ implies the axiom of multiple choice (cf. Th.* 7.7(i)*).*

Proof. (a): We know from Th. 7.5 that AC is equivalent to Tukey's lemma and we will use Tukey's lemma to prove that every vector space has a basis: Let V be a vector space over a field F. According to [Phi19a, Th. 5.17], a subset B of V is a basis of V if, and only if, B is a maximal linearly independent subset of V. Thus, letting $\mathcal{F}$ denote the set of all linearly independent subsets of V, we merely need to show that $\mathcal{F}$ contains a $\subsetneq$-maximal element B. However, $\mathcal{F}$ is of finite character, since it is immediate from [Phi19a, Def. 5.12(b)] that a subset X of V is linearly independent if, and only if, each finite $A \subseteq X$ is linearly independent. In consequence, Tukey's lemma of Th. 7.5(iv) yields a $\subsetneq$-maximal element B of $\mathcal{F}$, which is a basis of V.

(b): We follow the proof in [Bla84] (this is the only proof presented in this class that makes use of algebraic theory that is somewhat advanced, namely the theory of polynomials in infinitely many variables). Let $\mathcal{M}_0$ be a nonempty set of nonemtpy sets, where we first assume the elements of $\mathcal{M}_0$ to be disjoint, i.e. $M \cap N = \emptyset$ for $M \neq N$,

$M, N \in \mathcal{M}_0$. Set $A := \bigcup \mathcal{M}_0$, let F be an arbitrary field[26] and consider $F[(X_a)_{a \in A}]$, the ring of polynomials in the infinitely many variables X_a, $a \in A$ (another notation for this ring of polynomials is $F[(\mathbb{N}_0)^A_{\text{fin}}]$, cf. [Phi19b, Ex. C.8(c)]) – each element of $F[(X_a)_{a \in A}]$ is a linear combination of monomials X^ν, where, for each $\nu \in (\mathbb{N}_0)^A_{\text{fin}}$, i.e. for each $\nu : A \longrightarrow \mathbb{N}_0$ such that $A_\nu := \{a \in A : \nu(a) \neq 0\}$ is finite, $X^\nu = \prod_{a \in A_\nu} X_a^{\nu(a)}$ (the product is then, in fact, finite, with the convention that $X^\nu := X^0 := 1$ if $A_\nu = \emptyset$). According to [Phi19b, Cor. F.12], $F[(X_a)_{a \in A}]$ constitutes a unique factorization domain, i.e. each polynomial $f \in F[(X_a)_{a \in A}]$ admits a unique prime factorization. Moreover, according to [Phi19b, Prop. C.11], $F[(X_a)_{a \in A}]$ constitutes an integral domain and, in consequence, we may consider the field of *rational fractions* $F((X_a)_{a \in A})$, which is the smallest field containing the integral domain $F[(X_a)_{a \in A}]$, and can be constructed analogous to the way $\mathbb{Q}$ is constructed from $\mathbb{Z}$, i.e. the elements of $F((X_a)_{a \in A})$ are (equivalence classes of) fractions f/g with $f, g \in F[(X_a)_{a \in A}]$, $g \neq 0$ (cf. [Phi19b, Th. 7.39], [Phi19b, Ex. 7.40(b)], and [Phi19b, Def. and Rem. 7.41]). Moreover, since each $f \in F[(X_a)_{a \in A}]$ admits a unique prime factorization, we can write each $q \in F((X_a)_{a \in A})$ in *reduced form*, i.e. as $q = f/g$ with $f, g \in F[(X_a)_{a \in A}]$ and f, g mutually prime (one can always cancel all common prime factors in numerator and denominator). Next, if $M \in \mathcal{M}_0$ and $X^\nu \in F[(X_a)_{a \in A}]$ is a monomial, $\nu \in (\mathbb{N}_0)^A_{\text{fin}}$, then we define the M-degree of X^ν (denoted $\deg_M(X^\nu)$) as the sum of all exponents of factors X_a in X^ν with $a \in M$, i.e.

$$\deg_M(X^\nu) = \deg_M\left(\prod_{a \in A_\nu} X_a^{\nu(a)}\right) := \sum_{a \in M \cap A_\nu} \nu(a)$$

(once again, noting that the sum is, in fact, finite, as only finitely many of the $\nu(a)$ are nonzero). We call a polynomial $f \in F[(X_a)_{a \in A}]$ *M-homogeneous* of degree $n \in \mathbb{N}_0$ if, and only if, all monomials in f have M-degree n, i.e. if, and only if,

$$\left(f = \sum_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} f_\nu X^\nu \;\wedge\; \forall_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} f_\nu \in F\right) \quad\Rightarrow\quad \forall_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} \Big(f_\nu \neq 0 \;\Rightarrow\; \deg_M(X^\nu) = n\Big),$$

where, of course, the sum is finite, i.e. only finitely many f_ν are nonzero; we call a rational fraction $q \in F((X_a)_{a \in A})$ *M-homogeneous* of degree[27] $d \in \mathbb{Z}$ if, and only if, there exist $f, g \in F[(X_a)_{a \in A}]$, $g \neq 0$, such that $q = f/g$, g is M-homogeneous of degree n and f is M-homogeneous of degree $n + d$ (note that the definition of M-homogeneous for rational fractions is consistent with the corresponding definition for

[26]Instead of the field F, a so-called unique factorization domain (UFD) R would suffice (i.e. an integral domain that admits unique factorizations into prime elements): What we actually need is that the ring of polynomials $R[(X_a)_{a \in A}]$ is a UFD (which, according to [Phi19b, Cor. F.12], holds if R is a UFD); if R is an integral domain, so is $R[(X_a)_{a \in A}]$ by [Phi19b, Prop. C.11] and then we can form the corresponding field of rational fractions $R((X_a)_{a \in A})$.

[27]In fact, to conclude the proof, we will only need the cases $d = 0$ and $d = -1$.

polynomials, since $g = 1 = 1 \cdot X^0$ is M-homogeneous of degree 0 for each $M \in \mathcal{M}_0$). To illustrade these notions, consider the following example: Let $\mathcal{M}_0 := \{M_i : i \in \mathbb{N}_0\}$, where $M_i := \{2i, 2i+1\}$. Then the M_i are disjoint and $\bigcup \mathcal{M}_0 = \mathbb{N}_0$. For

$$f := X_0^2 + X_1^2 + X_0X_1X_5 + X_0^2X_6X_7^2, \quad g := X_1X_6X_7^2 + X_0X_6^3 + X_1X_7^3,$$

we have $f, g \in F[(X_i)_{i\in\mathbb{N}_0}]$ with f being M_0-homogeneous of degree 2, not M_2-homogeneous, not M_3-homogeneous, and M_i-homogeneous of degree 0 for each $i \in \mathbb{N}_0 \setminus \{0,2,3\}$; g being M_0-homogeneous of degree 1, M_3-homogeneous of degree 3, and M_i-homogeneous of degree 0 for each $i \in \mathbb{N}_0 \setminus \{0,3\}$. Now consider

$$q := \frac{f}{g}, \quad r := 1 = \frac{X_0 + X_1^3}{X_0 + X_1^3}.$$

Then q is M_0-homogeneous of degree 1 and M_i-homogeneous of degree 0 for each $i \in \mathbb{N}_0 \setminus \{0,2,3\}$; r is M_i-homogeneous of degree 0 for each $i \in \mathbb{N}_0$, even though $X_0 + X_1^3$ is not M_0-homogeneous. We now proceed with the proof for the general nonempty set of nonempty disjoint sets $\mathcal{M}_0$:

Claim 1. Let $M \in \mathcal{M}_0$ and $m, n \in \mathbb{N}_0$. Moreover, let $f, g \in F[(X_a)_{a\in A}]$.

(a) If f, g are both M-homogeneous of degree m, then so is $f + g$.

(b) If f is M-homogeneous of degree m and g is M-homogeneous of degree n, then fg is M-homogeneous of degree $m + n$.

Proof. For each $\nu \in (\mathbb{N}_0)^A_{\text{fin}}$, let $f_\nu, g_\nu \in F$ such that

$$f = \sum_{\nu\in(\mathbb{N}_0)^A_{\text{fin}}} f_\nu\, X^\nu \quad \wedge \quad g = \sum_{\nu\in(\mathbb{N}_0)^A_{\text{fin}}} g_\nu\, X^\nu. \tag{7.11}$$

If f is M-homogeneous of degree m and g is M-homogeneous of degree n, then

$$\forall_{\nu\in(\mathbb{N}_0)^A_{\text{fin}}} \begin{pmatrix} f_\nu \neq 0 & \Rightarrow & \deg_M(X^\nu) = m, \\ g_\nu \neq 0 & \Rightarrow & \deg_M(X^\nu) = n \end{pmatrix}. \tag{7.12}$$

(a): Assuming $m = n$, we compute

$$f + g = \sum_{\nu\in(\mathbb{N}_0)^A_{\text{fin}}} (f_\nu + g_\nu)\, X^\nu.$$

If $f_\nu + g_\nu \neq 0$, then $f_\nu \neq 0$ or $g_\nu \neq 0$. Thus, (7.12) implies $\deg_M(X^\nu) = m$, proving $f + g$ to be M-homogeneous of degree m.

(b): First, we consider the case of monomials X^{ν_1}, X^{ν_2} with $\nu_1, \nu_2 \in (\mathbb{N}_0)^A_{\text{fin}}$, then, according to the definition of polynomial multiplication (cf. [Phi19b, (C.4)]), $X^{\nu_1} X^{\nu_2} = X^{\nu_1+\nu_2}$, i.e., if $\deg_M(X^{\nu_1}) = m$ and $\deg_M(X^{\nu_2}) = n$, then

$$\begin{aligned}\deg_M(X^{\nu_1} X^{\nu_2}) &= \deg(X^{\nu_1+\nu_2}) = \deg_M\left(\prod_{a \in A_{\nu_1+\nu_2}} X_a^{(\nu_1+\nu_2)(a)}\right) = \sum_{a \in M \cap A_{\nu_1+\nu_2}} (\nu_1+\nu_2)(a) \\ &= \sum_{a \in M \cap A_{\nu_1}} \nu_1(a) + \sum_{a \in M \cap A_{\nu_2}} \nu_2(a) = \deg_M(X^{\nu_1}) + \deg_M(X^{\nu_2}) = m+n,\end{aligned} \tag{7.13}$$

as claimed. Now let f, g as in (7.11) with f M-homogeneous of degree m and g M-homogeneous of degree n. Once again using the definition of polynomial multiplication, we compute

$$fg = \sum_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} \left(\sum_{\nu_1, \nu_2 \in (\mathbb{N}_0)^A_{\text{fin}}:\, \nu_1+\nu_2=\nu} f_{\nu_1} g_{\nu_2} \right) X^\nu.$$

If $\sum_{\nu_1, \nu_2 \in (\mathbb{N}_0)^A_{\text{fin}}:\, \nu_1+\nu_2=\nu} f_{\nu_1} g_{\nu_2} \neq 0$, then there exist $\nu_1, \nu_2 \in (\mathbb{N}_0)^A_{\text{fin}}$ such that $f_{\nu_1} \neq 0$ and $g_{\nu_2} \neq 0$, implying $\deg_M(X^{\nu_1}) = m$ and $\deg_M(X^{\nu_2}) = n$. Then (7.13) yields $\deg_M(X^\nu) = \deg(X^{\nu_1+\nu_2}) = \deg_M(X^{\nu_1}) + \deg_M(X^{\nu_2}) = m+n$, proving fg to be M-homogeneous of degree $m+n$. ▲

Claim 2. The set

$$K := \left\{ q \in F((X_a)_{a \in A}) : \underset{M \in \mathcal{M}_0}{\forall}\ q \text{ is } M\text{-homogeneous of degree } 0 \right\}$$

constitutes a subfield of $F((X_a)_{a \in A})$.

Proof. According to [Phi19a, Th. 4.35(b)] and [Phi19a, Th. 4.17], we need to show that $K \neq \emptyset$ satisfies, for each $q, r \in K$:

$$-q \in K, \tag{7.14a}$$
$$q^{-1} \in K \text{ for } q \neq 0, \tag{7.14b}$$
$$q + r \in K, \tag{7.14c}$$
$$q \cdot r \in K. \tag{7.14d}$$

We already know from above that $1 \in K$, i.e. $K \neq \emptyset$. Assume $q = f/g$ and $r = f'/g'$ with $f, f', g, g' \in F[(X_a)_{a \in A}]$, $g, g' \neq 0$, such that both f, g are M-homogeneous of degree $n_q \in \mathbb{N}_0$ and both f', g' are M-homogeneous of degree $n_r \in \mathbb{N}_0$ for some arbitrary

$M \in \mathcal{M}_0$. Thus, if, for each $\nu \in (\mathbb{N}_0)^A_{\text{fin}}$ we have $f_\nu \in F$ such that $f = \sum_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} f_\nu \, X^\nu$, then

$$\forall_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} \Big(f_\nu \neq 0 \;\Rightarrow\; \deg_M(X^\nu) = n_q \Big).$$

Since $-q = (-f)/g$ and $-f = \sum_{\nu \in (\mathbb{N}_0)^A_{\text{fin}}} (-f_\nu) \, X^\nu$ and

$$-f_\nu \neq 0 \quad \Rightarrow \quad f_\nu \neq 0 \quad \Rightarrow \quad \deg_M(X^\nu) = n_q,$$

we see that $-q$ is still M-homogeneous of degree 0, proving $-q \in K$ and (7.14a). Since $q^{-1} = g/f$, it is immediate that q^{-1} is still M-homogeneous of degree 0, proving $q^{-1} \in K$ and (7.14b). To prove (7.14c), we compute

$$q + r = \frac{f}{g} + \frac{f'}{g'} = \frac{fg' + gf'}{gg'}.$$

In consequence of Claim 1(a),(b), both $fg' + gf'$ and gg' are M-homogeneous of degree $n_q + n_r$, showing $q + r$ to be M-homogeneous of degree 0, proving $q + r \in K$ and (7.14c). Finally, $qr = \frac{ff'}{gg'}$ and, since, once again, Claim 1(b) yields both ff' and gg' to be M-homogeneous of degree $n_q + n_r$, we know qr to be M-homogeneous of degree 0, proving $qr \in K$ and (7.14d). ▲

Since, by Claim 2, K is a subfield of $F((X_a)_{a \in A})$, we also know $F((X_a)_{a \in A})$ to be a vector space over K (cf. [Phi19a, Ex. 5.2(b)]). We let V be the vector subspace of $F((X_a)_{a \in A})$, spanned by K together with the monomials X_a, $a \in A$, i.e.

$$V := \Big\langle K \cup \{X_a : a \in A\} \Big\rangle.$$

In the remainder of the proof, we show how each basis B of V over K gives rise to a choice function $\varphi_0 : \mathcal{M}_0 \longrightarrow \mathcal{P}(A)$ that selects a finite subset from each $M \in \mathcal{M}_0$: As B is a basis of V over K, we can express each X_a, $a \in A$, as a unique linear combination of elements of B: More precisely, according to [Phi19a, Th. 5.19], for each $a \in A$, there exists a unique finite subset B_a of B and a unique function $c_a : B_a \longrightarrow K \setminus \{0\}$ such that

$$X_a = \sum_{b \in B_a} c_a(b) \, b.$$

In particular, if $M \in \mathcal{M}_0$ and $x, y \in M$, then, with suitable finite subsets B_x, B_y of B and functions $c_x : B_x \longrightarrow K \setminus \{0\}$, $c_y : B_y \longrightarrow K \setminus \{0\}$:

$$X_x = \sum_{b \in B_x} c_x(b) \, b, \quad X_y = \sum_{b \in B_y} c_y(b) \, b = X_x \cdot \frac{X_y}{X_x} = \sum_{b \in B_x} \frac{X_y \cdot c_x(b)}{X_x} \, b.$$

