W I A S

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A Quasistatic Crack Propagation Model Allowing for Cohesive Forces and Crack Reversibility

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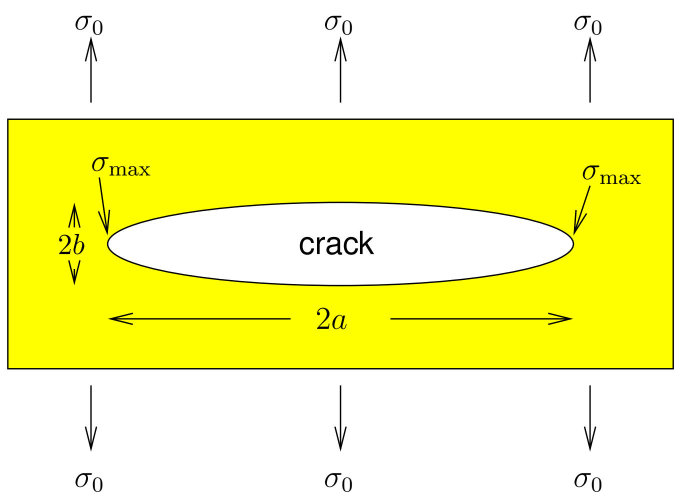
Griffith Theory: Drawbacks

A.A. GRIFFITH. Phil. Trans. of the Royal Soc. of London. Series A **221** (1921), 163–198.

Experimental Evidence: Material failure occurs below theoretical value of critical stress, critical stress varies with the size and geometry of the specimen (G.B. SINCLAIR. Appl. Mech. Rev. **57** (2004), 251–297).



Physical / Mathematical Issues: Griffith theory predicts stress singularity at crack tip:

$\sigma_{\max} = \frac{2\,a}{b}\,\sigma_0$ (load acts entirely on undeformed state),

$\sigma_{\max} = \frac{\epsilon}{\sigma_0}\ln\left(\cosh\frac{2\,\sigma_0}{\epsilon} + \frac{a}{b}\sinh\frac{2\,\sigma_0}{\epsilon}\right)$ (load applied incrementally),

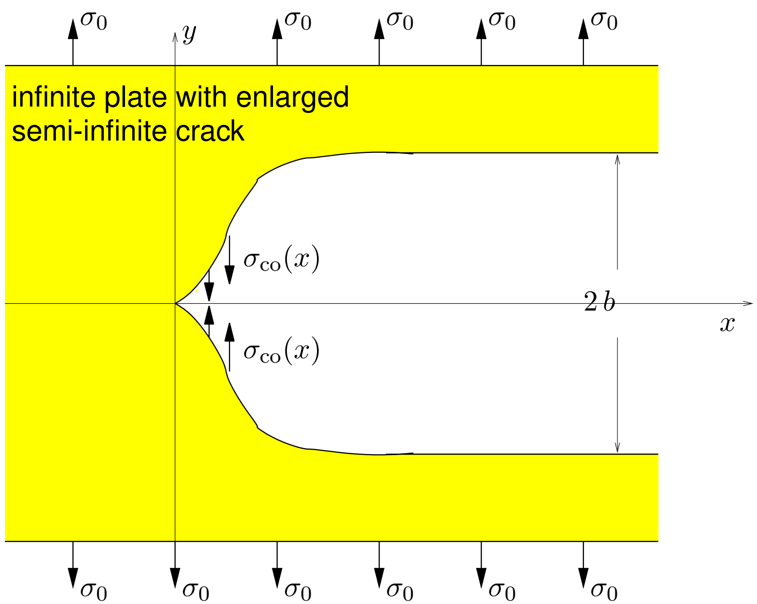
$\lim_{b\to 0}\sigma_{\max} = \infty$ in both cases.

