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A Finite Volume Scheme for Transient Nonlocal Conductive-Radiative Heat Transfer, Part 1: Formulation and Discrete Maximum Principle

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Selected Publications:

- O. Klein, P. Philip, J. Sprekels: *Modeling and simulation of sublimation growth of SiC bulk single crystals*, Interfaces and Free Boundaries 6 (2004), 295–314 (summary of modeling, numerical results).
- P. Philip: *Transient Numerical Simulation of Sublimation Growth of SiC Bulk Single Crystals. Modeling, Finite Volume Method, Results*, Thesis, Department of Mathematics, Humboldt University of Berlin, Germany, 2003 Report No. 22, Weierstrass Institute for Applied Analysis and Stochastics, Berlin (modeling & discrete existence (very general, very detailed), numerical results).
- O. Klein, P. Philip: *Transient conductive-radiative heat transfer: Discrete existence and uniqueness for a finite volume scheme*, Mathematical Models and Methods in Applied Sciences 15 (2005), 227–258 (simplified model of this talk: maximum principle, discrete existence).
- P. Philip: *Transient conductive-radiative heat transfer: Convergence of a finite volume scheme*, in preparation (simplified model of this talk: convergence, existence of weak solution).

Outline

Part 1:

- The Model: Domains, mathematical assumptions, transient nonlinear heat equations, nonlocal interface and boundary conditions
- Formulation of Finite Volume Scheme: Focus on discretization of nonlocal radiation terms, maximum principle
- Discrete Maximum Principle, Discrete Existence and Uniqueness

Part 2:

- Piecewise Constant Interpolation, Existence of Convergent Subsequence
  - A Priori Estimates I: Discrete norms, discrete continuity in time
  - Riesz-Fréchet-Kolmogorov Compactness Theorem, A Priori Estimates II: Space and time translate estimates
- Weak Solution: Formulation, discrete analogue, convergence of the discrete analogue to a weak solution

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Model: Nonlinear Transient Heat Conduction, Assumptions

$$\frac{\partial \varepsilon_m(\theta)}{\partial t} - \operatorname{div}(\kappa_m \nabla \theta) = f_m(t,x) \quad \text{in }]0,T[\times\Omega_m \qquad (m \in \{\mathrm{s},\mathrm{g}\}), \tag{1}$$

Ω_{s}: solid domain, Ω_{g}: gas domain, $\theta(t,x) \in \mathbb{R}_0^+$: absolute temperature, ε_m: internal energy, $\kappa_m \in \mathbb{R}^+$: thermal conductivity, f_m: heat source.

Assumptions:

(A-1) For $m \in \{\mathrm{s},\mathrm{g}\}$, $\varepsilon_m \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$, and there is $C_\varepsilon \in \mathbb{R}^+$ such that

$$\varepsilon_m(\theta_2) \geq (\theta_2 - \theta_1)\, C_\varepsilon + \varepsilon_m(\theta_1) \qquad (\theta_2 \geq \theta_1 \geq 0).$$

(A-1*) For $m \in \{\mathrm{s},\mathrm{g}\}$, ε_m is locally Lipschitz: For each $M \in \mathbb{R}_0^+$, there is $L_M \in \mathbb{R}_0^+$ such that $\quad |\varepsilon_m(\theta_2) - \varepsilon_m(\theta_1)| \leq L_M\, |\theta_2 - \theta_1| \qquad \big((\theta_2, \theta_1) \in [0,M]^2\big)$.

(A-2) For $m \in \{\mathrm{s},\mathrm{g}\}$: $\kappa_m \in \mathbb{R}^+$.

(A-3) For $m \in \{\mathrm{s},\mathrm{g}\}$: $f_m \in L^\infty(0,T,L^\infty(\Omega_m))$, $f_m \geq 0$ a.e.

Remark 1. ε_m is strictly increasing, unbounded with image $[\varepsilon_m(0), \infty[$, invertible on its image, and its inverse function ε_m^{-1} is C_ε^{-1}-Lipschitz.