Since $\deg_M(X_x) = \deg_M(X_y) = 1$, we have $\frac{X_y}{X_x} \in K$ and $\frac{X_y \cdot c_x(b)}{X_x} \in K$. Thus, the uniqueness of the representation with respect to the basis B then yields

$$B(M) := B_x = B_y \quad \wedge \quad \underset{b\in B_M}{\forall} \; c_y(b) = \frac{X_y \cdot c_x(b)}{X_x} \quad \Rightarrow \quad c(M,b) := \frac{c_y(b)}{X_y} = \frac{c_x(b)}{X_x},$$

i.e. the finite sets $B(M) \subseteq B$ and the fractions $c(M,b) \in F((X_a)_{a\in A})$ only depend on M (and b), not on particular elements of M. Since $c_x(b) \in K$ and $\deg_M(X_x) = 1$, $c(M,b)$ is M-homogeneous of degree -1. Thus, if $f(M,b), g(M,b) \in F[(X_a)_{a\in A}]$ are such that

$$c(M,b) = f(M,b)/g(M,b), \quad g(M,b) = \sum_{\nu\in(\mathbb{N}_0)^A_{\text{fin}}} g_\nu(M,b)\, X^\nu,$$

and $c(M,b)$ is in reduced form, then at least one X_a with $a \in M$ must occur in $g(M,b)$, i.e.

$$\varphi_0(M,b) := \left\{ a \in M : \underset{\nu\in(\mathbb{N}_0)^A_{\text{fin}}}{\exists} \Big(g_\nu(M,b) \neq 0 \wedge \nu(a) > 0 \Big) \right\} \neq \emptyset :$$

Indeed, otherwise (i.e. if $g(M,b)$ is M-homogeneous of degree 0), for each polynomial $h \in F[(X_a)_{a\in A}]$, $h = \sum_{\nu\in(\mathbb{N}_0)^A_{\text{fin}}} h_\nu\, X^\nu$, $h_\nu \in F$, if

$$\mu := \max \big\{ \deg_M(X^\nu) : \nu \in (\mathbb{N}_0)^A_{\text{fin}} \wedge h_\nu > 0 \big\},$$

then (7.13) implies that, for $h\, f(M,b)$ being M-homogeneous of degree m and $h\, g(M,b)$ being M-homogeneous of degree n, one has $n = \mu$ and $m \geq \mu$, in contradiction to $c(M,b)$ being M-homogeneous of degree -1. Thus, we can now define the desired choice function

$$\varphi_0 : \mathcal{M}_0 \longrightarrow \mathcal{P}(A), \quad \varphi_0(M) := \bigcup \big\{ \varphi_0(M,b) : b \in B(M) \big\} :$$

Since each $B(M)$ is nonempty and finite, and each $\varphi_0(M,b)$ is a nonempty finite subset of M, φ_0 is as required by AMC. It remains to consider a nonempty set $\mathcal{M}$ of nonemtpy sets. To apply the first part of the proof, as at the end of the proof of (iv) $\Rightarrow$ AC in Th. 7.5, let $\mathcal{M}_0 := \big\{ \{M\} \times M : M \in \mathcal{M} \big\}$. As the elements of $\mathcal{M}_0$ are disjoint, we have proved the existence of a choice function $\varphi_0 : \mathcal{M}_0 \longrightarrow \bigcup \mathcal{M}_0$, satisfying

$$\underset{M\in\mathcal{M}}{\forall} \Big(\varphi_0\big(\{M\} \times M\big) \subseteq \{M\} \times M \;\; \wedge \;\; 0 < \#\varphi_0\big(\{M\} \times M\big) < \infty \Big).$$

Thus, letting

$$\varphi : \mathcal{M} \longrightarrow \bigcup \mathcal{M}, \quad \varphi(M) := \big\{ a \in M : (M,a) \in \varphi_0(\{M\} \times M) \big\},$$

provides the required choice function on $\mathcal{M}$. ∎

7.3 Cardinal Arithmetic

Assuming AC, we know $\mathbf{V} = \mathbf{WO}$ from Th. 7.5(ii), and, thus, the cardinality $\#A \in \mathbf{Card}$ is defined for every set A (cf. (4.5) and Cor. 5.13(b)). While several definitions and results in the present section need AC, we will still always explicitly indicate, where this is the case.

Definition 7.9. Let $\kappa, \lambda \in \mathbf{Card}$ and, as before, let $+$ and $\cdot$ denote ordinal addition and multiplication, respectively.

(a) *Cardinal Addition:* $\kappa \oplus \lambda := \#(\kappa + \lambda)$.

(b) *Cardinal Multiplication:* $\kappa \otimes \lambda := \#(\kappa \cdot \lambda)$.

(c) *Cardinal Exponentiation:* Using AC, we define $\kappa^{\lambda}_{\text{card}} := \#(\kappa^{\lambda})$.

Proposition 7.10. *Let $\alpha, \beta \in \mathbf{ON}$. Then the following holds:*

(a) $\#(\alpha + \beta) = \#\alpha \oplus \#\beta = \#\big(\{0\} \times \alpha \cup \{1\} \times \beta\big)$.

(b) $\#(\alpha \cdot \beta) = \#\alpha \otimes \#\beta = \#(\alpha \times \beta)$.

Proof. Let $\kappa := \#\alpha$, $\lambda := \#\beta$, let $f_\kappa : \alpha \longrightarrow \kappa$ and $f_\lambda : \beta \longrightarrow \lambda$ be corresponding bijections.

(a): According to (4.33), there exist bijections $g_0 : \alpha + \beta \longrightarrow \{0\} \times \alpha \cup \{1\} \times \beta$ and $g_1 : \kappa + \lambda \longrightarrow \{0\} \times \kappa \cup \{1\} \times \lambda$; according to Def. 7.9(a), there exists a bijection $h : \kappa + \lambda \longrightarrow \kappa \oplus \lambda$. As we also have the bijection

$$f : \{0\} \times \alpha \cup \{1\} \times \beta \longrightarrow \{0\} \times \kappa \cup \{1\} \times \lambda, \quad f(x, \gamma) := \begin{cases} (x, f_\kappa(\gamma)) & \text{for } x = 0, \\ (x, f_\lambda(\gamma)) & \text{for } x = 1, \end{cases}$$

combining the bijections

$$\alpha + \beta \xrightarrow{g_0} \{0\} \times \alpha \cup \{1\} \times \beta \xrightarrow{f} \{0\} \times \kappa \cup \{1\} \times \lambda \xrightarrow{(g_1)^{-1}} \kappa + \lambda \xrightarrow{h} \kappa \oplus \lambda$$

proves (a).

(b): According to (4.35), there exist bijections $g_0 : \alpha \cdot \beta \longrightarrow \alpha \times \beta \approx \beta \times \alpha$ and $g_1 : \kappa \cdot \lambda \longrightarrow \kappa \times \lambda \approx \lambda \times \kappa$; according to Def. 7.9(b), there exists a bijection $h : \kappa \cdot \lambda \longrightarrow \kappa \otimes \lambda$. As we also have the bijection

$$f : \alpha \times \beta \longrightarrow \kappa \times \lambda, \quad f(\gamma, \delta) := (f_\kappa(\gamma), f_\lambda(\delta)),$$

combining the bijections

$$\alpha \cdot \beta \xrightarrow{g_0} \alpha \times \beta \xrightarrow{f} \kappa \times \lambda \xrightarrow{(g_1)^{-1}} \kappa \cdot \lambda \xrightarrow{h} \kappa \otimes \lambda$$

proves (b). ∎

Theorem 7.11. **(a)** Associativity of Cardinal Addition:

$$\forall_{\kappa,\lambda,\mu \in \mathbf{Card}} \quad (\kappa \oplus \lambda) \oplus \mu = \kappa \oplus (\lambda \oplus \mu).$$

(b) Commutativity of Cardinal Addition:

$$\forall_{\kappa,\lambda \in \mathbf{Card}} \quad \kappa \oplus \lambda = \lambda \oplus \kappa.$$

(c) Associativity of Cardinal Multiplication:

$$\forall_{\kappa,\lambda,\mu \in \mathbf{Card}} \quad (\kappa \otimes \lambda) \otimes \mu = \kappa \otimes (\lambda \otimes \mu).$$

(d) Commutativity of Cardinal Multiplication:

$$\forall_{\kappa,\lambda \in \mathbf{Card}} \quad \kappa \otimes \lambda = \lambda \otimes \kappa.$$

(e) Cardinal Distributivity:

$$\forall_{\kappa,\lambda,\mu \in \mathbf{Card}} \quad (\kappa \oplus \lambda) \otimes \mu = \kappa \otimes \mu \oplus \lambda \otimes \mu.$$

(f) Cardinal Exponentiation Laws: *One has*

$$\forall_{\kappa,\lambda \in \omega} \quad \kappa^{\lambda}_{\text{card}} = \kappa^{\lambda}_{\text{ord}}. \tag{7.15}$$

Moreover, if $\kappa, \lambda, \mu \in \omega$, then

$$\kappa^{\lambda \oplus \mu}_{\text{card}} = \kappa^{\lambda}_{\text{card}} \otimes \kappa^{\mu}_{\text{card}} \quad \wedge \quad \kappa^{\lambda \otimes \mu}_{\text{card}} = (\kappa^{\lambda}_{\text{card}})^{\mu}_{\text{card}}. \tag{7.16}$$

If one assumes AC, then (7.16) *even holds for each $\kappa, \lambda, \mu \in \mathbf{Card}$.*

(g) *Assume $\kappa, \kappa', \lambda, \lambda' \in \mathbf{Card}$ and $\kappa \leq \kappa'$ as well as $\lambda \leq \lambda'$. Then the following holds:*

$$\kappa \oplus \lambda \leq \kappa' \oplus \lambda', \tag{7.17a}$$

$$\kappa \otimes \lambda \leq \kappa' \otimes \lambda', \tag{7.17b}$$

$$\kappa^{\lambda}_{\text{card}} \leq (\kappa')^{\lambda'}_{\text{card}} \quad \textit{(assuming AC and unless } \kappa = \kappa' = \lambda = 0\textit{)}. \tag{7.17c}$$

Proof. Let $\kappa, \lambda, \mu, \kappa', \lambda' \in \mathbf{Card}$.

(a): By Def. 7.9(a) and Cor. 5.13(e), it suffices to show $(\kappa \oplus \lambda) \oplus \mu \approx \kappa \oplus (\lambda \oplus \mu)$. Thus, using Prop. 7.10(a), it suffices to show

$$A := \{0\} \times \big(\{0\} \times \kappa \cup \{1\} \times \lambda\big) \cup \{1\} \times \mu \approx B := \{0\} \times \kappa \cup \{1\} \times \big(\{0\} \times \lambda \cup \{1\} \times \mu\big). \quad (7.18)$$

To prove (7.18), we note that

$$f : A \longrightarrow B, \quad \begin{array}{c} f(0,(x,\alpha)) := \begin{cases} (0,\alpha) & \text{if } x = 0, \\ (1,(0,\alpha)) & \text{if } x = 1, \end{cases} \\ f(1,\alpha) := (1,(1,\alpha), \end{array}$$

is, clearly, well-defined and a bijection.

(b) is proved by the observation

$$\kappa \oplus \lambda \overset{\text{Prop. 7.10(a)}}{\approx} \{0\} \times \kappa \cup \{1\} \times \lambda \approx \{0\} \times \lambda \cup \{1\} \times \kappa \overset{\text{Prop. 7.10(a)}}{\approx} \lambda \oplus \kappa.$$

(c): Analogous to the proof of (a), it suffices to show $(\kappa \otimes \lambda) \otimes \mu \approx \kappa \otimes (\lambda \otimes \mu)$ and, thus, using Prop. 7.10(b), it suffices to show

$$A := (\kappa \times \lambda) \times \mu \approx B := \kappa \times (\lambda \times \mu). \quad (7.19)$$

To prove (7.19), we note that

$$f : A \longrightarrow B, \quad f\big((\alpha,\beta),\gamma\big) := \big(\alpha,(\beta,\gamma)\big),$$

is, clearly, well-defined and a bijection.

(d) is proved by the observation

$$\kappa \otimes \lambda \overset{\text{Prop. 7.10(b)}}{\approx} \kappa \times \lambda \approx \lambda \times \kappa \overset{\text{Prop. 7.10(b)}}{\approx} \lambda \otimes \kappa.$$

(e): From Prop. 7.10(a),(b), we know

$$(\kappa \oplus \lambda) \otimes \mu \approx \big(\{0\} \times \kappa \cup \{1\} \times \lambda\big) \times \mu \quad \wedge \quad \kappa \otimes \mu \oplus \lambda \otimes \mu \approx \{0\} \times (\kappa \times \mu) \cup \{1\} \times (\lambda \times \mu).$$

Thus, the bijection

$$f : \big(\{0\} \times \kappa \cup \{1\} \times \lambda\big) \times \mu \longrightarrow \{0\} \times (\kappa \times \mu) \cup \{1\} \times (\lambda \times \mu), \quad f\big((x,\alpha),\beta\big) := \big(x,(\alpha,\beta)\big),$$

proves $(\kappa \oplus \lambda) \otimes \mu \approx \kappa \otimes \mu \oplus \lambda \otimes \mu$ and (e).

(f): According to Prop. 5.8(c),(d), we have, for each $\kappa, \lambda, \mu \in \mathbf{Card}$ (and without using AC),

$$\kappa^{\lambda\times\mu} \approx (\kappa^\lambda)^\mu \tag{7.20}$$

and

$$\kappa^{\{0\}\times\lambda\cup\{1\}\times\mu} \approx \kappa^\lambda \times \kappa^\mu. \tag{7.21}$$

For $\kappa, \lambda \in \omega$, we now show via induction on $\lambda \in \omega$, that $\kappa^\lambda_{\mathrm{card}}$ is well-defined without AC, satisfying (7.15): The base case is provided by

$$\kappa^0_{\mathrm{card}} = \#(\kappa^0) = \#\{\emptyset\} = 1 = \kappa^0_{\mathrm{ord}}.$$

Now let $\kappa, \lambda \in \omega$, recall $\omega \subseteq \mathbf{Card}$ from Th. 5.11(c), and assume $\kappa^\lambda_{\mathrm{card}} = \kappa^\lambda_{\mathrm{ord}}$ via induction hypothesis. Then

$$\kappa^{\mathbf{S}(\lambda)} = \kappa^{\lambda+1} \approx \kappa^{\{0\}\times\lambda\cup\{1\}\times 1} \overset{(7.21)}{\approx} \kappa^\lambda \times \kappa \overset{\text{ind.hyp.}}{\approx} \kappa^\lambda_{\mathrm{ord}} \times \kappa \approx \kappa^\lambda_{\mathrm{ord}} \cdot \kappa = \kappa^{\mathbf{S}(\lambda)}_{\mathrm{ord}}.$$

Thus, $\kappa^{\mathbf{S}(\lambda)} \in \mathbf{WO}$ by Cor. 5.13(a) and

$$\kappa^{\mathbf{S}(\lambda)}_{\mathrm{card}} = \#(\kappa^{\mathbf{S}(\lambda)}) = \kappa^{\mathbf{S}(\lambda)}_{\mathrm{ord}},$$

completing the induction. Combining (7.15) with Th. 4.51 proves (7.16) for $\kappa, \lambda, \mu \in \omega$. If one assumes AC, then the expressions in (7.16) are well-defined for each $\kappa, \lambda, \mu \in \mathbf{Card}$ and, according to Def. 7.9(c) and Prop. 7.10(a),(b), (7.20) proves $\kappa^{\lambda\otimes\mu}_{\mathrm{card}} = (\kappa^\lambda_{\mathrm{card}})^\mu_{\mathrm{card}}$, whereas (7.21) proves $\kappa^{\lambda\oplus\mu}_{\mathrm{card}} = \kappa^\lambda_{\mathrm{card}} \otimes \kappa^\mu_{\mathrm{card}}$.