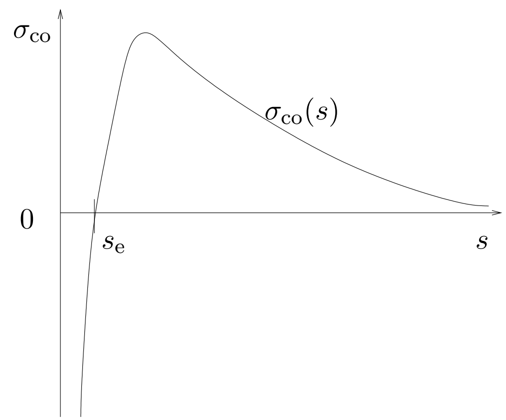
For $b \to 0$, Griffith theory predicts its own failure: The assumptions used in its derivation (elasticity, small deformations) no longer apply. Its predictions become nonphysical.

Limited Scope: Griffith theory can not predict location of crack initiation and crack path.

Cohesive Forces According to Barenblatt and Sinclair

G.I. BARENBLATT. Advan. Appl. Mech. **7** (1962), 55–129.
G.B. SINCLAIR. Appl. Mech. Rev. **57** (2004), 251–297.





Qualitative picture of dependence of cohesive stress σ_{co} on the separation s.

Qualitative properties of the cohesive stress-separation law (s_e: equilibrium separation):

Repulsion for $s < s_e$, nearly linear $\sigma_{co}(s) \approx k_e(s - s_e)$ for s close to s_e, then σ_{co} reaches maximum and gradually decays to 0 for large s.

Challenges:

(a) Compute realistic quantitative laws for $\sigma_{co}(s)$ for a given material from quantum mechanics.
(b) Accounting for the nonlinear law $\sigma_{co}(s)$ can render computation of the resulting stress field very difficult.

Predicting Crack Initiation and Path: Francfort-Marigo Theory

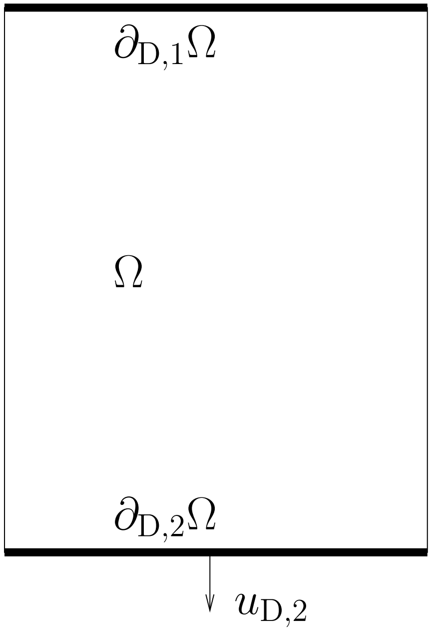
G.A. FRANCFORT, J.-J. MARIGO. J. Mech. Phys. Solids **46** (1998), 1319–1342.

Goal: Formulate the problem such that the location and path of a crack, in contrast to Griffith theory, does not need to be described a priori, but is part of the problem's solution.

Mathematical setting:

Let $\Omega \subseteq \mathbb{R}^N$, $N \in \{1,2,3\}$, be a body's uncracked reference configuration.

For each time $t \in [0,T]$, a strained and cracked configuration of the body is described by a displacement function $u : \Omega \longrightarrow \mathbb{R}^N$ together with a crack $\Gamma \subseteq \Omega$.



Example with Dirichlet boundary conditions:

Prescribe $u = u_{\mathrm{D}}$ on some part of the boundary $\partial\Omega$ of Ω. For example $u_{\mathrm{D},1}(t,x) = (0,0,0)$, $u_{\mathrm{D},2}(t,x) = (0,-t,0)$ (see figure).

Quasistatic Energy Minimization

The goal is to determine $u(t, x)$ by quasistatic energy minimization.

Quasistatic:

Assumption: There are two, decoupled time scales:

Slow Time Scale: Variation of boundary conditions and loading.

Fast Time Scale: The system instantaneously settles into an energy minimum for each time t of the slow time scale.