Model: Domain, Assumptions, Notation

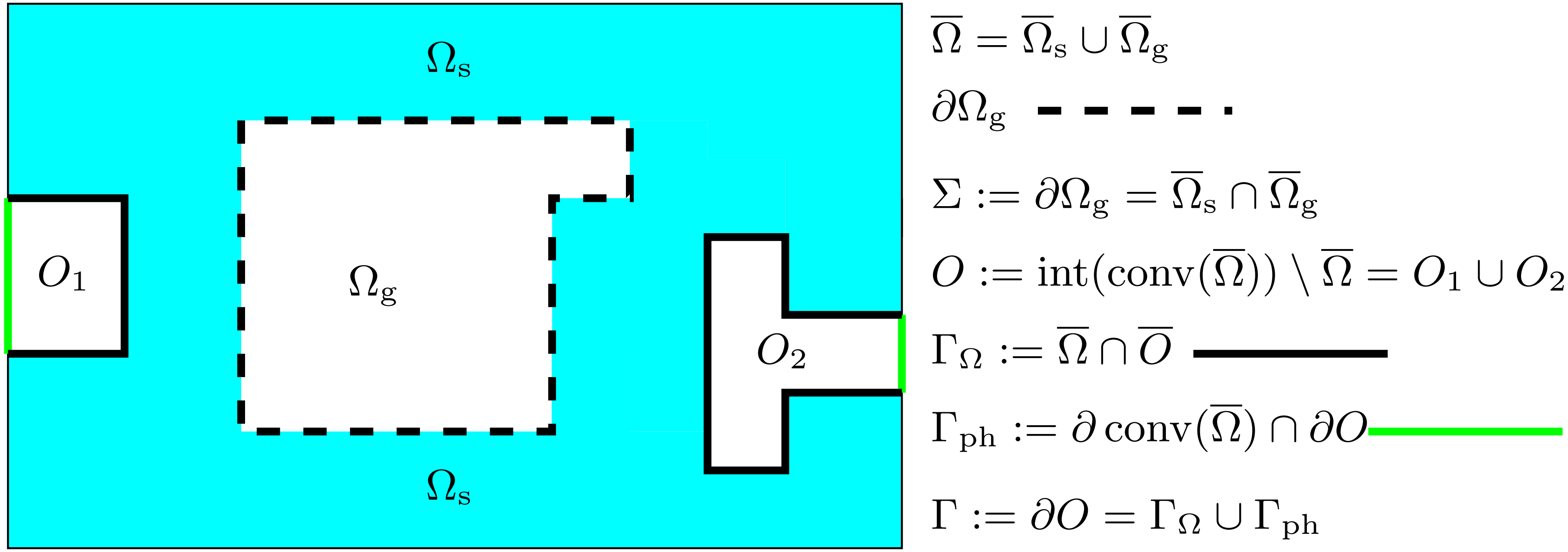

Figure 1: 2-d section through 3-d domain $\overline{\Omega}$. Open radiation regions O_1 and O_2 are artificially closed by the phantom closure Γ_{ph}. By (A-5), Ω_g is engulfed by Ω_s (not visible in 2-d section.

(A-4) $T \in \mathbb{R}^+$, $\overline{\Omega} = \overline{\Omega}_s \cup \overline{\Omega}_g$, $\Omega_s \cap \Omega_g = \emptyset$, and each of the sets Ω, Ω_s, Ω_g, is a nonvoid, polyhedral, bounded, and open subset of $\mathbb{R}^3$.

(A-5) Ω_g is enclosed by Ω_s, i.e. $\partial\Omega_s = \partial\Omega \,\dot\cup\, \partial\Omega_g$, where $\dot\cup$ denotes a disjoint union. Thus, $\Sigma := \partial\Omega_g = \overline{\Omega}_s \cap \overline{\Omega}_g$, and $\partial\Omega = \partial\Omega_s \setminus \Sigma$.

Model: Nonlocal Interface Conditions Modeling Diffuse-Gray Radiation

Continuity of the temperature at Σ:

$$\theta(t,\cdot)\restriction_{\overline{\Omega}_{\mathrm{s}}} = \theta(t,\cdot)\restriction_{\overline{\Omega}_{\mathrm{g}}} \quad \text{on } \Sigma \quad (t \in [0,T]).$$

Continuity of the heat flux at Σ:

$$(\kappa_{\mathrm{g}} \nabla \theta)\restriction_{\overline{\Omega}_{\mathrm{g}}} \bullet \mathbf{n}_{\mathrm{g}} + R(\theta) - J(\theta) = (\kappa_{\mathrm{s}} \nabla \theta)\restriction_{\overline{\Omega}_{\mathrm{s}}} \bullet \mathbf{n}_{\mathrm{g}} \quad \text{on } \Sigma. \tag{2}$$

R: radiosity, J: irradiation, $\mathbf{n}_{\mathrm{g}}$: unit normal vector pointing from gas to solid.

$$R(\theta) = \sigma\, \epsilon(\theta)\, \theta^4 + \big(1 - \epsilon(\theta)\big)\, J(\theta). \tag{3}$$

(A-6) $\sigma \in \mathbb{R}^+$: Boltzmann radiation constant, $\epsilon : \mathbb{R}_0^+ \longrightarrow]0,1]$ is continuous: emissivity of solid surface.