(g): In each case, by Cor. 5.13(e), it suffices to show there exists an injective function from the cardinal on the left of (7.17) into the corresponding cardinal on the right of (7.17). For (7.17a), the existence of such an injective function follows from Prop. 7.10(a) and

$$\{0\} \times \kappa \cup \{1\} \times \lambda \subseteq \{0\} \times \kappa' \cup \{1\} \times \lambda'.$$

Likewise, for (7.17b), the existence of an injective function $f : \kappa \otimes \lambda \longrightarrow \kappa' \otimes \lambda'$ follows from Prop. 7.10(b) and

$$\kappa \times \lambda \subseteq \kappa' \times \lambda'.$$

For (7.17b), we assume that $\kappa = \kappa' = \lambda = 0$ does not hold and start by considering some special cases: If $\kappa = \lambda = 0$, then $\kappa^\lambda_{\mathrm{card}} = 1$ and $\kappa' > 0$ implies $1 \leq (\kappa')^{\lambda'}_{\mathrm{card}}$ such that (7.17c) holds. If $\kappa = 0$ and $\lambda > 0$, then $\kappa^\lambda_{\mathrm{card}} = 0$ and (7.17c) holds. If $\kappa > 0$, then $\kappa' > 0$ as well. Then $\lambda = 0$ implies $\kappa^\lambda_{\mathrm{card}} = 1 \leq (\kappa')^{\lambda'}_{\mathrm{card}}$, i.e. (7.17c) holds again. Thus, we may now assume $\kappa, \lambda, \kappa', \lambda' > 0$. According to Def. 7.9(c), it suffices to provide an injective function $\phi : \kappa^\lambda \longrightarrow (\kappa')^{\lambda'}$. To this end, if $f \in \kappa^\lambda$, define

$$\phi(f) : \lambda' \longrightarrow \kappa', \quad \phi(f)(\alpha) := \begin{cases} f(\alpha) & \text{for } \alpha \in \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ϕ is well-defined. Moreover, if $f, g \in \kappa^\lambda$ with $f \neq g$, then there exists $\alpha \in \lambda$ with $f(\alpha) \neq g(\alpha)$, implying $\phi(f)(\alpha) \neq \phi(g)(\alpha)$ and showing $\phi(f) \neq \phi(g)$. Thus, ϕ is, indeed, injective, thereby completing the proof. ■

Proposition 7.12. **(a)** *One has*

$$\forall_{\kappa,\lambda \in \mathbf{Card}} \quad \Big(\big(0 < \min\{\kappa,\lambda\} \ \wedge\ \omega \leq \max\{\kappa,\lambda\}\big) \quad \Rightarrow \quad \kappa \oplus \lambda = \kappa \otimes \lambda = \max\{\kappa,\lambda\} \Big),$$

where the first equality also holds if $\min\{\kappa,\lambda\} = 0$.

(b) *Assuming AC:*

$$\forall_{\kappa \in \mathbf{Card}} \quad 2^\kappa_{\text{card}} = \#\mathcal{P}(\kappa).$$

(c) *Assuming AC and using the notation of Def.* 5.15*:*

$$\forall_{\alpha \in \mathbf{ON}} \quad 2^{\omega_\alpha}_{\text{card}} \geq \omega_{\mathbf{S}(\alpha)}.$$

(d) *Assuming AC:*

$$\forall_{\kappa,\lambda \in \mathbf{Card}} \quad \Big(\big(2 \leq \kappa \leq 2^\lambda_{\text{card}} \ \wedge\ \omega \leq \lambda\big) \quad \Rightarrow \quad \kappa^\lambda_{\text{card}} = 2^\lambda_{\text{card}}.$$

Proof. (a): By possibly switching their names, we may assume $\kappa \leq \lambda$. Then, by assumption, $\omega \leq \lambda$ and Th. 5.17 yields $\lambda \approx \lambda \times \lambda$. Thus, according to (5.5a), it now suffices to show

$$\lambda \preccurlyeq \kappa \oplus \lambda \preccurlyeq \lambda \times \lambda \tag{7.22}$$

and, for $0 < \kappa$,

$$\lambda \preccurlyeq \kappa \otimes \lambda \preccurlyeq \lambda \times \lambda. \tag{7.23}$$

To prove (7.22), note

$$\lambda \preccurlyeq \{0\} \times \kappa \cup \{1\} \times \lambda \overset{\text{Prop. 7.10(a)}}{\approx} \kappa \oplus \lambda \overset{(*)}{\preccurlyeq} \lambda \times \lambda,$$

where $(*)$ holds due to $\{0\} \times \kappa \cup \{1\} \times \lambda \subseteq \lambda \times \lambda$. To prove (7.23) for $0 < \kappa$, note

$$\lambda \approx \{0\} \times \lambda \overset{0<\kappa}{\preccurlyeq} \kappa \times \lambda \overset{\text{Prop. 7.10(b)}}{\approx} \kappa \otimes \lambda \preccurlyeq \lambda \times \lambda.$$

(b) holds, as $2^\kappa \approx \mathcal{P}(\kappa)$ by Th. 5.7(b).

(c) holds, since

$$\omega_{\mathbf{S}(\alpha)} \overset{(5.7b)}{=} (\omega_\alpha)^+ \overset{\text{Def. 5.15}}{=} \min\{\kappa \in \mathbf{Card} : \kappa \not\preccurlyeq \omega_\alpha\}$$

and, by Th. 5.7(a),(b), $\omega_\alpha \prec 2^{\omega_\alpha}$ and, thus, $2^{\omega_\alpha}_{\text{card}} = \#(2^{\omega_\alpha}) \not\preceq \omega_\alpha$.

(d): If $2 \leq \kappa \leq 2^\lambda_{\text{card}}$ and $\omega \leq \lambda$, then

$$2^\lambda_{\text{card}} \overset{(7.17c)}{\leq} \kappa^\lambda_{\text{card}} \overset{(7.17c)}{\leq} (2^\lambda_{\text{card}})^\lambda_{\text{card}} \overset{(7.16)}{=} 2^{\lambda\otimes\lambda}_{\text{card}} \overset{(a)}{=} 2^\lambda_{\text{card}},$$

thereby establishing the case. ■

Definition and Remark 7.13. **(a)** The *Continuum Hypothesis (CH)* is the statement

$$2^\omega_{\text{card}} = \omega_1;$$

the *Generalized Continuum Hypothesis (GCH)* is the statement

$$\forall_{\alpha\in\mathbf{ON}} \quad 2^{\omega_\alpha}_{\text{card}} = \omega_{\mathbf{S}(\alpha)}.$$

Since $\omega = \omega_0$, CH is, indeed, a special case of GCH. The name comes from the set of real numbers $\mathbb{R}$ being, at least traditionally, often referred to as the *continuum*, where, even without AC, one can show $\mathbb{R} \approx \mathcal{P}(\omega) \approx 2^\omega$ (cf. [Phi16a, Th. F.2]) and then Prop. 7.12(b) shows $\#\mathbb{R} = 2^\omega_{\text{card}}$, such that CH is equivalent to saying that $\#\mathbb{R}$ (i.e. the cardinality of the continuum) is the smallest cardinal larger than ω. While, according to Prop. 7.12(c), ZFC implies one of the inequalities of CH (and even of GCH) and, historically, many mathematicians thought CH (or even GCH) might be provable in ZFC, it is now known that CH and GCH are independent of ZFC. Working in ZF, one can define the class **L**, sometimes called *Gödel's class of constructible sets*, which turns out to be a model of ZFC + GCH, proving that the consistency of ZF implies the consistency of ZFC + GCH (see, e.g., [Kun13, Sec. II.6] – the original proof of this result was, actually, already provided by Gödel [Göd40]). Proving that the consistency of ZF also implies the consistency of ZFC + ¬CH is more difficult, all known proofs (to my knowledge) requiring the technique of *forcing*, where the orginal proof goes back to Cohen [Coh63, Coh64]. Using forcing, one can then even show that ZFC is consistent with every axiom of the form $2^\omega_{\text{card}} = \omega_\alpha$ as long as it does not violate König's Th. 7.23(c) below (e.g., $2^\omega_{\text{card}} = \omega_1$ (i.e. CH) or $2^\omega_{\text{card}} = \omega_7$ or $2^\omega_{\text{card}} = \omega_{\omega+1}$ or $2^\omega_{\text{card}} = \omega_{\omega_1}$, but $2^\omega_{\text{card}} \neq \omega_\omega$ (cf. Ex. 7.24); see, e.g., [Kun13, Cor. IV.3.14] and [Kun13, Sec. IV.5]).

(b) Assuming AC, analogous to the definition of the function $\alpha \mapsto \aleph_\alpha = \omega_\alpha$ of (5.7) (which, however, did not make use of AC), we now define, for each $\alpha \in \mathbf{ON}$ and each limit ordinal λ,

$$\beth_0 := \omega, \tag{7.24a}$$

$$\beth_{\mathbf{S}(\alpha)} := 2^{\beth_\alpha}_{\text{card}}, \tag{7.24b}$$

$$\beth_\lambda := \sup\{\beth_\gamma : \gamma < \lambda\} \tag{7.24c}$$

(the symbol $\beth$ is called *beth*). Then, clearly, for CH and GCH introduced in (a):

$$\Big(\text{CH} \;\Leftrightarrow\; \beth_1 = \aleph_1\Big) \quad\wedge\quad \Big(\text{GCH} \;\Leftrightarrow\; \forall_{\alpha\in\mathbf{ON}}\, \beth_\alpha = \aleph_\alpha\Big).$$

To justify, using Cor. 4.30(a), that (7.24) defines a unique function $\mathbf{F}:\,\mathbf{ON}\longrightarrow\mathbf{Card}$, $\mathbf{F}(\alpha)=\beth_\alpha$, let $x_0:=\omega$ and $\mathbf{H}:\,\mathbf{V}\longrightarrow\mathbf{V}$,

$$\mathbf{H}(x) := \begin{cases} 2^{x(\alpha)}_{\text{card}} & \text{if } x \text{ is a function with } \operatorname{dom}(x)=\mathbf{S}(\alpha),\ \alpha\in\mathbf{ON},\\ & x(\alpha)\in\mathbf{Card},\\ \bigcup\{x(\gamma):\,\gamma<\lambda\} & \text{if } x \text{ is a function with } \operatorname{dom}(x)=\lambda,\ \lambda \text{ a limit ordinal},\\ 0 & \text{otherwise.}\end{cases}$$

Then Cor. 4.30(a) provides a unique function $\mathbf{F}:\,\mathbf{ON}\longrightarrow\mathbf{V}$ with $\mathbf{F}(0)=x_0=\omega$ and $\forall_{\xi\in\mathbf{ON}\setminus\{0\}}\ \mathbf{F}(\xi)=\mathbf{H}\big(\mathbf{F}\restriction_\xi\big)$. We use transfinite induction, using Cor. 4.26(b), to show that $\mathbf{F}$ maps into $\mathbf{Card}$ and satisfies (7.24): $\mathbf{F}(0)=\omega\in\mathbf{Card}$ provides the base case. If $\alpha\in\mathbf{ON}$ and $\mathbf{F}(\alpha)=\beth_\alpha\in\mathbf{Card}$, then

$$\beth_{\mathbf{S}(\alpha)} = \mathbf{F}(\mathbf{S}(\alpha)) = \mathbf{H}\big(\mathbf{F}\restriction_{\mathbf{S}(\alpha)}\big) = 2^{\mathbf{F}(\alpha)}_{\text{card}} = 2^{\beth_\alpha}_{\text{card}} \in \mathbf{Card},$$

yielding (7.24b); and, for each limit ordinal λ, assuming $\mathbf{F}(\gamma)=\beth_\gamma\in\mathbf{Card}$ for each $\gamma\in\lambda$,

$$\beth_\lambda = \mathbf{F}(\lambda) = \mathbf{H}\big(\mathbf{F}\restriction_\lambda\big) = \bigcup\{\mathbf{F}(\gamma):\,\gamma<\lambda\} = \bigcup\{\beth_\gamma:\,\gamma<\lambda\}\in\mathbf{Card},$$

yielding (7.24c), and completing the induction.

Proposition 7.14. *Assume AC.*

(a) *For each $\alpha,\beta\in\mathbf{ON}$, $\alpha<\beta$ implies $\beth_\alpha<\beth_\beta$.*

(b) *The function*

$$\mathbf{f}:\,\mathbf{ON}\longrightarrow\mathbf{Card},\quad \mathbf{f}(\alpha):=\beth_\alpha,$$

is a normal function and, for each nonempty set X of ordinals,

$$\beth_{\sup X} = \sup\{\beth_\alpha:\,\alpha\in X\}. \tag{7.25}$$

Moreover,

$$\forall_{\alpha\in\mathbf{ON}}\quad \exists_{\beta\in\mathbf{ON}}\quad \Big(\alpha\le\beta \;\wedge\; \beth_\beta=\beta\Big). \tag{7.26}$$

Proof. (a): Exercise.

(b): For each $\alpha \in \mathbf{ON}$, $\beth_\alpha \in \mathbf{Card}$, as was already shown in Def. and Rem. 7.13(b). Moreover, $\mathbf{f}$ is normal, since $\mathbf{f}$ is continuous by (7.24c) and $\mathbf{f}$ is strictly isotone by (a). Then (7.25) holds due to Prop. 4.49 and (7.26) is immediate from Th. 4.52. ∎

A notion of importance in connexion with cardinal arithmetic is the notion of *cofinality*. Before defining cofinality in Def. 7.16, we still provide a result that estimates the size of unions under the assumption of AC – while it is of interest in its own right (including, e.g., the statement that countable unions of countable sets are countable), it will also be used in the proofs of Th. 7.18(f) and Th. 7.21(a) below.