Quasistatic Evolution:

Find $t \mapsto u(t)$ such that $u(t)$ satisfies an energy balance (energy spent in crack increase must equal the work of the external forces) and has minimal energy among all admissible displacement fields $v \in \mathrm{AD}(t)$.

The choice for $\mathrm{AD}(t)$ is not obvious. Tentative choice:

$$\mathrm{AD}(t) := \big\{ u \in BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) : \ u_\mathrm{D}(t) = u \text{ on } \partial_\mathrm{D} u \big\}.$$

Allowing for Reversible Cracks -1-

For each time $t \in [0, T]$, reversibility is described by a reversibility function

$$r : \Omega \longrightarrow \{0, 1\}, \quad r(x) = \begin{cases} 1 & \text{if there is an irreversible crack at } x, \\ 0 & \text{otherwise.} \end{cases}$$

Irreversibility is triggered where a crack has opened more than a threshold value a_{th}.

Cracks are now defined in terms of u and r:
$\Gamma(u, r) := r^{-1}\{1\} \cup \left\{x \in J_u : \big([u](x)\big) \bullet n_{J_u}(x) > 0\right\}$.

Given $u(t)$, r can be defined in terms of u as a memory function:

$$r_u(t, x) = \begin{cases} 0 & \text{if } \big([u](t, x)\big) \bullet n_{J_{u(t)}}(x) < a_{\text{th}} \text{ for all } s \leq t, \\ 1 & \text{otherwise.} \end{cases}$$

Allowing for Reversible Cracks -2-

Due to the reversibility function, the formulation of the minimality condition at t makes use of the function u already defined for times smaller than t:

Let $v \in \mathrm{AD}(t)$ be an admissible displacement field at time t, and let
$u : [0, t[\longrightarrow BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$
be given. Then u can be extended to t by v:

$$u_v : [0, t] \longrightarrow BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N),$$

$$u_v(s) := \begin{cases} u(s) & \text{for } s < t, \\ v & \text{for } s = t. \end{cases}$$

Thereby, v also gives rise to a reversibility function r^v:

$$r^v : [0, t] \longrightarrow \{0, 1\}, \; r^v := r_{u_v}.$$

Minimality condition at time t

$u(t)$ needs to satisfy

(1) $u(t) \in \mathrm{AD}(t)$, (2) $\mathcal{E}(t)(u_{u(t)}) \leq \mathcal{E}(t)(u_v)$ for each $v \in \mathrm{AD}(t)$,

where the total energy is given by

$\mathcal{E}(t)(u) = \mathcal{E}_\mathrm{b}(u) - \mathcal{F}(t)(u) + \mathcal{E}_\mathrm{cr}\big(\Gamma(u, r_u)\big)$:

$\mathcal{E}_\mathrm{b}(u)$ is the strain energy of the bulk, $\mathcal{E}_\mathrm{b}(u) := \int_\Omega W\big(x, \nabla\, u(x)\big)\,\mathrm{d}x$
with a suitable material function $W : \Omega \times \mathbb{R}^{N^2} \longrightarrow \mathbb{R}_0^+$;

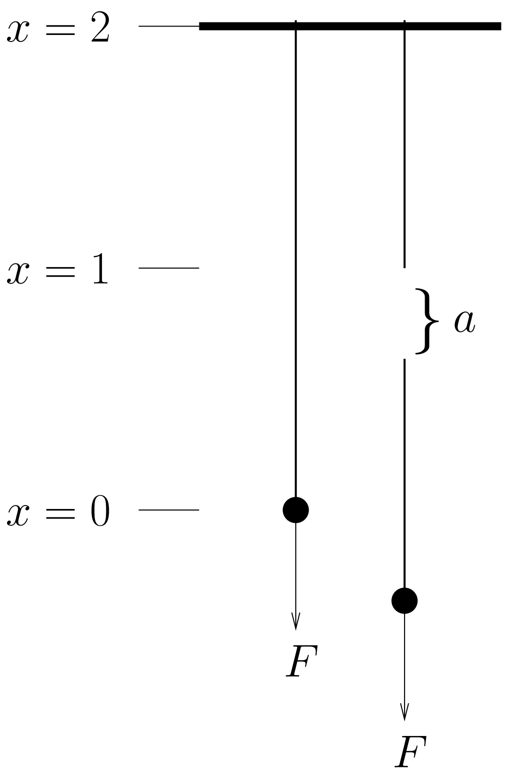
$\mathcal{F}(t)(u)$ is the energy due to body and surface forces;