Model: Nonlocal Radiation Operator (1)

$$J(\theta) = K(R(\theta)), \tag{4}$$

$$K(\rho)(x) := \int_\Sigma \Lambda(x,y)\,\omega(x,y)\,\rho(y)\,\mathrm{d}y \quad \text{(a.e. } x \in \Sigma), \tag{5}$$

$\Lambda(x,y) \in \{0,1\}$: visibility factor, ω: view factor defined by

$$\omega(x,y) := \frac{\big(\mathbf{n}_\mathrm{g}(y) \bullet (x-y)\big)\,\big(\mathbf{n}_\mathrm{g}(x) \bullet (y-x)\big)}{\pi\big((y-x) \bullet (y-x)\big)^2} \quad \big(\text{a.e. } (x,y) \in \Sigma^2,\ x \neq y\big). \tag{6}$$

Lemma 2. (Tiihonen 1997: Eur. J. App. Math. 8, Math. Meth. in Appl. Sci. 20)

$\Lambda(x,y)\,\omega(x,y) \geq 0$ *a.e.*, $\Lambda(x,\cdot)\,\omega(x,\cdot)$ *is in* $L^1(\Sigma)$ *for a.e.* $x \in \Sigma$.

Conservation of radiation energy: $\int_\Sigma \Lambda(x,y)\,\omega(x,y)\,\mathrm{d}y = 1$ *for a.e.* $x \in \Sigma$.

K *is a positive compact operator from* $L^p(\Sigma)$ *into itself for each* $p \in [1,\infty]$, *and* $\|K\| = 1$.

For every measurable $S \subseteq \Sigma$, *the function* $x \mapsto \int_S \Lambda(x,y)\,\omega(x,y)\,\mathrm{d}y$ *is in* $L^\infty(\Sigma)$.

Model: Nonlocal Radiation Operator (2)

Combining (3) and (4) provides nonlocal equation for $R(\theta)$:

$$R(\theta) - \big(1 - \epsilon(\theta)\big)\, K\big(R(\theta)\big) = \sigma\, \epsilon(\theta)\, \theta^4 \tag{7}$$

or

$$G_\theta\big(R(\theta)\big) = \sigma\, \epsilon(\theta)\, \theta^4, \quad G_\theta(\rho) := \rho - \big(1 - \epsilon(\theta)\big)\, K(\rho). \tag{8}$$

Lemma 3. (Laitinen & Tiihonen 2001: Quart. Appl. Math. 59)

If $\epsilon : \mathbb{R}_0^+ \longrightarrow]0, 1]$ *is a Borel function, and if* $\theta : \Sigma \longrightarrow \mathbb{R}_0^+$ *is measurable, then, for each* $p \in [1, \infty]$, *the operator* G_θ *maps* $L^p(\Sigma)$ *into itself and has a positive inverse.*

Lemma 3 allows to state (7) as: $\quad R(\theta) = G_\theta^{-1}\big(E(\theta)\big)$.

From (7) and (4): $\quad R(\theta) - J(\theta) = -\epsilon(\theta)\,\big(K\big(R(\theta)\big) - \sigma\, \theta^4\big)$,

such that (2) becomes

$$(\kappa_{\mathrm{g}}\, \nabla\, \theta){\restriction}_{\overline{\Omega}_{\mathrm{g}}} \bullet \mathbf{n}_{\mathrm{g}} - \epsilon(\theta)\,\big(K\big(R(\theta)\big) - \sigma\, \theta^4\big) = (\kappa_{\mathrm{s}}\, \nabla\, \theta){\restriction}_{\overline{\Omega}_{\mathrm{s}}} \bullet \mathbf{n}_{\mathrm{g}} \quad \text{on } \Sigma. \tag{9}$$

Model: Nonlocal Outer Boundary Conditions, Initial Condition

$$\kappa_{\mathrm{s}} \nabla \theta \bullet \mathbf{n}_{\mathrm{s}} + R_{\Gamma}(\theta) - J_{\Gamma}(\theta) = 0 \quad \text{on } \Gamma_{\Omega} \tag{10}$$

in analogy with (2); $\mathbf{n}_{\mathrm{s}}$: outer unit normal to solid.

$$\kappa_{\mathrm{s}} \nabla \theta \bullet \mathbf{n}_{\mathrm{s}} - \epsilon(\theta) \left(K_{\Gamma}(R_{\Gamma}(\theta)) - \sigma \, \theta^4 \right) = 0 \quad \text{on } \Gamma_{\Omega}, \tag{11}$$

where K_{Γ} is defined analogous to K.

On $\partial\Omega \setminus \Gamma_{\Omega}$:

$$\kappa_{\mathrm{s}} \nabla \theta \bullet \mathbf{n}_{\mathrm{s}} - \sigma \, \epsilon(\theta) \, (\theta_{\mathrm{ext}}^4 - \theta^4) = 0 \quad \text{on } \partial\Omega \setminus \Gamma_{\Omega}. \tag{12}$$

Initial Condition: $\theta(0, \cdot) = \theta_{\mathrm{init}}$, where

(A-7) $\theta_{\mathrm{init}} \in L^{\infty}(\Omega, \mathbb{R}_0^+)$.