Theorem 7.15. *Assume AC and let $\kappa \in \mathbf{Card}$, $\omega \leq \kappa$. If $\mathcal{M}$ is a set with $\#\mathcal{M} \leq \kappa$ and $\#M \leq \kappa$ for each $M \in \mathcal{M}$, then $\#\bigcup \mathcal{M} \leq \kappa$.*

Proof. If $\mathcal{M} = \emptyset$, then $\#\mathcal{M} = \#\bigcup \mathcal{M} = 0 < \kappa$, in accordance with the statement of the theorem. Thus, we now assume $\mathcal{M} \neq \emptyset$. Moreover, without loss of generality, we may also assume $\emptyset \notin \mathcal{M}$. According to Cor. 5.13(d),(e), there exists a surjective function $f : \kappa \longrightarrow \mathcal{M}$ and

$$\underset{M \in \mathcal{M}}{\forall} \quad F(M) := \big\{(g : \kappa \longrightarrow M) : g \text{ is surjective}\big\} \neq \emptyset.$$

Using AC, for each $M \in \mathcal{M}$, we choose $g_M \in F(M)$. Now, we define

$$h : \kappa \times \kappa \longrightarrow \bigcup \mathcal{M}, \quad h(\alpha, \beta) := g_{f(\alpha)}(\beta).$$

If $M \in \mathcal{M}$ and $x \in M$, then there exist $\alpha_M, \beta_m \in \kappa$ such that $f(\alpha_M) = M$ and $g_M(\beta_m) = m$, i.e. $h(\alpha_M, \beta_m) = g_{f(\alpha_M)}(\beta_m) = g_M(\beta_m) = m$, showing h to be surjective. Since $\kappa \times \kappa \approx \kappa$ by Th. 5.17, Cor. 5.13(d),(e) prove $\#\bigcup \mathcal{M} \leq \kappa$. ∎

Definition 7.16. Let $\alpha, \beta \in \mathbf{ON}$.

(a) If $0 < \alpha, \beta$, then a function $f : \alpha \longrightarrow \beta$ maps α *cofinally* into β if, and only if, $\sup(f(\alpha)) = \max \beta$ or $\sup(f(\alpha)) = \beta$ (where max and sup are taken with respect to the usual order on $\mathbf{ON}$).

(b) We define $\operatorname{cf}(0) := 0$ and, for $0 < \beta$,

$$\operatorname{cf}(\beta) := \min\left\{\alpha \in \mathbf{ON} : \underset{f \in \beta^\alpha}{\exists} f \text{ maps } \alpha \text{ cofinally into } \beta\right\}.$$

We call $\operatorname{cf}(\beta)$ the *cofinality* of β.

(c) β is called *regular* if, and only if, it is a limit ordinal with $\mathrm{cf}(\beta) = \beta$; otherwise β is called *singular*.

Lemma 7.17. *Let $\alpha, \beta \in \mathbf{ON}$.*

(a) *If $0 < \alpha, \beta$ and $f \in \beta^\alpha$, then the following statements are equivalent:*

(i) *f maps α cofinally into β.*

(ii) *There does not exist $\mu \in \beta$ such that $f(\gamma) < \mu$ for each $\gamma \in \alpha$.*

(iii) *It holds true that*

$$\begin{aligned}&\big(\beta \textit{ is a successor ordinal and } \sup(f(\alpha)) = \max\beta\big)\\ \vee\quad &\big(\beta \textit{ is a limit ordinal and } \sup(f(\alpha)) = \beta\big).\end{aligned}$$

(b) $\mathrm{cf}(\beta) \leq \beta$.

(c) *If β is a successor ordinal, then $\mathrm{cf}(\beta) = 1$.*

(d) *If $\beta \neq 0$, then there exists a strictly isotone function $f : \mathrm{cf}(\beta) \longrightarrow \beta$, mapping $\mathrm{cf}(\beta)$ cofinally into β.*

(e) $\mathrm{cf}(\beta)$ *is only nontrivial if β is a limit ordinal: More precisely, one has*

$$\begin{aligned}&\mathrm{cf}(\beta) = 0 \quad \textit{if} \quad \beta = 0,\\ &\mathrm{cf}(\beta) = 1 \quad \textit{if} \quad \beta \textit{ is a successor ordinal},\\ &\mathrm{cf}(\beta) = \min\big\{\mathrm{type}(X) :\, 0 \neq X \subseteq \beta \,\wedge\, \sup X = \beta\big\} \quad \textit{if} \quad \beta \textit{ is a limit ordinal.}\end{aligned}$$

(f) *If α is a limit ordinal and $f : \alpha \longrightarrow \beta$ is strictly isotone, mapping α cofinally into β, then β is a limit ordinal with $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.*

Proof. (a): Suppose β to be a successor ordinal. Then, according to Prop. 3.38(f), $\beta = \mathbf{S}(\sup\beta)$ and, in particular, $\sup\beta = \max\beta \in \beta$. Thus,

$$\text{(i)} \quad \overset{\text{Def. 7.16(a)}}{\Leftrightarrow} \quad \sup(f(\alpha)) = \max\beta \quad \Leftrightarrow \quad \text{(iii)} \quad \Leftrightarrow \quad \text{(ii)}.$$

If β is a limit ordinal, then, according to Prop. 3.38(f), $\sup\beta = \beta \notin \beta$. Thus,

$$\text{(i)} \quad \overset{\text{Def. 7.16(a)}}{\Leftrightarrow} \quad \sup(f(\alpha)) = \beta \quad \Leftrightarrow \quad \text{(iii)} \quad \Leftrightarrow \quad \text{(ii)}.$$

(b) holds, since $\mathrm{Id} : \beta \longrightarrow \beta$ maps β cofinally into β.

(c): If $\beta = \mathbf{S}(\gamma)$, then, clearly, $f: 1 = \{0\} \longrightarrow \beta$, $f(0) := \gamma$, is mapping 1 cofinally into β.

(d): If $\beta > 0$, then, according to Def. 7.16(b), $\alpha := \mathrm{cf}(\beta) > 0$ and there exists $g: \alpha \longrightarrow \beta$, mapping α cofinally into β. We now use transfinite recursion to define $f: \alpha \longrightarrow \beta$ by

$$f(0) := g(0) \quad \wedge \quad \forall_{0<\gamma\in\alpha} \; f(\gamma) := \max\big\{g(\gamma), \sup\{\mathbf{S}(f(\xi)) : \xi \in \gamma\}\big\} : \tag{7.27}$$

If β is a successor ordinal, then, by (c), $\alpha = 1$ and $f = g$ is (trivially) strictly isotone. Thus, let β be a limit ordinal. To see that f is strictly isotone note that, if $\xi \in \gamma \in \alpha$, then (7.27) implies $f(\xi) < \mathbf{S}(f(\xi)) \leq f(\gamma)$. Thus, f is an isomorphism onto its image:

$$f: \alpha \cong f(\alpha) \subseteq \mathbf{ON}.$$

For f to be well-defined, we need $f(\alpha) \subseteq \beta$. Seeking a contradiction, suppose $f(\alpha) \not\subseteq \beta$ and let $\mu := \min\{\gamma \in \alpha : \beta \leq f(\gamma)\}$ and $\sigma := \sup\{f(\xi) : \xi \in \mu\}$. Then $\sigma = \beta$, since, otherwise, by (7.27), $f(\mu) = \sup\{\mathbf{S}(f(\xi)) : \xi \in \mu\} \leq \mathbf{S}(\sigma) \in \beta$ (as β is a limit ordinal), contradicting $f(\mu) \notin \beta$ (also $f(\mu) = g(\mu) \in \beta$ is excluded by $f(\mu) \notin \beta$). Since $\sigma = \beta$ means f maps μ cofinally into β, we obtain $\mathrm{cf}(\beta) \leq \mu < \alpha$, in contradiction to the assumption $\alpha = \mathrm{cf}(\beta)$. Thus, we have $f(\alpha) \subseteq \beta$, as desired. Finally, since $g(\gamma) \leq f(\gamma)$ for each $\gamma \in \alpha$ is immediate from (7.27), f maps α cofinally into β.

(e): We already know $\mathrm{cf}(0) = 0$ by Def. 7.16(b) and $\mathrm{cf}(\beta) = 1$ for each successor ordinal β by (c). Now let β be a limit ordinal. Letting

$$A := \left\{\alpha \in \mathbf{ON} : \exists_{f\in\beta^\alpha} \; f \text{ maps } \alpha \text{ cofinally into } \beta\right\},$$
$$B := \Big\{\mathrm{type}(X) : 0 \neq X \subseteq \beta \wedge \sup X = \beta\Big\},$$

according to Def. 7.16(b), it suffices to show $A = B$. Thus, let $\alpha \in A$. Then, according to (d), there exists an isomorphism $g: \alpha \longrightarrow X := g(\alpha) \subseteq \beta$, mapping α cofinally into β. In consequence, $\sup X = \beta$ by (a) and $\alpha = \mathrm{type}(X) \in B$, showing $A \subseteq B$. Conversely, if $\alpha = \mathrm{type}(X) \in B$ with $0 \neq X \subseteq \beta$ and $\sup X = \beta$, then there exists an isomorphism $f: \alpha \longrightarrow X$ which maps α cofinally into β due to $\sup X = \beta$ and (a), showing $\alpha \in A$ and $B \subseteq A$.

(f): As a limit ordinal, according to Prop. 3.38(f), α does not have a max. As f maps α cofinally into β and is strictly isotone, β does not have a max, either, i.e. β is a limit ordinal, again, by Prop. 3.38(f). Now, let $g_\alpha: \mathrm{cf}(\alpha) \longrightarrow \alpha$ and $g_\beta: \mathrm{cf}(\beta) \longrightarrow \beta$ such that g_α maps $\mathrm{cf}(\alpha)$ cofinally into α and g_β maps $\mathrm{cf}(\beta)$ cofinally into β. Then, $h_\beta := f \circ g_\alpha : \mathrm{cf}(\alpha) \longrightarrow \beta$ maps $\mathrm{cf}(\alpha)$ cofinally into β (showing $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$): Indeed, if $\delta \in \beta$, then there exists $\gamma \in \alpha$ with $\delta \le f(\gamma)$. Moreover, there exists $\xi \in \mathrm{cf}(\alpha)$ with $\gamma \le g_\alpha(\xi)$. Thus, the isotonicity of f implies $\delta \le f(\gamma) \le f(g_\alpha(\xi)) = h_\beta(\xi)$. It remains to show $\mathrm{cf}(\alpha) \le \mathrm{cf}(\beta)$. To this end, define

$$h_\alpha : \mathrm{cf}(\beta) \longrightarrow \alpha, \quad h_\alpha(\xi) := \min\big\{\gamma \in \alpha : g_\beta(\xi) < f(\gamma)\big\},$$

and note that h_α maps $\mathrm{cf}(\beta)$ cofinally into α (thereby proving $\mathrm{cf}(\alpha) \le \mathrm{cf}(\beta)$, as desired): Indeed, if $\delta \in \alpha$, then there exists $\xi_\delta \in \mathrm{cf}(\beta)$ with $f(\delta) \le g_\beta(\xi_\delta)$. Thus, if $\gamma \in \alpha$ with $g_\beta(\xi_\delta) < f(\gamma)$, then $\delta < \gamma$, showing $\delta < h_\alpha(\xi_\delta)$. ■

Theorem 7.18. *Let λ be a limit ordinal.*

(a) *If $A \subseteq \lambda$ and $\sup A = \lambda$, then $\mathrm{type}(A)$ is a limit ordinal and $\mathrm{cf}(\lambda) = \mathrm{cf}(\mathrm{type}(A))$.*

(b) $\mathrm{cf}\big(\mathrm{cf}(\lambda)\big) = \mathrm{cf}(\lambda)$, *i.e. $\mathrm{cf}(\lambda)$ is regular according to Def.* 7.16(c) *(in particular, $\mathrm{cf}(\lambda)$ is a limit ordinal).*

(c) $\omega \le \mathrm{cf}(\lambda) \le \#\lambda \le \lambda$ *(and, assuming AC, all inequalities can be strict, see Ex.* 7.20(e) *below)*[28].

(d) *If λ is regular, then $\lambda \in$ **Card** (in particular, $\mathrm{cf}(\lambda) \in$ **Card**).*

(e) *If $\lambda = \omega$, then it is a regular cardinal.*

(f) *If λ is a so-called* infinite successor cardinal, *i.e. $\lambda = \omega_\alpha$ with α a successor ordinal, then, assuming AC, λ is a regular cardinal.*

(g) *If λ is a so-called* limit cardinal, *i.e. $\lambda = \omega_\alpha$, where α is a limit ordinal, then $\mathrm{cf}(\lambda) = \mathrm{cf}(\alpha)$.*

Proof. (a): Let $\alpha := \mathrm{type}(A)$ and let $f : \alpha \longrightarrow \mathrm{type}(A)$ be an isomorphism. Then f is strictly isotone and, since $\lambda = \sup A = \sup f(\alpha) \subseteq \lambda$, f maps α cofinally into λ. Moreover, $\lambda = \sup A$ also shows that A (and, thus, α) can not have a maximum, such that α must be a limit ordinal as well. According to Lem. 7.17(f), we obtain $\mathrm{cf}(\lambda) = \mathrm{cf}(\alpha)$.

(b): According to Lem. 7.17(d), there exists a strictly isotone function $f : \mathrm{cf}(\lambda) \longrightarrow \lambda$, mapping $\mathrm{cf}(\lambda)$ cofinally into λ. Thus, letting $A := f(\mathrm{cf}(\lambda))$, f is an isomorphism $f : \mathrm{cf}(\lambda) \cong A \subseteq \lambda$, showing $\mathrm{type}(A) = \mathrm{cf}(\lambda)$. Since $A \subseteq \lambda$ and $\sup A = \lambda$ by Lem. 7.17(d), we can apply (a) to obtain $\mathrm{cf}(\lambda) = \mathrm{cf}(\mathrm{type}(A)) = \mathrm{cf}(\mathrm{cf}(\lambda))$, thereby establishing the case ((a) also yields $\mathrm{type}(A) = \mathrm{cf}(\lambda)$ to be a limit ordinal).

[28]To my knowlege, it is unknown if one can proof in ZF that ordinals λ with $\mathrm{cf}(\lambda) > \omega$ exist; however, cf. [Car13].

(c): $\omega \leq \mathrm{cf}(\lambda)$ holds, since $\mathrm{cf}(\lambda)$ is a limit ordinal by (c). Since there exists a bijection $f : \#\lambda \longrightarrow \lambda$, which, as it is surjective, maps $\#\lambda$ cofinally into λ, $\mathrm{cf}(\lambda) \leq \#\lambda$ is immediate from Def. 7.16(b). Finally, $\#\lambda \leq \lambda$ is immediate from Cor. 5.13(b).

(d): If λ is regular, then $\lambda = \mathrm{cf}(\lambda)$ and, by (c),

$$\mathrm{cf}(\lambda) \leq \#\lambda \leq \lambda = \mathrm{cf}(\lambda) \quad \Rightarrow \quad \lambda = \mathrm{cf}(\lambda) = \#\lambda.$$

(e): Since $\mathrm{cf}(\omega)$ is a limit ordinal by (b), and $\mathrm{cf}(\omega) \leq \omega$ by Lem. 7.17(b), $\mathrm{cf}(\omega) = \omega$ is already proved.

(f): Let $\lambda = \omega_\alpha$, $\alpha = \mathbf{S}(\beta)$, $\beta \in \mathbf{ON}$. Moreover, let $\emptyset \neq X \subseteq \lambda$ with $\mathrm{type}(X) < \lambda$. Applying Lem. 7.17(e), we need to show $\sup X < \lambda$. Since $\mathrm{type}(X) < \lambda \in \mathbf{Card}$, we know $\#\,\mathrm{type}(X) < \lambda$, i.e. $\#\,\mathrm{type}(X) \leq \omega_\beta$ (since $\lambda = (\omega_\beta)^+$, cf. (5.7b) and Prop. 5.16(a)). Furthermore, since $X \subseteq \lambda$, $\#\gamma < \lambda$ for each $\gamma \in X$, implying $\#\gamma \leq \omega_\beta$ for each $\gamma \in X$ (since $\alpha = \mathbf{S}(\beta)$ – this is where we use, in an essential way, that λ is a successor cardinal). Thus, as $\sup X = \bigcup X$ and we assume AC, we can now use Th. 7.15 to conclude $\#\sup X = \#\bigcup X \leq \omega_\beta < \lambda$, showing $\sup X < \lambda$, as needed.