$\mathcal{E}_\mathrm{cr}\big(\Gamma(u, r_u)\big)$ is the crack energy:
$\mathcal{E}_\mathrm{cr}(\Gamma) = \int_\Gamma \kappa\big(x, n_\Gamma(x), [u](x), r(x)\big)\,\mathrm{d}\mathcal{H}^{N-1}(x)$,
where $\kappa : \Omega \times \mathbb{S}^{N-1} \times \mathbb{R}^N \times \{0, 1\} \longrightarrow \mathbb{R}_0^+ \cup \{\infty\}$ is a material function, describing the material's toughness, n_Γ is the unit normal vector on the crack.

The dependence on $r(x)$ can account for crack reversibility:
Cohesive forces should play no role once the crack has become irreversible:
κ depends nontrivially on the third variable if, and only if, the fourth variable is 0.

Global Versus Local Energy Minimization

Example: Global minimization fails (const. body force, F. & M. 1998, Sec. 5.2):



$$\Omega := \{x \in \mathbb{R} : 0 < x < 2\}, \quad \partial_{\rm D}\Omega := \{2\},$$
$$u_{\rm D} : [0,T] \longrightarrow L^\infty(\partial_{\rm D}\Omega, \mathbb{R}), \quad u_{\rm D}(t)(2) := 0,$$
$$W : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}_0^+, \quad W(x,\xi) := \xi^2/2,$$
$$F : [0,T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t,x,z) := -t\, z,$$
$$\kappa : \Omega \times \{-1,1\} \times \mathbb{R}^N \times \{0,1\} \longrightarrow \mathbb{R}_0^+ \cup \{\infty\},$$

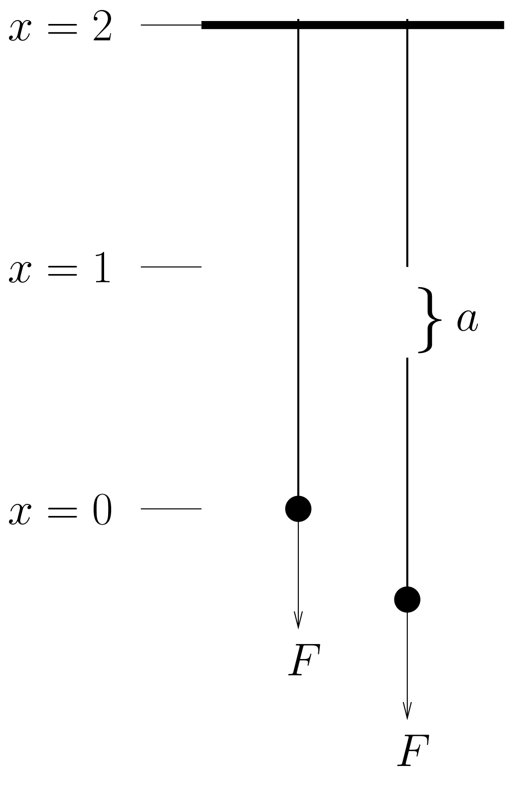
$$\kappa(x,n,z,0) := \begin{cases} \infty & \text{for } z \bullet n \leq -a_{\rm th}, \\ \text{continuous} & \text{for } -a_{\rm th} < z \bullet n \leq a_{\rm th}, \\ \kappa_{\rm th} > 0 & \text{for } a_{\rm th} \leq z \bullet n, \end{cases}$$