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Finite Volume Scheme: Discretization of Time and Space Domain

Time domain discretization: $0 = t_0 < \cdots < t_N = T$, $N \in \mathbb{N}$;

time steps: $k_\nu := t_\nu - t_{\nu-1}$;   fineness: $k := \max\{k_\nu : \nu \in \{1, \dots, n\}\}$.

Admissible space domain discretization $\mathcal{T}$ satisfies:

(DA-1) $\mathcal{T} = (\omega_i)_{i \in I}$ forms a finite partition of Ω, and, for each $i \in I$, ω_i is a nonvoid, polyhedral, connected, and open subset of Ω.

From $\mathcal{T}$ define discretizations of Ω_s and Ω_g: For $m \in \{\mathrm{s}, \mathrm{g}\}$ and $i \in I$, let

$$\omega_{m,i} := \omega_i \cap \Omega_m, \quad I_m := \left\{j \in I : \omega_{m,j} \neq \emptyset\right\}, \quad \mathcal{T}_m := (\omega_{m,i})_{i \in I_m}.$$

(DA-2) For each $i \in I$: $\partial_\mathrm{reg}\omega_{\mathrm{s},i} \cap \Sigma = \partial_\mathrm{reg}\omega_{\mathrm{g},i} \cap \Sigma$, where ∂_reg denotes the regular boundary of a polyhedral set, i.e. the parts of the boundary, where a unique outer unit normal vector exists, $\partial_\mathrm{reg}\emptyset := \emptyset$.

Fineness: $h := \max\{\mathrm{diam}(\omega_i) : i \in I\}$.

Finite Volume Scheme: Discretization Points

Associate a discretization point $x_i \in \overline{\omega}_i$ with each control volume ω_i (discrete unknown $\theta_{\nu,i}$ can be interpreted as $\theta_\nu(x_i)$). Further regularity assumptions can be expressed in terms of the x_i:

(DA-3) For each $m \in \{\mathrm{s}, \mathrm{g}\}$, $i \in I_m$, the set $\overline{\omega}_{m,i}$ is star-shaped with respect to the discretization point x_i, i.e., for each $x \in \overline{\omega}_{m,i}$, the line segment $\mathrm{conv}\{x, x_i\}$ lies entirely in $\overline{\omega}_{m,i}$. In particular, if $\omega_{\mathrm{s},i} \neq \emptyset$ and $\omega_{\mathrm{g},i} \neq \emptyset$, then $x_i \in \overline{\omega}_{\mathrm{s},i} \cap \overline{\omega}_{\mathrm{g},i}$.

(DA-4) For each $i \in I$: If $\lambda_2(\overline{\omega}_i \cap \Gamma_\Omega) \neq 0$, then $x_i \in \overline{\omega}_i \cap \Gamma_\Omega$; and, if $\lambda_2\big(\overline{\omega}_i \cap (\partial\Omega \setminus \Gamma_\Omega)\big) \neq 0$, then $x_i \in \overline{\omega}_i \cap \overline{\partial\Omega \setminus \Gamma_\Omega}$.

$$\mathrm{nb}_m(i) := \{j \in I_m \setminus \{i\} : \lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j}) \neq 0\},$$
$$\mathrm{nb}(i) := \{j \in I \setminus \{i\} : \lambda_2(\partial\omega_i \cap \partial\omega_j) \neq 0\}.$$