(g): Let α be a limit ordinal, $\lambda = \omega_\alpha$, and $A := \{\omega_\beta : \beta \in \alpha\}$. Then $A \subseteq \lambda$, $\sup A = \lambda$ by (5.7c), and $\mathrm{type}(A) = \alpha$ (since, clearly, $f : \alpha \longrightarrow A$, $f(\xi) := \omega_\xi$, is an isomorphism). Applying (a), we obtain $\mathrm{cf}(\lambda) = \mathrm{cf}(\mathrm{type}(A)) = \mathrm{cf}(\alpha)$. ■

Corollary 7.19. *For each* $\beta \in \mathbf{ON}$, $\mathrm{cf}(\beta) \in \mathbf{Card}$. *More precisely:*

(a) $\mathrm{cf}(0) = 0$.

(b) $\mathrm{cf}(\beta) = 1$ *if* β *is a successor ordinal.*

(c) $\mathrm{cf}(\beta)$ *is a regular cardinal if* β *is a limit ordinal.*

Proof. $\mathrm{cf}(0) = 0$ holds by Def. 7.16(b) and $\mathrm{cf}(\beta) = 1$ for each successor ordinal β holds by Lem. 7.16(c). If β is a limit ordinal, then $\mathrm{cf}(\beta)$ is a regular cardinal by Th. 7.18(b),(d). ■

Example 7.20. (a) One has $\mathrm{cf}(\omega_\omega) \overset{\text{Th. 7.18(g)}}{=} \mathrm{cf}(\omega) \overset{\text{Th. 7.18(e)}}{=} \omega$.

(b) If $\alpha, \lambda \in \mathbf{ON}$, where λ is a limit ordinal, then $\alpha + \lambda$ is a limit ordinal (by Lem. 4.48(a)) and $\mathrm{cf}(\alpha + \lambda) = \mathrm{cf}(\lambda)$ (e.g., $\mathrm{cf}(\omega_1 + \omega) = \mathrm{cf}(\omega + \omega) = \omega$): Indeed, letting $A := \{\alpha + \gamma : \gamma < \lambda\}$, we have $A \subseteq \alpha + \lambda$ with $\sup A = \bigcup A = \alpha + \lambda$ by (4.27c), and, applying Th. 7.18(a), we obtain $\mathrm{cf}(\alpha + \lambda) = \mathrm{cf}(\mathrm{type}(A)) = \mathrm{cf}(\lambda)$, as, clearly, $f : \lambda \longrightarrow A$, $f(\gamma) := \alpha + \gamma$, is an isomorphism.

(c) If λ is a limit ordinal with $\lambda < \omega_1$, then $\#\lambda = \omega$ and, thus, by Th. 7.18(c), $\omega \leq \mathrm{cf}(\lambda) \leq \#\lambda = \omega$, yielding $\mathrm{cf}(\lambda) = \omega$. Thus, for example, $\mathrm{cf}(\omega^\omega_{\mathrm{ord}}) = \omega$, since $\#(\omega^\omega_{\mathrm{ord}}) = \omega$ by Th. 5.18, and $\omega^\omega_{\mathrm{ord}}$ is a limit ordinal, since $\alpha \in \omega^\omega_{\mathrm{ord}}$ means $\alpha \in \omega^n_{\mathrm{ord}}$ for some $n \in \omega$ and, thus,

$$\mathbf{S}(\alpha) = \alpha + 1 < \alpha + \alpha = \alpha \cdot 2 < \alpha \cdot \omega < \omega^n_{\mathrm{ord}} \cdot \omega = \omega^{n+1}_{\mathrm{ord}} < \omega^\omega_{\mathrm{ord}}.$$

(d) Assuming AC, we have

$$\omega_1 \overset{\text{Th. 7.18(f)}}{=} \mathrm{cf}(\omega_1) \overset{\text{(b)}}{=} \mathrm{cf}(\omega_1 \cdot 2).$$

(e) Assuming AC, we have, for $\lambda := \omega_{\omega_1} + \omega_{\omega_1} = \omega_{\omega_1} \cdot 2$,

$$\omega < \omega_1 \overset{\text{(d)}}{=} \mathrm{cf}(\omega_1) \overset{\text{Th. 7.18(g)}}{=} \mathrm{cf}(\omega_{\omega_1}) \overset{\text{(b)}}{=} \mathrm{cf}(\lambda) \overset{\text{Prop. 5.16(b)}}{<} \omega_{\omega_1} \overset{(5.12)}{=} \#\lambda < \lambda.$$

—

The following Th. 7.21(a) provides a generalization of Th. 7.15: Given $\kappa \in \mathbf{Card}$, $\omega \leq \kappa$, as in Th. 7.15, one obtains the statement of Th. 7.15 by applying Th. 7.21(a) with $\theta := \kappa^+$ and observing κ^+ to be regular by Th. 7.18(f).

Theorem 7.21. *Assume AC and let $\theta \in \mathbf{Card}$, $\omega \leq \theta$.*

(a) *If θ is regular, $\mathcal{M}$ is a set with $\#\mathcal{M} < \theta$, and $\#M < \theta$ for each $M \in \mathcal{M}$, then $\#\bigcup \mathcal{M} < \theta$.*

(b) *If $\lambda := \mathrm{cf}(\theta) < \theta$, then there exists a set $\mathcal{M}$ of subsets of θ such that $\#\mathcal{M} = \lambda$, $\bigcup \mathcal{M} = \theta$, and $\#M < \theta$ for each $M \in \mathcal{M}$.*

Proof. Note that θ is a limit ordinal by Th. 5.11(b).

(a): Assume $\mathrm{cf}(\theta) = \theta$ and let $X := \{\#M : M \in \mathcal{M}\}$. Then $\#M < \theta$ for each $M \in \mathcal{M}$ implies $X \subseteq \theta$ and $\#\mathcal{M} < \theta$ implies $\#X < \theta$ and $\mathrm{type}(X) < \theta$. Since $\mathrm{cf}(\theta) = \theta$, Lem. 7.17(e) yields $\sup X < \theta$. Thus $\kappa := \max\{\sup X, \#\mathcal{M}\} < \theta$ (note $\kappa \in \mathbf{Card}$ by Th. 5.11(d)). If $\kappa < \omega$, then $\#\bigcup \mathcal{M} < \omega \leq \theta$. If $\omega \leq \kappa$, $\#\bigcup \mathcal{M} \leq \kappa < \theta$ by Th. 7.15.

(b): Since $\lambda = \mathrm{cf}(\theta)$, by Lem. 7.17(e), there exists $\mathcal{M} \subseteq \theta$ such that $\lambda = \mathrm{type}(\mathcal{M})$ and $\bigcup \mathcal{M} = \sup \mathcal{M} = \theta$. As $\lambda = \mathrm{type}(\mathcal{M})$ yields $\#\mathcal{M} = \lambda$ and $\mathcal{M} \subseteq \theta$ means $\#M < \theta$ for each $M \in \mathcal{M}$, the proof is complete. ■

In preparation for König's Th. 7.23(a), we now define sums and products of arbitrarily many cardinals:

Definition and Remark 7.22. Let $I \neq \emptyset$ be an index set and let $(\kappa_i)_{i\in I} \in \mathbf{Card}^I$ be a family of cardinals.

(a) Let $(A_i)_{i\in I}$ be a disjoint family of sets (i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$) such that $\#A_i = \kappa_i$ for each $i \in I$ (e.g., one can set $A_i := \kappa_i \times \{i\}$ for each $i \in I$) and define

$$\bigoplus_{i\in I} \kappa_i := \#\bigcup\{A_i : i \in I\}, \qquad \bigoplus_{i\in\emptyset} \kappa_i := 0.$$

Clearly, the above definition does not depend on the choice of the disjoint family $(A_i)_{i\in I}$: If $(B_i)_{i\in I}$ is an arbitrary disjoint family of sets such that $\#B_i = \kappa_i$ for each $i \in I$, then, for each $i \in I$, there exists a bijection $f_i : A_i \longrightarrow B_i$, yielding the bijection

$$f : \bigcup\{A_i : i \in I\} \longrightarrow \bigcup\{B_i : i \in I\}, \quad f(x) := f_i(x) \text{ for } x \in A_i.$$

(b) Define

$$\bigotimes_{i\in I} \kappa_i := \#\prod_{i\in I} \kappa_i, \qquad \bigotimes_{i\in\emptyset} \kappa_i := 1.$$

Theorem 7.23 (König)**.** *Assume AC.*

(a) *If I is an index set and $(\kappa_i)_{i\in I}, (\lambda_i)_{i\in I} \in \mathbf{Card}^I$ are families of cardinals, then*

$$\left(\underset{i\in I}{\forall} \; \kappa_i < \lambda_i\right) \quad \Rightarrow \quad \bigoplus_{i\in I} \kappa_i < \bigotimes_{i\in I} \lambda_i.$$

(b) *If $\kappa \in \mathbf{Card}$, then*

$$\omega \leq \kappa \quad \Rightarrow \quad \kappa < \kappa_{\text{card}}^{\text{cf}(\kappa)}.$$

(c) *If $\kappa, \lambda \in \mathbf{Card}$, then*

$$2 \leq \kappa \,\wedge\, \omega \leq \lambda \quad \Rightarrow \quad \lambda < \text{cf}(\kappa_{\text{card}}^{\lambda}).$$

Proof. (a): If $I = \emptyset$, then we have the true statement $\bigoplus_{i\in I} \kappa_i = 0 < 1 = \bigotimes_{i\in I} \lambda_i$. Thus, we now assume $I \neq \emptyset$. As in Def. and Rem. 7.22(a), let $(A_i)_{i\in I}$ be a disjoint family of sets such that $\#A_i = \kappa_i$ for each $i \in I$. We need to show

$$\mathcal{A} := \bigcup\{A_i : i \in I\} \prec \prod_{i\in I} \lambda_i. \tag{7.28}$$

As we assume, for each $i \in I$, $\kappa_i < \lambda_i$, we can choose, for each $i \in I$ (using AC) an injection $f_i : A_i \longrightarrow \lambda_i$. Due to $\kappa_i < \lambda_i$, f_i is not surjective, i.e. $B_i := \lambda_i \setminus f(A_i) \neq \emptyset$

and we can set $b_i := \min B_i$. To construct an injective function $g : \mathcal{A} \longrightarrow \prod_{i\in I} \lambda_i$, we first define, for each $i \in I$,

$$g_i : \mathcal{A} \longrightarrow \lambda_i, \quad g_i(x) := \begin{cases} f_i(x) & \text{if } x \in A_i, \\ b_i & \text{otherwise,} \end{cases}$$

followed by

$$g : \mathcal{A} \longrightarrow \prod_{i\in I} \lambda_i, \quad g(x) := \big(g_i(x)\big)_{i\in I}.$$

Clearly, g maps into $\prod_{i\in I} \lambda_i$. Moreover, if $x, y \in \mathcal{A}$ with $x, y \in A_i$, $x \neq y$, then $g_i(x) = f_i(x) \neq f_i(y) = g_i(y)$, since f_i is injective, showing $g(x) \neq g(y)$; if $x \in A_i$, $y \in A_j$ with $i \neq j$, then $g_i(x) = f_i(x) \neq b_i = g_i(y)$, again showing $g(x) \neq g(y)$, thereby establishing g to be injective. To finish the proof of (7.28), it remains to show no function $h : \mathcal{A} \longrightarrow \prod_{i\in I} \lambda_i$ can be surjective. Indeed, if $h : \mathcal{A} \longrightarrow \prod_{i\in I} \lambda_i$, and, for each $j \in I$, $\pi_j : \prod_{i\in I} \lambda_i \longrightarrow \lambda_j$, $\pi_j\big((\alpha_i)_{i\in I}\big) := \alpha_j$, is the canonical projection, then, $P_j := (\pi_j \circ h{\restriction}_{A_j}) : A_j \longrightarrow \lambda_j$ can not be surjective, we can let $\beta_j := \min \lambda_j \setminus P_j(A_j)$ and $b := (\beta_i)_{i\in I} \in \prod_{i\in I} \lambda_i$. Then $b \notin h(\mathcal{A})$: Indeed, if $x \in \mathcal{A}$, then there exists a unique $i \in I$ with $x \in A_i$, implying $\pi_i(h(x)) = P_i(x) \in P_i(A_i) \subseteq \lambda_i$, whereas $\pi_i(b) = \beta_i \in \lambda_i \setminus P_i(A_i)$. Thus, $b \neq h(x)$, thereby establishing the case.

(b): As κ is a limit ordinal, according to Lem. 7.17(e), there exists $X \subseteq \kappa$ with $\#X = \mathrm{cf}(\kappa)$ and $\bigcup X = \sup X = \kappa$. Thus,

$$\begin{aligned} \kappa &= \bigcup X \leq \bigoplus_{\alpha\in X} \#\alpha \leq \bigoplus_{\alpha\in X} \#\kappa = \#(X \times \kappa) = \#(\mathrm{cf}(\kappa) \times \kappa) \\ &\overset{\text{Prop. 7.10(b)}}{=} \mathrm{cf}(\kappa) \otimes \kappa \overset{\text{Prop. 7.12(a)}}{=} \kappa, \end{aligned}$$

implying $\kappa = \bigoplus_{\alpha\in X} \#\alpha$. On the other hand, $\alpha < \kappa$ for each $\alpha \in X$ and, hence, (a) yields

$$\kappa = \bigoplus_{\alpha\in X} \#\alpha < \bigotimes_{\alpha\in X} \#\kappa = \#(\kappa^X) = \#(\kappa^{\mathrm{cf}(\kappa)}) = \kappa_{\mathrm{card}}^{\mathrm{cf}(\kappa)},$$

as desired.