$\kappa(x,n,z,1) := \kappa_{\rm th}, \quad a_{\rm th} := 1.$ Let $a > 1$ and consider

$$u_a : [0,T] \longrightarrow BV^\infty(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R}), \quad u_a := \begin{cases} u_a(t,x) := 0 & \text{for } 1 < x < 2, \\ u_a(t,x) := -a & \text{for } 0 < x < 1. \end{cases}$$

Global Versus Local Energy Minimization -2-

Example where global minimization fails (constant body force):



$$W(x,\xi) := \xi^2/2, \quad F(t,x,z) := -t\, z, \quad \kappa(x,n,z,1) := \kappa_{\text{th}},$$

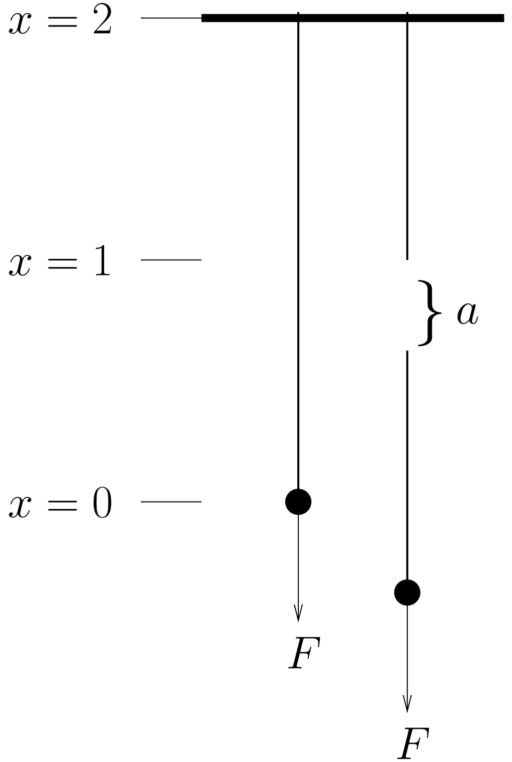
$$u_a := \begin{cases} u_a(t,x) := 0 & \text{for } 1 < x < 2, \\ u_a(t,x) := -a & \text{for } 0 < x < 1. \end{cases}$$

$$\begin{aligned} \mathcal{E}(t)(u_a) &= \int_{\Gamma(u_a,r_{u_a})} \kappa(x, n_{\Gamma(u_a,r_{u_a})}(x), [u_a](x), r_{u_a}(t,x))\, \mathrm{d}\mathcal{H}^0(x) \\ &\quad + \int_\Omega W\big(x, \nabla u_a(x)\big)\, \mathrm{d}x - \int_\Omega F\big(t, x, u_a(x)\big)\, \mathrm{d}x \\ &= \int_{\{x=1\}} \kappa(x,1,a,1)\, \mathrm{d}\mathcal{H}^0(x) \\ &\quad + \int_\Omega (\nabla u_a(t,x))^2/2\, \mathrm{d}x - \int_0^1 t\, a\, \mathrm{d}x \\ &= \kappa_{\text{th}} - t\, a. \end{aligned}$$

Thus, for each $t > 0$, one has $\lim_{a\to\infty} \mathcal{E}(t)(u_a) = -\infty \quad \Rightarrow$ failure for arbitrarily small positive load.

Global Versus Local Energy Minimization -3-

Constant body force with local energy minimization:



$$W(x,\xi) := \xi^2/2, \quad F(t,x,z) := -t\,z, \quad a_{\rm th} := 1,$$

$$\kappa(x,n,z,1) := \kappa_{\rm th},$$

$$\kappa(x,n,z,0) := \begin{cases} \infty & \text{for } z \bullet n \le -a_{\rm th}, \\ \text{cont. } \kappa_{\rm j}(z \bullet n) & \text{for } -a_{\rm th} < z \bullet n \le a_{\rm th}, \\ \kappa_{\rm th} > 0 & \text{for } a_{\rm th} \le z \bullet n. \end{cases}$$