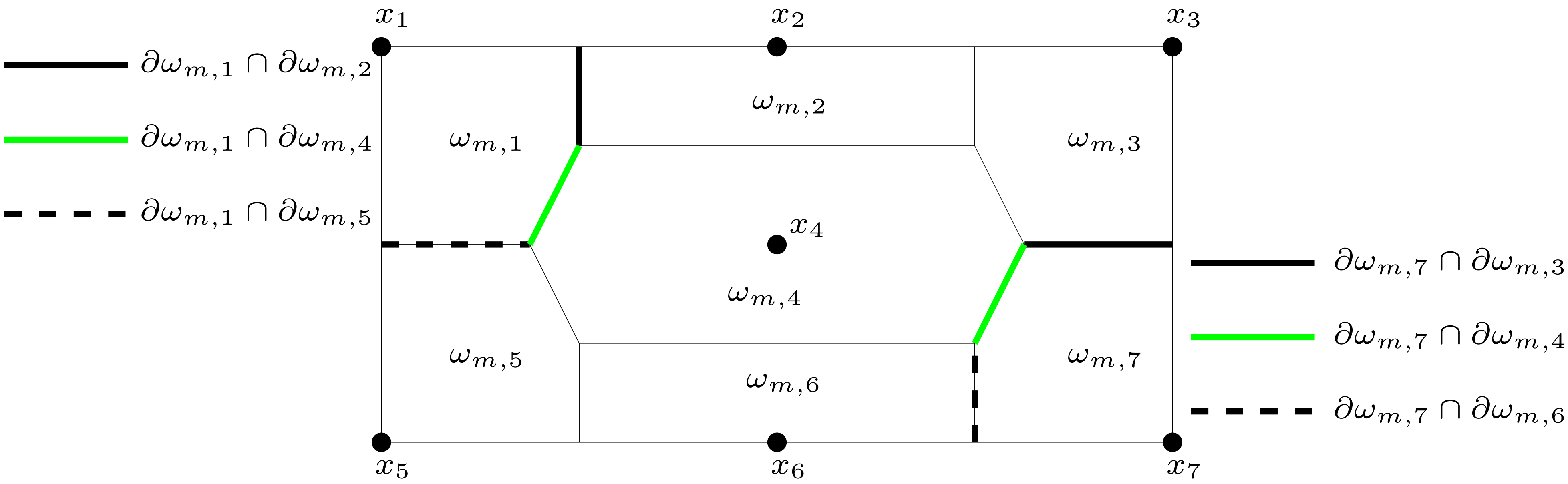


Figure 2: Illustration of conditions (DA-4) (with $\Gamma_\Omega = \emptyset$) and (DA-5) as well as of the partition of $\partial\omega_{m,i} \cap \Omega_m$ according to (15). One has $\mathrm{nb}_m(1) = \{2, 4, 5\}$ and $\mathrm{nb}_m(7) = \{3, 4, 6\}$.

Finite Volume Scheme: Neighbors (2)

(DA-5) For each $i \in I$, $j \in \mathrm{nb}(i)$: $x_i \neq x_j$ and $\frac{x_j - x_i}{\|x_i - x_j\|_2} = \mathbf{n}_{\omega_i}\restriction_{\partial\omega_i \cap \partial\omega_j}$, where $\|\cdot\|_2$ denotes Euclidian distance, and $\mathbf{n}_{\omega_i}\restriction_{\partial\omega_i \cap \partial\omega_j}$ is the restriction of the normal vector $\mathbf{n}_{\omega_i}$ to the interface $\partial\omega_i \cap \partial\omega_j$. Thus, the line segment joining neighboring vertices x_i and x_j is always perpendicular to $\partial\omega_i \cap \partial\omega_j$.

Decomposing the boundary of control volumes $\omega_{m,i}$:

$$\partial\omega_{m,i} = \big(\partial\omega_{m,i} \cap \Omega_m\big) \;\cup\; \big(\partial\omega_{m,i} \cap \partial\Omega\big) \;\cup\; \big(\partial\omega_{m,i} \cap \Sigma\big),$$

$$\partial\omega_{m,i} \cap \Omega_m = \bigcup_{j \in \mathrm{nb}_m(i)} \partial\omega_{m,i} \cap \partial\omega_{m,j}.$$

Finite Volume Scheme: Discretizing the Heat Equation

Integrating (1) over $[t_{\nu-1}, t_\nu] \times \omega_{m,i}$, applying the Gauss-Green integration theorem, and using implicit time discretization yields

$$k_\nu^{-1} \int_{\omega_{m,i}} \big(\varepsilon_m(\theta_\nu) - \varepsilon_m(\theta_{\nu-1})\big) - \int_{\partial\omega_{m,i}} \kappa_m \nabla \theta_\nu \cdot \mathbf{n}_{\omega_{m,i}} = k_\nu^{-1} \int_{t_{\nu-1}}^{t_\nu} \int_{\omega_{m,i}} f_m, \quad (13)$$

where $\theta_\nu := \theta(t_\nu, \cdot)$, and $\mathbf{n}_{\omega_{m,i}}$ denotes the outer unit normal vector to $\omega_{m,i}$.

Approximating integrals by quadrature formulas, e.g.

$$\int_{\omega_{m,i}} \big(\varepsilon_m(\theta_\nu) - \varepsilon_m(\theta_{\nu-1})\big) \approx \big(\varepsilon_m(\theta_{\nu,i}) - \varepsilon_m(\theta_{\nu-1,i})\big)\, \lambda_3(\omega_{m,i}).$$

Decompositions of $\partial\omega_{m,i}$, interface and boundary conditions are used on the $\nabla \theta_\nu$ term.

$\theta_{\nu,i}$ becomes the unknown $u_{\nu,i}$ in the scheme.