(c): We give two proofs. The first one is based on (b), whereas the second one is based on Th. 7.21(b) (perhaps, this is preferable if one is only interested in (c)). For both proofs, letting $\theta := \kappa_{\mathrm{card}}^{\lambda}$, note that

$$\theta_{\mathrm{card}}^{\lambda} \overset{(7.16)}{=} \kappa_{\mathrm{card}}^{\lambda\otimes\lambda} \overset{\text{Prop. 7.12(a)}}{=} \theta. \tag{7.29}$$

For the first proof, seeking a contradiction to (b), assume $\mathrm{cf}(\theta) \leq \lambda$. Then, by (7.17c),

$$\theta_{\mathrm{card}}^{\mathrm{cf}(\theta)} \leq \theta_{\mathrm{card}}^{\lambda} \overset{(7.29)}{=} \theta,$$

in contradiction to (b) (applied with κ replaced by θ). For the second proof, we let $f : \theta \longrightarrow \theta^\lambda$ be bijective (using that $\theta^\lambda \approx \theta^\lambda_{\text{card}} \approx \theta$ by (7.29)). Thus, setting, for each $\alpha \in \theta$, $f_\alpha := f(\alpha)$, we have $\theta^\lambda = \{f_\alpha : \alpha \in \theta\}$. Once again, seeking a contradiction, assume $\text{cf}(\theta) \leq \lambda$. Then

$$\text{cf}(\theta) \leq \lambda \overset{\text{Th. 5.7(a),(b)}}{<} 2^\lambda_{\text{card}} = \kappa^\lambda_{\text{card}} \overset{\text{Prop. 7.12(d)}}{=} \theta$$

and, by Th. 7.21(b), there exist sets M_α, $\alpha \in \lambda$, such that $\theta = \bigcup \mathcal{M}$, where $\mathcal{M} = \{M_\alpha : \alpha \in \lambda\}$ and $\# M_\alpha < \theta$ for each $\alpha \in \lambda$. We now define

$$g : \lambda \longrightarrow \theta, \quad g(\beta) := \min\big(\theta \setminus \{f_\alpha(\beta) : \alpha \in M_\beta\}\big)$$

(note that g is well-defined, since, for each $\beta \in \lambda$, $\#\{f_\alpha(\beta) : \alpha \in M_\beta\} \leq \# M_\beta < \theta$, implying $\theta \setminus \{f_\alpha(\beta) : \alpha \in M_\beta\} \neq \emptyset$). We claim $g \neq f_\alpha$ for each $\alpha \in \theta$: Indeed, if $\alpha \in \theta$, then let $\beta := \min\{\gamma \in \lambda : \alpha \in M_\beta\}$. Then, since $\alpha \in M_\beta$, $g(\beta) \neq f_\alpha(\beta)$, proving $g \neq f_\alpha$. Thus, $g \in \theta^\lambda \setminus \{f_\alpha : \alpha \in \theta\}$, in contradiction to $\theta^\lambda = \{f_\alpha : \alpha \in \theta\}$. ■

Example 7.24. Applying Th. 7.23(c) with $\kappa := 2$ and $\lambda := \omega$, we obtain $\omega < \text{cf}(2^\omega_{\text{card}})$. In consequence, $2^\omega_{\text{card}} \neq \omega_\omega$, since $\text{cf}(\omega_\omega) = \omega$ by Ex. 7.20(a) (also cf. Def. and Rem. 7.13(a)).

Definition and Remark 7.25. (a) A regular cardinal $\kappa > \omega$ is called *weakly inaccessible* (resp. *strongly inaccessible*) if, and only if, $\lambda^+ < \kappa$ (resp. $2^\lambda_{\text{card}} < \kappa$) for each $\lambda < \kappa$ (the definition of strongly inaccessible makes use of AC). Thus, assuming AC, every strongly inaccessible cardinal κ is also weakly inaccessible (since $\lambda < \lambda^+ \leq 2^\lambda_{\text{card}}$) and, under the additional assumption of GCH of Def. and Rem. 7.13(a), $\kappa \in \mathbf{Card}$ is weakly inaccessible if, and only if, it is strongly inaccessible.

(b) Questions regarding the existence of inaccessible cardinals are logically subtle. One can show that it is consistent with ZFC + GCH that no weakly inaccessible cardinals exist (see [Kun80, Cor. VI.4.13], where it is described how one can obtain a corresponding model). It is even possible that ZFC implies that inaccessible cardinals do not exist, however, this does not seem to be the expectation of most set theorists. Unfortunately, if ZFC does, indeed, happen to be consistent with the existence of weakly (or even strongly) inaccessible cardinals, then, in consequence of Gödel's second incompleteness theorem (see, e.g., [Kun12, Th. IV.5.32]), this consistency can not be proved in ZFC, unless ZFC is itself inconsistent (see, e.g., [Kun80, Sec. IV.§10] or [Jec06, Th. 12.12]). On the other hand, if the existence of weakly (or even strongly) inaccessible cardinals *is* consistent with ZFC, then the same holds true for a number of related statements: According to [Kun13, Lem. IV.3.17], the following statements are equivalent (as statements in the metatheory):

(i) “ZFC + GCH + there exists a strongly inaccessible cardinal” is consistent.

(ii) “ZFC + there exists a weakly inaccessible cardinal” is consistent.

(iii) “ZFC + 2^{ω}_{card} is weakly inaccessible” is consistent.

(iv) “ZFC + there exists a weakly inaccessible cardinal $\kappa < 2^{\omega}_{\text{card}}$” is consistent.

A Ordinal Topology

We know the class of ordinals **ON** to be well-ordered by $\leq$ and strictly well-ordered by $<$, where, on **ON**, $\leq$ is identical to $\subseteq$ and $<$ is identical to $\in$. According to [Phi16b, Ex. 1.52(b)], each nonempty *set* with a total order $\leq$ is endowed with the induced *order topology*. As in the standard definition provided in [Phi16b, Def. 1.1], we consider a topology $\mathcal{T}$ to be a suitable subset of the power set $\mathcal{P}(X)$ of a *set* X. In consequence, we do not have an order topology on the proper class **ON**. However, we do have an order topology $\mathcal{T}_\alpha$ on each ordinal $\alpha \in \mathbf{ON}$. The purpose of the present section is to study the order topologies on ordinals α, investigating basic topological properties, depending on α.

Notation A.1. For each $\alpha \in \mathbf{ON}$, we let $\mathcal{T}_\alpha$ denote the order topology on α.

Notation A.2. For each $\alpha, \beta \in \mathbf{ON}$, we introduce the following notation for intervals:

$$\begin{aligned}
]\alpha, \beta[&:= I_{\alpha,\beta} := \{\xi \in \mathbf{ON} : \alpha < \xi < \beta\}s,\\
I_{<\beta} &:= \{\xi \in \mathbf{ON} : \xi < \beta\} = \beta,\\
[\alpha, \beta] &:= \{\xi \in \mathbf{ON} : \alpha \leq \xi \leq \beta\},\\
]\alpha, \beta] &:= \{\xi \in \mathbf{ON} : \alpha < \xi \leq \beta\},\\
[\alpha, \beta[&:= \{\xi \in \mathbf{ON} : \alpha \leq \xi < \beta\}.
\end{aligned}$$

Remark A.3. Let $\beta \in \mathbf{ON}$. It is immediate from [Phi16b, Ex. 1.52(b)] and Not. A.2 that a base for $\mathcal{T}_\beta$ is given by

$$\mathcal{B}_\beta := \{\beta\} \cup \{I_{<\gamma} : \gamma \in \beta\} \cup \{I_{\gamma,\delta} : \gamma, \delta \in \beta\} \cup \{I_{\gamma,\beta} : \gamma \in \beta\}. \tag{A.1}$$

Lemma A.4. *Let $\alpha, \beta \in \mathbf{ON}$, $\beta \in \alpha$. Then $\mathcal{T}_\beta = \mathcal{T}_\alpha\restriction_\beta$, where $\mathcal{T}_\alpha\restriction_\beta$ denotes the relative topology on β, induced by $\mathcal{T}_\alpha$.*

Proof. A base for $\mathcal{T}_\beta$ is given by (A.1); a base for $\mathcal{T}_\alpha\restriction_\beta$ is, according to [Phi16b, Prop. 1.54(c)], given by

$$\mathcal{B}_{\alpha,\beta} := \{\alpha \cap \beta\} \cup \{I_{<\gamma} \cap \beta : \gamma \in \alpha\} \cup \{I_{\gamma,\delta} \cap \beta : \gamma, \delta \in \alpha\} \cup \{I_{\gamma,\alpha} \cap \beta : \gamma \in \alpha\}.$$

As $\alpha \cap \beta = \beta$ as well as

$$\forall_{\gamma,\delta\in\alpha} \quad \begin{aligned} I_{<\gamma} \cap \beta &= \begin{cases} \beta & \text{for } \beta \leq \gamma, \\ I_{<\gamma} & \text{for } \gamma < \beta, \end{cases} \\ I_{\gamma,\delta} \cap \beta &= I_{\gamma,\delta\cap\beta}, \\ I_{\gamma,\alpha} \cap \beta &= I_{\gamma,\beta}, \end{aligned}$$

we have $\mathcal{B}_\beta = \mathcal{B}_{\alpha,\beta}$, proving $\mathcal{T}_\beta = \mathcal{T}_\alpha{\restriction}_\beta$. ■

Lemma A.5. *Let $\alpha, \beta \in$ **ON** and $\beta \in \alpha$. Then the following statements* (i) – (iii) *are equivalent:*

(i) *$\beta = 0$ or β is a successor ordinal.*

(ii) *With respect to $\mathcal{T}_\alpha$, β is an isolated point.*

(iii) *$\{\beta\} \in \mathcal{T}_\alpha$.*

Proof. The equivalence between (ii) and (iii) is immediate from the definition of an isolated point (cf. [Phi16b, Def. 1.32(f)], applied with $X := A := \alpha$).

"(i) ⇒ (iii)": If $\beta = 0$, then, either $\alpha = \{0\} = 1$ is open or $\alpha > 1$ and $\{0\} = I_{<1}$ is open. If $\beta > 0$ is a successor ordinal, then let $\gamma \in \alpha$ with $\beta = \mathbf{S}(\gamma)$. If $\beta = \max \alpha$, then $\{\beta\} = I_{>\gamma}$ is open; otherwise $\{\beta\} = I_{\gamma,\mathbf{S}(\beta)}$ is open.

"(ii) ⇒ (i)": If (i) does not hold, then β is a limit ordinal and, if $U \subseteq \alpha$ is a neighborhood of β, then there exists $\gamma \in \beta$ such that $I_{\gamma,\mathbf{S}(\beta)} \subseteq U$ (if β is not the max of α) or $I_{>\gamma} \subseteq U$ (if $\beta = \max \alpha$). In both cases, $\mathbf{S}(\gamma) \in U \setminus \{\beta\}$, proving β to be a cluster point of α (and, thus, not an isolated point). ■

Proposition A.6. **(a)** *If **X** is a class with a strict well-order $<$, then there does not exist a strictly decreasing sequence in **X** (i.e. every decreasing sequence in **X** must be finally constant).*

(b) *Let $(\alpha_k)_{k\in\omega}$ be a strictly isotone sequence in **ON**, $\beta := \sup\{\alpha_k : k \in \omega\}$. Then $\lim_{k\to\infty} \alpha_k = \beta$ (with respect to $\mathcal{T}_\gamma$ for each $\gamma \in$ **ON** with $\beta < \gamma$), but $(\alpha_k)_{k\in\omega}$ is divergent in $(\beta, \mathcal{T}_\beta)$.*

Proof. (a): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in **X**. Since $<$ is a strict well-order, the set $A := \{x_n : n \in \mathbb{N}\}$ has a minimum $m := \min A$. If $(x_n)_{n\in\mathbb{N}}$ is decreasing, then let $N := \min\{n \in \mathbb{N} : x_n = m\}$. Then $x_n = x_N = m$ for each $n \geq N$, showing the sequence is finally constant and, in particular, not strictly decreasing.

(b): Let $\alpha \in \beta$. Since $\alpha < \beta = \sup\{\alpha_k : k \in \omega\}$, there exists $N \in \omega$ such that $\alpha < \alpha_N$. Then, for each $k > N$, since $(\alpha_k)_{k\in\omega}$ is strictly isotone, $\alpha_k \in I_{\alpha,\mathbf{S}(\beta)}$, showing $\lim_{k\to\infty} \alpha_k = \beta$ (and that $(\alpha_k)_{k\in\omega}$ does not converge to any $\alpha \in \beta$). ■

Proposition A.7. *Let $\alpha, \tilde{\alpha} \in \mathbf{ON}$ and consider $F : (\alpha, \mathcal{T}_\alpha) \longrightarrow (\tilde{\alpha}, \mathcal{T}_{\tilde{\alpha}})$. Let $\beta \in \alpha$.*

(a) *If $\beta = 0$ or β is a successor ordinal, then F is always continuous in β.*

(b) *F is continuous in β if, and only if,*

$$\underset{\tilde{\gamma}\in F(\beta)}{\forall} \quad \underset{\gamma\in\beta}{\exists} \quad F\big(]\gamma,\beta]\big) \subseteq]\tilde{\gamma}, F(\beta)]. \tag{A.2}$$

(c) *If $F(\beta)$ is a successor ordinal, then F is continuous in β if, and only if,*

$$\underset{\gamma\in\beta}{\exists} \quad \underset{\xi\in]\gamma,\beta]}{\forall} \quad F(\xi) = F(\beta). \tag{A.3}$$

(d) *If F is strictly isotone and β is a limit ordinal, then F is continuous in β if, and only if,*

$$F(\beta) = \bigcup\{F(\gamma) : \gamma \in \beta\} \tag{A.4}$$

(however, in general, the condition is neither necessary nor sufficient for F to be continuous in β, see Ex. A.8(a),(b),(c) *below).*

(e) *If F is strictly monotone, then F is continuous if, and only if,*

$$\underset{\lambda\in\alpha}{\forall} \quad \Big(\lambda \text{ limit ordinal}^{29} \;\Rightarrow\; F(\lambda) = \bigcup\{F(\gamma) : \gamma \in \lambda\}\Big). \tag{A.5}$$

Proof. According to [Phi16b, Def. 2.1(a)], F is continuous in β if, and only if, for each neighborhood U of $F(\beta)$, there exists a neighborhood V of β with $F(V) \subseteq U$.

(a): Lemma A.5 yields $\{\beta\} \in \mathcal{T}_\alpha$. Thus, if $U \subseteq \tilde{\alpha}$ is a neighborhood of $F(\beta)$, then $\{\beta\}$ is a neighborhood of β such that $F(\{\beta\}) = \{F(\beta)\} \subseteq U$, proving F to be continuous in β.

(b): Let F be continuous in β. Making use of the base of $\mathcal{T}_\alpha$ given by Rem. A.3, we know, for each $\tilde{\gamma}$, that $]\tilde{\gamma}, F(\beta)]$ is a neighborhood of $F(\beta)$, and that, thus, there exists $\gamma \in \beta$ such that $F\big(]\gamma,\beta]\big) \subseteq]\tilde{\gamma}, F(\beta)]$, i.e. (A.2) holds true. Conversely, we now assume (A.2). If U is a neighborhood of $F(\beta)$, then there exists $\tilde{\gamma} \in F(\beta)$ such that $]\tilde{\gamma}, F(\beta)] \subseteq U$.

[29] One can not expect the condition to hold at successor ordinals: If $F = \mathrm{Id}$ and $\beta = \mathbf{S}(\gamma) \in \alpha$, then $\sup\{F(\xi) : \xi \in \beta\} = \max \beta = \gamma < \beta = F(\beta)$.

Choosing $\gamma \in \beta$ according to (A.2) and noting $V :=]\gamma, \beta]$ to be a neighborhood of β, one obtains

$$F(V) = F\big(]\gamma, \beta]\big) \subseteq]\tilde{\gamma}, F(\beta)] \subseteq U,$$

proving F to be continuous in β.

(c): Let $\tilde{\gamma} \in F(\beta)$ be such that $\mathbf{S}(\tilde{\gamma}) = F(\beta)$. Then $]\tilde{\gamma}, F(\beta)] = \{F(\beta)\}$. If F is continuous in β, then, by (A.2), there exists $\gamma \in \beta$ with $F\big(]\gamma, \beta]\big) \subseteq \{F(\beta)\}$, thereby proving (A.3). Conversely, (A.3) implies (A.2), since the γ of (A.3) will work in (A.2) for each $\tilde{\gamma} \in F(\beta)$.