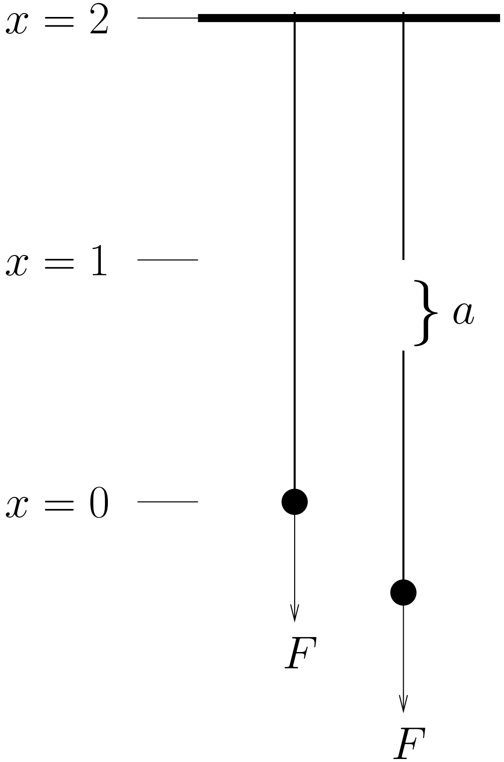
For $t \in [0,T]$, let $u_{\rm e}(t) : \Omega \longrightarrow \mathbb{R}$, $u_{\rm e}(t) \le 0$, be the solution for the "perfectly elastic" limit of the material, i.e. $u_{\rm e}(t)$ is the (global) minimizer of

$$\mathcal{E}_{\rm e}(t)(u) := \int_\Omega \frac{1}{2}\left(\nabla u(x)\right) \bullet \left(\nabla u(x)\right) {\rm d}x + \int_\Omega t\,u(x)\,{\rm d}x\,.$$

Consider a crack at $y \in \Omega$: $u_{\rm e}(t) + \phi_{a,b,y}$, where $\phi_{a,b,y}(x) := \begin{cases} b & \text{for } y < x < 2, \\ -a & \text{for } 0 < x < y. \end{cases}$

Global Versus Local Energy Minimization -4-

Constant body force with local energy minimization:



$$u_{\mathrm{D}}(t)(2) := 0, \quad W(x,\xi) := \xi^2/2,$$

$$F(t,x,z) := -t\,z, \quad \kappa(x,n,z,0) := \kappa_{\mathrm{j}}(n\,z),$$

$$\phi_{a,b,y}(x) := \begin{cases} b & \text{for } y < x < 2, \\ -a & \text{for } 0 < x < y, \end{cases} \quad \Rightarrow \quad b = 0.$$

$$\phi_{a,y} := \phi_{a,0,y}, \quad \nabla\,\phi_{a,y} = 0, \quad \nabla(u_{\mathrm{e}}(t) + \phi_{a,y}) = \nabla(u_{\mathrm{e}}(t)).$$

$$\mathcal{E}(t)\big(u_{\mathrm{e}}(t) + \phi_{a,y}\big) = \kappa_{\mathrm{j}}(a) + \int_\Omega \frac{1}{2}\big(\nabla\,u_{\mathrm{e}}(t,x)\big) \bullet \big(\nabla\,u_{\mathrm{e}}(t,x)\big)\,\mathrm{d}x$$
$$+ \int_\Omega t\,\big(u_{\mathrm{e}}(t,x) + \phi_{a,y}(x)\big)\,\mathrm{d}x$$
$$= \mathcal{E}_{\mathrm{e}}(t)\big(u_{\mathrm{e}}(t)\big) + \kappa_{\mathrm{j}}(a) - t\,a\,y.$$

$\mathcal{E}(t)\big(u_{\mathrm{e}}(t)\big)$ is a local min if, and only if, $\kappa_{\mathrm{j}}(a) - t\,a\,y$ has local min at 0.

Result: At critical $t > 0$, crack appears at $x = 2$ (the physically expected result).