Finite Volume Scheme: First Glimpse at the Actual Scheme

One is seeking a nonnegative solution $(\mathbf{u}_0, \dots, \mathbf{u}_N)$, $\mathbf{u}_\nu = (u_{\nu,i})_{i \in I}$, to

$$u_{0,i} = \theta_{\mathrm{init},i} \qquad (i \in I), \tag{14a}$$

$$\mathcal{H}_{\nu,i}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu) = 0 \qquad (i \in I, \quad \nu \in \{1, \dots, n\}), \tag{14b}$$

$$\mathcal{H}_{\nu,i} : (\mathbb{R}_0^+)^I \times (\mathbb{R}_0^+)^I \longrightarrow \mathbb{R},$$

$$\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}) = k_\nu^{-1} \sum_{m \in \{\mathrm{s},\mathrm{g}\}} \big(\varepsilon_m(u_i) - \varepsilon_m(\tilde{u}_i)\big)\, \lambda_3(\omega_{m,i}) \tag{15a}$$

$$- \sum_{m \in \{\mathrm{s},\mathrm{g}\}} \kappa_m \sum_{j \in \mathrm{nb}_m(i)} \frac{u_j - u_i}{\|x_i - x_j\|_2}\, \lambda_2\big(\partial\omega_{m,i} \cap \partial\omega_{m,j}\big) \tag{15b}$$

$$+ \sigma\, \epsilon(\tilde{u}_i)\, u_i^4\, \lambda_2\big(\partial\omega_{\mathrm{s},i} \cap \Gamma_\Omega\big) - \sum_{\alpha \in J_{\Omega,i}} \epsilon(u_{\nu-1,i}) \int_{\zeta_\alpha} K_\Gamma(R_\Gamma(\theta_{\nu-1}, \theta_\nu)) \tag{15c}$$

$$+ \sigma\, \epsilon(\tilde{u}_i)\, (u_i^4 - \theta_{\mathrm{ext}}^4)\, \lambda_2\big(\partial\omega_{\mathrm{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)\big) \tag{15d}$$

$$+ \sigma\, \epsilon(\tilde{u}_i)\, u_i^4\, \lambda_2\big(\omega_i \cap \Sigma\big) - \sum_{\alpha \in J_{\Sigma,i}} \epsilon(u_{\nu-1,i}) \int_{\zeta_\alpha} K(R(\theta_{\nu-1}, \theta_\nu)) \tag{15e}$$

$$- \sum_{m \in \{\mathrm{s},\mathrm{g}\}} f_{m,\nu,i}\, \lambda_3(\omega_{m,i}). \tag{15f}$$

Finite Volume Scheme: Approximation of Source Term and Initial Condition

(AA-1) For each $m \in \{\mathrm{s}, \mathrm{g}\}$, $\nu \in \{0, \dots, N\}$, and $i \in I$,

$$f_{m,\nu,i} \approx \frac{\int_{t_{\nu-1}}^{t_\nu} \int_{\omega_{m,i}} f_m}{k_\nu \, \lambda_3(\omega_{m,i})}$$

is a suitable approximation of the source term, satisfying

$$\left(f_{m,\nu,i} \, k_\nu \, \lambda_3(\omega_{m,i}) - \int_{t_{\nu-1}}^{t_\nu} \int_{\omega_{m,i}} f_m \right) \to 0 \quad \text{for} \quad k_\nu \ \mathrm{diam}(\omega_{m,i}) \to 0.$$

(AA-2) For each $i \in I$,

$$\theta_{\mathrm{init},i} \approx \frac{\int_{\omega_i} \theta_{\mathrm{init}}}{\lambda_3(\omega_i)}$$

is a suitable approximation of the initial temperature distribution, satisfying

$$\left(\theta_{\mathrm{init},i} \, \lambda_3(\omega_i) - \int_{\omega_i} \theta_{\mathrm{init}} \right) \to 0 \quad \text{for} \quad \mathrm{diam}(\omega_i) \to 0.$$

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (1)

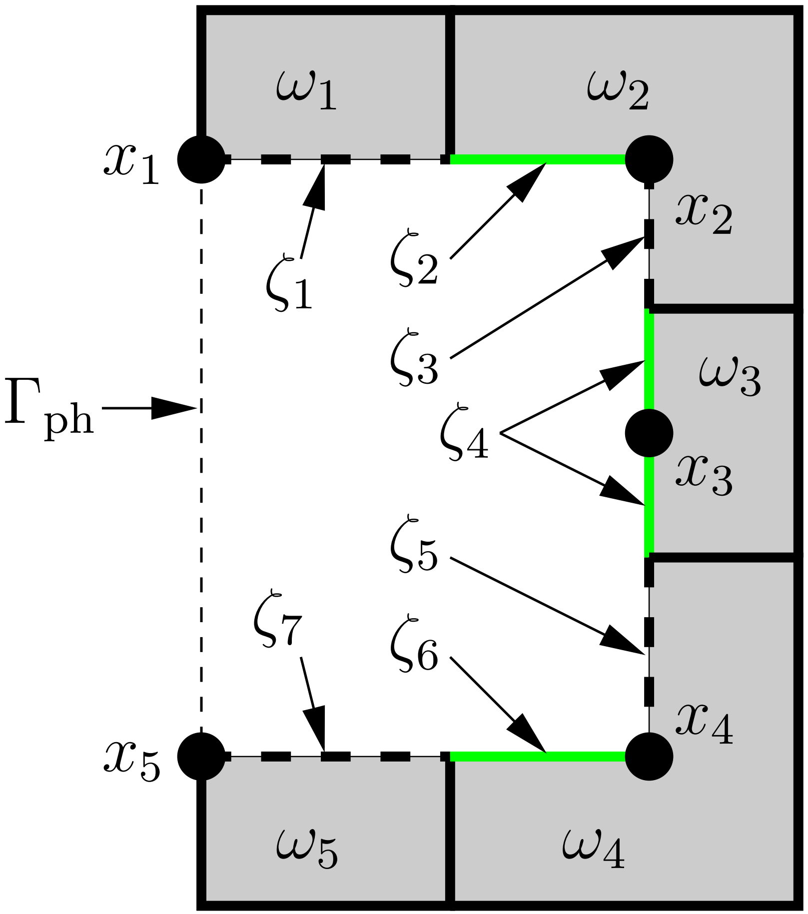
(DA-6) $(\zeta_\alpha)_{\alpha\in I_\Omega}$ and $(\zeta_\alpha)_{\alpha\in I_\Sigma}$ are finite partitions of Γ_Ω and Σ, respectively, where $I_\Omega \cap I_\Sigma = \emptyset$ and, for each $\alpha \in I_\Omega$ (resp. $\alpha \in I_\Sigma$), the boundary element ζ_α is a nonvoid, polyhedral, connected, and (relatively) open subset of Γ_Ω (resp. Σ), lying in a 2-dimensional affine subspace of $\mathbb{R}^3$.