(d): If $\gamma < \beta$, then the strict isotonicity of F yields $F(\gamma) < F(\beta)$, i.e. $F(\gamma) \subsetneq F(\beta)$ by Prop. 3.32. Thus, $\supseteq$ always holds in (A.4), just due to the isotonicity of F. Now assume β to be a limit ordinal. If F is continuous in β and $\tilde{\gamma} \in F(\beta)$, then, by (A.2), there exists $\xi \in \beta$ such that $\tilde{\gamma} < F(\mathbf{S}(\xi)) \leq F(\beta)$. As β is a limit ordinal, $\mathbf{S}(\xi) < \beta$ and the strict isotonicity of F yields $\tilde{\gamma} < F(\mathbf{S}(\xi)) < F(\beta)$ and $\tilde{\gamma} \in \bigcup\{F(\gamma) : \gamma < \beta\}$, thereby proving (A.4). Conversely, we now assume (A.4) holds. Then, if $\tilde{\gamma} \in F(\beta)$, there exists $\gamma \in \beta$ with $\tilde{\gamma} \in F(\gamma)$. Thus, due to the strict isotonicity of F,

$$\xi \in]\gamma, \beta] \quad \Rightarrow \quad \tilde{\gamma} < F(\gamma) < F(\xi) \leq F(\beta) \quad \Rightarrow \quad F(\xi) \in]\tilde{\gamma}, F(\beta)],$$

thereby proving (A.2) and the continuity of F in β.

(e): If F is strictly antitone, then Prop. A.6(a) implies $\alpha < \omega$ (otherwise, $(F(k)_{k\in\omega}$ would yield a strictly decreasing sequence in $\tilde{\alpha}$). Then F is always continuous by (a) and (A.5) holds trivially, as there does not exist any limit ordinal in α. Now assume F to be strictly isotone. If F is continuous, then (d) and (A.4) imply (A.5). Conversely, if (A.5) holds, then F is continuous in all limit ordinals $\lambda \in \alpha$ by (d) and continuous in all remaining ordinals in α by (a), i.e. F is continuous. ■

Example A.8. (a) Let $\alpha, \tilde{\alpha} \in \mathbf{ON}$ and consider an antitone function $F : (\alpha, \mathcal{T}_\alpha) \longrightarrow (\tilde{\alpha}, \mathcal{T}_{\tilde{\alpha}})$. We show that F is then always continuous and that it satisfies (A.5) if, and only if, $\alpha \in \omega$. Indeed, we already noticed in the proof of Prop. A.7(e) that $\alpha \in \omega$ implies F to be continuous and to trivially satisfy (A.5), as, in this case, α does not contain any limit ordinals. If $\omega \leq \alpha$, then, by Prop. A.6(a), F needs to be finally constant: More precisely,

$$\underset{n\in\omega}{\exists} \quad \underset{\tilde{\beta}\in\tilde{\alpha}}{\exists} \quad \underset{n<\beta\in\alpha}{\forall} \quad F(\beta) = \tilde{\beta}.$$

Thus,

$$\underset{n<\beta\in\alpha}{\forall} \quad \underset{\tilde{\gamma}\in F(\beta)=\tilde{\beta}}{\forall} \quad F\big(]n, \beta]\big) = \{F(\beta)\} \subseteq]\tilde{\gamma}, F(\beta)],$$

showing (A.2) to hold with $\gamma := n$, proving F to be continuous (in each $\beta > n$ by Prop. A.7(b) and in each $\beta \leq n$, as these β are not limit ordinals). On the other hand, if $\lambda \in \alpha$ is a limit ordinal (e.g., $\lambda = \omega$), then $n < \lambda$ and the strict antitonicity of F yields $F(n) > F(\lambda) = \tilde{\beta}$ and (A.5) fails to hold.

(b) We have already seen in (a) that antitone functions can be continuous (at a limit ordinal β) without satisfying (A.4) or (A.5). Indeed, there are many other examples: Let $\alpha \in \mathbf{ON}$, $\omega < \alpha$, and $m, n \in \omega$. Define

$$F : \alpha \longrightarrow \alpha, \quad F(\xi) := \begin{cases} \mathbf{S}(m) & \text{for } \xi = m, \\ m & \text{for } n < \xi \leq \omega, \\ \xi & \text{otherwise.} \end{cases}$$

Then F is continuous, as it is continuous in 0 and all successor ordinals by Prop. A.7(a), it is continuous in ω by Prop. A.7(c), and it is continuous is all limit ordinals $\beta \in \alpha \setminus \{\omega\}$, since, clearly, (A.2) holds at such limit ordinals β.

(c) The following function $F : \mathbf{S}(\omega) \longrightarrow \mathbf{S}(\omega)$ satisfies (A.4) (for $\beta = \omega$) and (A.5), but is not continuous (at ω): Let

$$F(\xi) := \begin{cases} 0 & \text{for } \xi \in \omega,\ \xi \text{ odd}, \\ \xi & \text{otherwise.} \end{cases}$$

Then F is not continuous at ω, since (A.2) does not hold: If $\tilde{n} \in \omega = F(\omega)$, then, for each $n \in \omega$, there exists $m \in \omega$, $n < m$, m odd, such that $F(m) = 0 \notin]\tilde{n}, \omega]$. On the other hand,

$$\omega = F(\omega) = \bigcup \{F(n) : n \in \omega\}$$

is true: Indeed, $F(n) \leq \omega$ for each $n \in \omega$ is immediate, whereas $\omega = \sup\{F(n) : n \in \omega\} = \sup\{n \in \omega : n \text{ even}\}$ is then also clear.

Theorem A.9. *Let $\alpha \in \mathbf{ON}$ and consider the topological space $(\alpha, \mathcal{T}_\alpha)$. Recall that ω_1 denotes the smallest uncountable cardinal.*

(a) *$(\alpha, \mathcal{T}_\alpha)$ is completely normal (i.e. T_1 and T_5).*

(b) *$(\alpha, \mathcal{T}_\alpha)$ is discrete (i.e. $\mathcal{T}_\alpha = \mathcal{P}(\alpha)$) if, and only if, $\alpha \leq \omega$.*

(c) *$(\alpha, \mathcal{T}_\alpha)$ is connected if, and only if, $\alpha \in \{0, 1\}$.*

(d) *$(\alpha, \mathcal{T}_\alpha)$ is separable (i.e. there exists a countable dense subset of α) if, and only if, $\alpha < \omega_1$ (the proof that the space is not separable for $\alpha \geq \omega_1$ makes use of AC).*

(e) *$(\alpha, \mathcal{T}_\alpha)$ is C_2 (i.e. there exists a countable base for $\mathcal{T}_\alpha$) if, and only if, $\alpha < \omega_1$ (the proof that the space is not C_2 for $\alpha \geq \omega_1$ makes use of AC).*

(f) *$(\alpha, \mathcal{T}_\alpha)$ is C_1 (i.e. each $\beta \in \alpha$ has a countable local base) if, and only if, $\alpha \leq \omega_1$ (the proof that the space is not C_1 for $\alpha > \omega_1$ makes use of AC).*

(g) *$(\alpha, \mathcal{T}_\alpha)$ is compact if, and only if, α is not a limit ordinal.*

(h) *$(\alpha, \mathcal{T}_\alpha)$ is locally compact (i.e. every $\beta \in \alpha$ has a compact neighborhood).*

(i) *$(\alpha, \mathcal{T}_\alpha)$ is sequentially compact (i.e. every sequence in α has a subsequence that converges in α) if, and only if, $\operatorname{cf}(\alpha) \neq \omega$. In particular, one has the following noteworthy special cases:*

(i) *For $\alpha < \omega_1$, $(\alpha, \mathcal{T}_\alpha)$ is sequentially compact if, and only if, α is not a limit ordinal.*

(ii) *If α is a successor ordinal, then $(\alpha, \mathcal{T}_\alpha)$ is sequentially compact.*

(iii) *If $\alpha = \omega_1$ or $\alpha = \omega_1 \cdot 2$, then $(\alpha, \mathcal{T}_\alpha)$ is sequentially compact (this makes use of AC).*

(j) *$(\alpha, \mathcal{T}_\alpha)$ is metrizable if, and only if, $\alpha < \omega_1$ (the proof that the space is not metrizable for $\alpha \geq \omega_1$ makes use of AC).*

Proof. (a): As $\mathcal{T}_\alpha$ is the order topology on α, this is immediate from [Phi16b, Th. D.16].

(b): For $\alpha \in \{0, 1\}$, $(\alpha, \mathcal{T}_\alpha)$ is trivially discrete. If $1 < \alpha \leq \omega = [0, \omega[$ then $\{0\} = I_{<1}$ is open and, for each $0 < \beta \in \alpha$ such that $\mathbf{S}(\beta) \in \alpha$, $\{\beta\} =]\beta - 1, \mathbf{S}(\beta)[$ is open, and, for $\alpha < \omega$, $\{\max \alpha\} = I_{>\max \alpha - 1}$ is open, showing $(\alpha, \mathcal{T}_\alpha)$ to be discrete for each $\alpha \leq \omega$. On the other hand, if $\alpha > \omega$, then $\omega \in \alpha$ and $\{\omega\}$ contains no nonempty open interval $I_{<\beta}$, $I_{>\beta}$, or $I_{\beta,\gamma}$ with $\beta, \gamma \in \alpha$, showing, since the open intervals form a base of $\mathcal{T}_\alpha$, that $\{\omega\}$ is not open and $(\alpha, \mathcal{T}_\alpha)$ is not discrete.

(c): For $\alpha \in \{0, 1\}$, $(\alpha, \mathcal{T}_\alpha)$ is trivially connected. If $\alpha \geq 2$, then $\alpha = I_{<1} \,\dot\cup\, I_{>0} = \{0\} \,\dot\cup\,]0, \alpha[$, showing α to be the disjoint union of nonempty open sets, i.e. $(\alpha, \mathcal{T}_\alpha)$ is not connected.

(d): If $\alpha < \omega_1$, then α is countable and, thus, separable. Now assume $\omega_1 \leq \alpha$. Then $\omega_1 = [0, \omega_1[\subseteq \alpha$. If $A \subseteq \alpha$ is countable, then $A \cap \omega_1$ is countable and $\gamma := \sup(A \cap \omega_1) = \bigcup(A \cap \omega_1)$ is countable by Th. 7.15 (which makes use of AC). In consequence, $\gamma < \omega_1$ and $]\gamma, \gamma + 2[$ is an example of a nonempty open set contained in $\alpha \setminus A$, showing A is not dense in α and $(\alpha, \mathcal{T}_\alpha)$ is not separable.

(e): If $\alpha < \omega_1$, then the set

$$\mathcal{B} := \{\alpha\} \cup \{I_{<\beta} : \beta \in \alpha\} \cup \{I_{>\beta} : \beta \in \alpha\} \cup \{I_{\beta,\gamma} : \beta, \gamma \in \alpha\}$$

is countable. Since $\mathcal{B}$ is a base of $\mathcal{T}_\alpha$, $(\alpha, \mathcal{T}_\alpha)$ is C_2. Conversely, if $\omega_1 \leq \alpha$, then, by (d), $(\alpha, \mathcal{T}_\alpha)$ is not separable and, thus, not C_2 by [Phi16b, Prop. 1.51].

(f): If $\alpha < \omega_1$, then $(\alpha, \mathcal{T}_\alpha)$ is C_2 by (e) and, thus, C_1 by [Phi16b, Lem. 1.47(b)]. If $\alpha = \omega_1$, then $\{\{0\}\}$ forms a local base at 0 (since $\{0\}$ is open) and, for each $0 < \beta \in \alpha$, $\mathcal{B}(\beta) := \{I_{\gamma,\mathbf{S}(\beta)} : \gamma < \beta\}$ forms a countable local base at β: $\mathcal{B}(\beta)$ is countable, since $\beta < \omega_1$; and, if $\beta \in O \in \mathcal{T}_\alpha$, then there exist $\gamma, \delta \in \alpha$ such that $\beta \in I_{\gamma,\delta}$ (i.e. $\gamma < \beta < \delta$), implying $\beta \in I_{\gamma,\mathbf{S}(\beta)} \subseteq I_{\gamma,\delta}$ and $\mathcal{B}(\beta)$ to be a local base at β. On the other hand, assuming $\alpha > \omega_1$, $\omega_1 \in \alpha$ does not have a countable local base: Let $(O_n)_{n\in\omega}$ be a countable family of open sets such that $\omega_1 \in O_n$ for each $n \in \omega$. Since the open intervals form a base for $\mathcal{T}_\alpha$, for each $n \in \omega$, there exists $\beta_n < \omega_1$ such that $]\beta_n, \omega_1] \subseteq O_n$ (where $]\beta_n, \omega_1] = I_{>\beta_n}$ for $\alpha = \mathbf{S}(\omega_1)$ and $]\beta_n, \omega_1] = I_{\beta_n,\mathbf{S}(\omega_1)}$ for $\alpha > \mathbf{S}(\omega_1)$). Then

$$\gamma := \sup\{\beta_n : n \in \omega\} = \bigcup\{\beta_n : n \in \omega\} < \omega_1$$

by Th. 7.15 (since γ is a countable union of countable sets). In consequence $G :=]\gamma, \omega_1]$ is an open set with $\omega_1 \in G$ that does not contain any of the sets O_n as a subset, proving $\{O_n : n \in \omega\}$ not to be a local base at ω_1.

(g): If $\alpha = 0$, then $(\alpha, \mathcal{T}_\alpha)$ is, trivially, compact. If α is a successor ordinal, then there exists $\beta \in \alpha$ with $\alpha = \mathbf{S}(\beta)$. Then $\alpha = [0, \beta]$ is bounded with $0 = \min \alpha$, $\beta = \max \alpha$. If $\emptyset \neq A \subseteq \alpha$, then $\min A = \bigcap A$ and $\sup A = \bigcup A$ both exist, showing $(\alpha, \leq)$ to be complete. Thus, according to [Phi16b, Th. D.17], $(\alpha, \mathcal{T}_\alpha)$ is compact. Conversely, if α is a limit ordinal, then α does not have a max, i.e. $(\alpha, \mathcal{T}_\alpha)$ is not compact by [Phi16b, Th. D.17] (one can also directly note that, if α is a limit ordinal, then the sets $I_{<\beta}$ with $\beta < \alpha$ provide an open cover of α without a finite subcover).

(h): Let $\beta \in \alpha$. If α is not a limit ordinal, then, by (g), α is a compact neighborhood of β. If α is a limit ordinal, then, again using (g) (and also Lem. A.4), $[0, \beta+2] = \beta+3 < \alpha$ is a compact neighborhood of β.

(i): Suppose $\mathrm{cf}(\alpha) = \omega$. Then, by Lem. 7.17(d), there exists a strictly isotone function $f : \omega \longrightarrow \alpha$, mapping ω cofinally into α. Thus, letting $\alpha_k := f(k)$ for each $k \in \omega$, the sequence $(\alpha_k)_{k\in\omega}$ is strictly isotone with $\alpha = \sup\{\alpha_k : k \in \omega\}$. Since every subsequence of $(\alpha_k)_{k\in\omega}$ is still a strictly isotone sequence in α with supremum α, Prop. A.6(b) yields that $(\alpha_k)_{k\in\omega}$ does not have a subsequence that converges in α, showing $(\alpha, \mathcal{T}_\alpha)$ is not sequentially compact. Conversely, suppose $\mathrm{cf}(\alpha) \neq \omega$. Since $\alpha = 0$ is, trivially, sequentially compact, let $\alpha > 0$. Let $(\alpha_k)_{k\in\omega}$ be a sequence in α, $\beta := \sup\{\alpha_k : k \in \omega\}$. We let $\beta_0 := \beta$. Then $f : \omega \longrightarrow \beta$, $f(k) := \alpha_{\phi_n(k)}$, is cofinal in β. Since $\mathrm{cf}(\alpha) \neq \omega$, β must be a successor ordinal (with $\beta = \max\{\alpha_k : k \in \omega\} < \alpha$) or a limit ordinal $\beta < \alpha$ (with $\mathrm{cf}(\beta) = \omega$). In each case, $\beta_0 = \beta < \alpha$. For $n \in \omega$, we inductively assume we already have constructed ordinals $\beta_0, \dots, \beta_n \in \alpha$ such that $\beta_n < \dots < \beta_0 < \alpha$, $M_n := \{k \in \omega : \alpha_k > \beta_n\}$ is finite, $N_n := \{k \in \omega : \alpha_k \leq \beta_n\} = \omega \setminus M_n$ (which is then infinite), and $\beta_n = \sup\{\alpha_k : k \in N_n\}$. Letting $\phi_n : \omega \longrightarrow N_n$ be a bijection, the function $f_n : \omega \longrightarrow \beta_n$, $f_n(k) := \alpha_{\phi_n(k)}$, is cofinal in β_n. If β_n is a limit ordinal, then we obtain a subsequence $(\alpha_{k_j})_{j\in\omega}$ of $(\alpha_k)_{k\in\omega}$ with $\lim_{j\to\infty}\alpha_{k_j}=\beta_n<\alpha$: As $\mathrm{cf}(\beta_n)=\omega$, there exists, by Lem. 7.17(d), a strictly isotone function $g_n:\omega\longrightarrow\beta_n$, mapping ω cofinally into β_n, i.e. $\beta_n=\sup\{g_n(i):\, i\in\omega\}$. Thus, if we let $k_0:=\phi_n(0)$, and, inductively, for $n\in\omega$,

$$k_{j+1}:=\min\Big\{k\in\omega:\, k>k_j\,\wedge\,\alpha_k>\max\{g_n(j),\alpha_{k_j}\}\Big\}$$

(k_{j+1} is well-defined, since $\max\{g_n(j),\alpha_{k_j}\}<\beta_n$), then $(\alpha_{k_j})_{j\in\omega}$ is a strictly isotone subsequence of $(\alpha_k)_{k\in\omega}$ with $\beta_n=\sup\{g_n(i):\, i\in\omega\}=\sup\{\alpha_{k_j}:\, j\in\omega\}$, and Prop. A.6(b) yields $\lim_{j\to\infty}\alpha_{k_j}=\beta_n<\alpha$. Now assume β_n to be a successor ordinal, i.e. $\beta_n=\max\{\alpha_k:\, k\in N_n\}<\alpha$. If the set $\{k\in N_n:\,\alpha_k=\beta_n\}$ is infinite, then, clearly, we obtain a (constant) subsequence $(\alpha_{k_j})_{j\in\omega}$ with $\lim_{j\to\infty}\alpha_{k_j}=\beta_n$. If the set $\{k\in N_n:\,\alpha_k=\beta_n\}$ is finite, then so is $M_{n+1}:=\{k\in\omega:\,\alpha_k\geq\beta_n\}=M_n\cup\{k\in N_n:\,\alpha_k=\beta_n\}$ and, in consequence, $N_{n+1}:=\omega\setminus M_{n+1}=\{k\in\omega:\,\alpha_k<\beta_n\}$ is infinite. Since β_n is a successor ordinal, $\beta_{n+1}:=\sup\{\alpha_k:\, k\in N_{n+1}\}<\beta_n$. That $M_{n+1}=\{k\in\omega:\,\alpha_k>\beta_{n+1}\}$ and $N_{n+1}:=\{k\in\omega:\,\alpha_k\leq\beta_{n+1}\}$ is then also clear. If, for each $n\in\mathbb{N}$, β_n is a successor ordinal with $\{k\in N_n:\,\alpha_k=\beta_n\}$ finite, then we obtain a strictly decreasing sequence $(\beta_n)_{n\in\omega}$ in α, which is impossible by Prop. A.6(a). Thus, for some $n\in\omega$, the recursion must come to an end, i.e. β_n must be a limit ordinal or a successor ordinal with $\{k\in N_n:\,\alpha_k=\beta_n\}$ infinite. In both cases, $(\alpha_k)_{k\in\omega}$ has a convergent subsequence, proving $(\alpha,\mathcal{T}_\alpha)$ to be sequentially compact. It remains to deduce the special cases (i) – (iii). For (i), we use that, by Cor. 7.19 and Ex. 7.20(c), we know, for $\alpha<\omega_1$, that $\mathrm{cf}(\alpha)=\omega$ if, and only if, α is a limit ordinal. Alternatively, one can prove (i) without using cofinalities by noting that $\alpha<\omega_1$ means $(\alpha,\mathcal{T}_\alpha)$ is C_2 by (e) and, thus, sequentially compact if, and only if, it is compact by [Phi16b, Th. E.5]. As, by (g), $(\alpha,\mathcal{T}_\alpha)$ is compact if, and only if, α is not a limit ordinal, we have another proof of (i). For (ii), we note that, if α is a successor ordinal, then Cor. 7.19(b) yields $\mathrm{cf}(\alpha)=1\neq\omega$. For (iii), we note that, by Ex. 7.20(d) (which uses AC), $\mathrm{cf}(\omega_1\cdot 2)=\mathrm{cf}(\omega_1)=\omega_1\neq\omega$.

(j): Let $\alpha<\omega_1$. As a consequence of (a) and (e), $(\alpha,\mathcal{T}_\alpha)$ is regular (i.e. T_1 and T_3) and C_2 and, thus, metrizable by the Urysohn metrization theorem [Phi16b, Cor. D.37]. However, the proof of [Phi16b, Cor. D.37] makes use of AC (e.g. when choosing the indices i_B in the proof of [Phi16b, Prop. D.33(a)] and when concluding that $\mathcal{B}_0$ is countable in the proof of [Phi16b, Prop. D.33(b)]). Alternatively, to avoid AC, one can prove metrizability of $(\alpha,\mathcal{T}_\alpha)$ by making use of [Phi16b, Th. D.21(c)], which does not use AC, and states that every countable space with an order topology is homeomorphic to a subspace of $\mathbb{Q}$ (with its usual topology) and, thus, metrizable. On the other hand, if $\alpha>\omega_1$, then $(\alpha,\mathcal{T}_\alpha)$ is not C_1 by (f) and, thus, not metrizable by [Phi16b, Rem. 1.39(a)]. If $\alpha=\omega_1$, then $(\alpha,\mathcal{T}_\alpha)$ is sequentially compact by (i), but not compact (according to (g)). In consequence, $(\alpha,\mathcal{T}_\alpha)$ is not metrizable by [Phi16b, Th. 3.20]. ■

Example A.10. We consider the space $(T,\mathcal{T})$, called the *Tychonoff plank*, where $T=$

$[0,\omega_1]\times[0,\omega] = \mathbf{S}(\omega_1)\times\mathbf{S}(\omega)$ and $\mathcal{T}$ is the product topology obtained from $\mathcal{T}_{\mathbf{S}(\omega_1)}$ and $\mathcal{T}_{\mathbf{S}(\omega)}$.

(a) $(T,\mathcal{T})$ is compact by [Phi16b, Th. 3.31], since $(\mathbf{S}(\omega_1),\mathcal{T}_{\mathbf{S}(\omega_1)})$ and $(\mathbf{S}(\omega),\mathcal{T}_{\mathbf{S}(\omega)})$ are both compact by Th. A.9(g).

(b) $(T,\mathcal{T})$ is sequentially compact: Let $((\alpha_k,\beta_k))_{k\in\omega}$ be a sequence in T. According to Th. A.9(i)(ii), $(\mathbf{S}(\omega_1),\mathcal{T}_{\mathbf{S}(\omega_1)})$ and $(\mathbf{S}(\omega),\mathcal{T}_{\mathbf{S}(\omega)})$ are both sequentially compact. Thus, $(\alpha_k)_{k\in\omega}$ has a subsequence $(\alpha_{k_j})_{j\in\omega}$ such that $\lim_{j\to\infty}\alpha_{k_j} = \alpha\in\mathbf{S}(\omega_1)$ and $(\beta_{k_j})_{j\in\omega}$ has a subsequence $(\beta_{k_{j_n}})_{n\in\omega}$ such that $\lim_{n\to\infty}\beta_{k_{j_n}} = \beta\in\mathbf{S}(\omega)$. Then, letting $(\gamma_n,\delta_n) := (\alpha_{k_{j_n}},\beta_{k_{j_n}})$, we obtain $\lim_{n\to\infty}(\gamma_n,\delta_n) = (\alpha,\beta)\in T$: Indeed, if $(\alpha,\beta)\in O\in\mathcal{T}$, then there exist $O_1\in\mathcal{T}_{\mathbf{S}(\omega_1)}$ and $O_2\in\mathcal{T}_{\mathbf{S}(\omega)}$ with $(\alpha,\beta)\in O_1\times O_2\subseteq O$ and there exists $N\in\omega$ such that, for each $n>N$, $(\gamma_n,\delta_n)\in O_1\times O_2\subseteq O$.

(c) $(T,\mathcal{T})$ normal (T_1 and T_4), but not T_5: Since $(\mathbf{S}(\omega_1),\mathcal{T}_{\mathbf{S}(\omega_1)})$ and $(\mathbf{S}(\omega),\mathcal{T}_{\mathbf{S}(\omega)})$ are T_1, T_2, T_3, so is $(T,\mathcal{T})$ by [Phi16b, Prop. 3.5(b)]. As $(T,\mathcal{T})$ is also compact by (a), and T_2 and compact imply T_4 by [Phi16b, Prop. 3.30], $(T,\mathcal{T})$ is, indeed, normal. To show that $(T,\mathcal{T})$ is not T_5, we show that there exists a subspace that is not T_4 (then it is also not T_5 and, as T_5 is inherited by subspaces according to [Phi16b, Prop. 3.5(b)], $(T,\mathcal{T})$ can not be T_5). The subspace we will show not to be T_4 is the so-called *deleted Tychonoff plank* $(T_\infty,\mathcal{T}_\infty)$, where $T_\infty := T\setminus\{(\omega_1,\omega)\}$ and $\mathcal{T}_\infty$ is the subspace topology on T_∞. Define

$$A := \big\{(\omega_1,n):\ n\in\omega\big\},\quad B := \big\{(\alpha,\omega):\ \alpha\in\omega_1\big\}.$$

We will show that both A and B are $\mathcal{T}_\infty$-closed subsets of T_∞ that can not be separated via disjoint $\mathcal{T}_\infty$-open sets: While $A,B\subseteq T_\infty$ is immediate, A and B are $\mathcal{T}_\infty$-closed, since

$$T_\infty\setminus A = \omega_1\times\mathbf{S}(\omega)\in\mathcal{T}_\infty,\qquad T_\infty\setminus B = \mathbf{S}(\omega_1)\times\omega$$

(note $\mathbf{S}(\omega),\omega = I_{<\omega}\in\mathcal{T}_{\mathbf{S}(\omega)}$ and $\mathbf{S}(\omega_1),\omega_1 = I_{<\omega_1}\in\mathcal{T}_{\mathbf{S}(\omega_1)}$). Now let $O_A,O_B\in\mathcal{T}_\infty$ such that $A\subseteq O_A$ and $B\setminus O_B$. The goal is to prove $O_A\cap O_B\neq\emptyset$. Let $(\omega_1,n)\in A$. Since $\{n\}\in\mathcal{T}_{\mathbf{S}(\omega)}$, there exists $O\in\mathcal{T}_{\mathbf{S}(\omega_1)}$ such that $\omega_1\in O$ and $O\times\{n\}\subseteq O_A$ and we can let

$$\alpha_n := \min\Big\{\alpha\in\omega_1 :]\alpha,\omega_1]\subseteq O\Big\}\in\omega_1.$$

In particular, we then have $(\omega_1,n)\in]\alpha_n,\omega_1]\times\{n\}\subseteq O_A$. Let $\bar\alpha := \sup\{\alpha_n:\ n\in\omega\}$. Then $\bar\alpha\in\omega_1$, since $\bar\alpha$ is countable. If $(\alpha,n)\in]\bar\alpha,\omega_1]\times\omega$, then $(\alpha,n)\in]\alpha_n,\omega_1]\times\{n\}\subseteq O_A$, showing $]\bar\alpha,\omega_1]\times\omega\subseteq O_A$. On the other hand, $(\mathbf{S}(\bar\alpha),\omega)\in B$, i.e. there exists $n\in\omega$ such that $\{\mathbf{S}(\bar\alpha)\}\times]n,\omega]\in O_B$, implying $(\mathbf{S}(\bar\alpha),\mathbf{S}(n))\in O_A\cap O_B$ and $O_A\cap O_B\neq\emptyset$.

(d) Assuming AC, $(T, \mathcal{T})$ is neither C_1 nor C_2: According to [Phi16b, Lem. 1.47(b)], it suffices to show that $(T, \mathcal{T})$ is not C_1. Seeking a contradiction, assume $(T, \mathcal{T})$ to be C_1. Then, for each $\alpha \in \mathbf{S}(\omega_1)$, $(\alpha, 0)$ has a countable local base. Thus, as $\{0\} \in \mathcal{T}_{\mathbf{S}(\omega)}$, there exists a countable set $\mathcal{B} \subseteq \mathcal{T}_{\mathbf{S}(\omega_1)}$ such that $\{O \times \{0\} : O \in \mathcal{B}$ is a local base at $(\alpha, 0)$. But then $\mathcal{B}$ is a local base at α in the space $(\mathbf{S}(\omega_1), \mathcal{T}_{\mathbf{S}(\omega_1)})$, in contradiction to $(\mathbf{S}(\omega_1), \mathcal{T}_{\mathbf{S}(\omega_1)})$ not being a C_1-space by Th. A.9(f) (which made use of AC).

(e) Assuming AC, $(T, \mathcal{T})$ is not separable: Let $\pi : T \longrightarrow \mathbf{S}(\omega_1)$, $\pi(\alpha, \beta) := \alpha$, denote the projection onto $\mathbf{S}(\omega_1)$. If $A \subseteq \alpha$ is countable, then $\pi(A)$ is a countable subset of $\mathbf{S}(\omega_1)$. According to Th. A.9(d) (which makes use of AC), $\pi(A)$ is not dense in $\mathbf{S}(\omega_1)$, i.e. there exists $\emptyset \neq B \in \mathcal{T}_{\mathbf{S}(\omega_1)}$ such that $B \subseteq \mathbf{S}(\omega_1) \setminus \pi(A)$. Then $B \times \{0\} \in \mathcal{T}$ and $B \times \{0\} \subseteq T \setminus A$, showing A is not dense in T.

(f) Assuming AC, $(T, \mathcal{T})$ is not metrizable, as it is not C_1 by (d) and we know every metric space to be C_1 by [Phi16b, Rem. 1.39(a)].

(g) $(T, \mathcal{T})$ is not connected:

$$T = \Big(\{0\} \times \{0\}\Big) \;\dot{\cup}\; \Big(\big(\mathbf{S}(\omega_1) \times I_{>0}\big) \cup \big(I_{>0} \times \mathbf{S}(\omega)\big)\Big),$$

where $\{0\} \times \{0\} \in \mathcal{T}$ and $\big(\mathbf{S}(\omega_1) \times I_{>0}\big) \cup \big(I_{>0} \times \mathbf{S}(\omega)\big) \in \mathcal{T}$.

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