On both Γ_Ω and Σ, the boundary elements are supposed to be compatible with the control volumes ω_i:

(DA-7) For each $\alpha \in I_\Omega$ (resp. $\alpha \in I_\Sigma$), there is a unique $i(\alpha) \in I$ such that $\zeta_\alpha \subseteq \partial\omega_{i(\alpha)} \cap \Gamma_\Omega$ (resp. $\zeta_\alpha \subseteq \partial\omega_{\mathrm{s},i(\alpha)} \cap \Gamma_\Sigma$). Moreover, for each $\alpha \in I_\Omega \,\dot\cup\, I_\Sigma$: $x_{i(\alpha)} \in \overline{\zeta}_\alpha$.

Definition and Remark 4. For each $i \in I$, define $J_{\Omega,i} := \{\alpha \in I_\Omega :\ \lambda_2(\zeta_\alpha \cap \partial\omega_i) \neq 0\}$ and $J_{\Sigma,i} := \{\alpha \in I_\Sigma :\ \lambda_2(\zeta_\alpha \cap \partial\omega_{\mathrm{s},i}) \neq 0\}$. It then follows from (DA-1), (DA-6), and (DA-7), that $(\zeta_\alpha \cap \partial\omega_i)_{\alpha\in J_{\Omega,i}}$ is a partition of $\partial\omega_i \cap \Gamma_\Omega = \partial\omega_{\mathrm{s},i} \cap \Gamma_\Omega$ and that $(\zeta_\alpha \cap \partial\omega_{\mathrm{s},i})_{\alpha\in J_{\Sigma,i}}$ is a partition of $\partial\omega_{\mathrm{s},i} \cap \Sigma = \overline{\omega}_i \cap \Sigma$. Moreover, (A-5) implies that at most one of the two sets $J_{\Omega,i}$, $J_{\Sigma,i}$ can be nonvoid.

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (2)



$$i(1) = 1,\ i(2) = i(3) = 2,$$
$$i(4) = 3,\ i(5) = i(6) = 4,\ i(7) = 5$$

$$J_{\Omega,1} = \{1\},\ J_{\Omega,2} = \{2,3\},$$
$$J_{\Omega,3} = \{4\},\ J_{\Omega,4} = \{5,6\},\ J_{\Omega,5} = \{7\}$$

Figure 3: Magnification of the open radiation region O_1 and of the adjacent part of Ω_{s}. It illustrates the partitioning of Γ_Ω into the ζ_α. In particular, it illustrates the compatibility condition (DA-7) as well as Def. and Rem. 4.

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (3)

Goal: Discretize $\int_{\zeta_\alpha} K(R(\theta_{\nu-1}, \theta_\nu))$.

$R(\theta_{\nu-1}, \theta_\nu)$ is approximated by a constant value $R_{\Sigma,\alpha}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu)$ on each boundary element ζ_α, $\alpha \in I_\Sigma$, where $\mathbf{u}_{\nu-1} := (\theta_{\nu-1,i(\beta)})_{\beta \in I_\Sigma}$, $\mathbf{u}_\nu := (\theta_{\nu,i(\beta)})_{\beta \in I_\Sigma}$.

Recalling $K(R)(x) = \int_\Sigma \Lambda(x,y)\,\omega(x,y)\,R(y)\,\mathrm{d}y$, one has

$$\int_{\zeta_\alpha} K(R(\theta_{\nu-1}, \theta_\nu)) \approx \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu)\,\Lambda_{\alpha,\beta} \qquad (\alpha \in I_\Sigma), \tag{16}$$

where

$$\Lambda_{\alpha,\beta} := \int_{\zeta_\alpha \times \zeta_\beta} \Lambda\,\omega, \qquad \Lambda_{\alpha,\beta} = \Lambda_{\beta,\alpha} \geq 0, \qquad \sum_{\beta \in I_\Sigma} \Lambda_{\alpha,\beta} = \lambda_2(\zeta_\alpha). \tag{17}$$

Using (16) allows to write (7) in the integrated and discretized form

$$\begin{aligned} &R_{\Sigma,\alpha}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu)\,\lambda_2(\zeta_\alpha) - \big(1 - \epsilon(\theta_{\nu-1,i(\alpha)})\big) \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu)\,\Lambda_{\alpha,\beta} \\ &= \sigma\,\epsilon(\theta_{\nu-1,i(\alpha)})\,\theta^4_{\nu,i(\alpha)}\,\lambda_2(\zeta_\alpha) \qquad (\alpha \in I_\Sigma). \end{aligned} \tag{18}$$

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (4)

$$R_{\Sigma,\alpha}(\mathbf{u}_{\nu-1},\mathbf{u}_\nu)\,\lambda_2(\zeta_\alpha) - \big(1-\epsilon(\theta_{\nu-1,i(\alpha)})\big)\sum_{\beta\in I_\Sigma} R_{\Sigma,\beta}(\mathbf{u}_{\nu-1},\mathbf{u}_\nu)\,\Lambda_{\alpha,\beta}$$
$$= \sigma\,\epsilon(\theta_{\nu-1,i(\alpha)})\,\theta^4_{\nu,i(\alpha)}\,\lambda_2(\zeta_\alpha) \qquad (\alpha\in I_\Sigma),$$

can be written in matrix form:

$$\mathbf{G}_\Sigma(\mathbf{u}_{\nu-1})\,\mathbf{R}_\Sigma(\mathbf{u}_{\nu-1},\mathbf{u}_\nu) = \mathbf{E}_\Sigma(\mathbf{u}_{\nu-1},\mathbf{u}_\nu), \tag{19}$$

$$\mathbf{R}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma}\times(\mathbb{R}_0^+)^{I_\Sigma}\longrightarrow(\mathbb{R}_0^+)^{I_\Sigma}, \qquad \mathbf{R}_\Sigma(\tilde{\mathbf{u}},\mathbf{u}) = \big(R_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{u})\big)_{\alpha\in I_\Sigma},$$
$$\mathbf{E}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma}\times(\mathbb{R}_0^+)^{I_\Sigma}\longrightarrow(\mathbb{R}_0^+)^{I_\Sigma}, \qquad \mathbf{E}_\Sigma(\tilde{\mathbf{u}},\mathbf{u}) = \big(E_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{u})\big)_{\alpha\in I_\Sigma},$$
$$E_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{u}) := \sigma\,\epsilon(\tilde{u}_\alpha)\,u_\alpha^4\,\lambda_2(\zeta_\alpha),$$

$$\mathbf{G}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma}\longrightarrow\mathbb{R}^{I_\Sigma^2}, \qquad \mathbf{G}_\Sigma(\tilde{\mathbf{u}}) = \big(G_{\Sigma,\alpha,\beta}(\tilde{\mathbf{u}})\big)_{(\alpha,\beta)\in I_\Sigma^2},$$
$$G_{\Sigma,\alpha,\beta}(\tilde{\mathbf{u}}) := \begin{cases} \lambda_2(\zeta_\alpha) - \big(1-\epsilon(\tilde{u}_\alpha)\big)\,\Lambda_{\alpha,\beta} & \text{for } \alpha=\beta,\\ -\big(1-\epsilon(\tilde{u}_\alpha)\big)\,\Lambda_{\alpha,\beta} & \text{for } \alpha\neq\beta. \end{cases}$$

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (5): Invertibility

Lemma 5. *The following holds for each* $\mathbf{u} \in (\mathbb{R}_0^+)^{I_\Sigma}$*:*

(a) *For each* $\alpha \in I_\Sigma$*:* $\sum_{\beta \in I_\Sigma \setminus \{\alpha\}} |G_{\Sigma,\alpha,\beta}(\mathbf{u})| \leq (1 - \epsilon(u_\alpha))\, G_{\Sigma,\alpha,\alpha}(\mathbf{u}) < G_{\Sigma,\alpha,\alpha}(\mathbf{u})$. *In particular,* $\mathbf{G}_\Sigma(\mathbf{u})$ *is strictly diagonally dominant.*

(b) $\mathbf{G}_\Sigma(\mathbf{u})$ *is an M-matrix, i.e.* $\mathbf{G}_\Sigma(\mathbf{u})$ *is invertible,* $\mathbf{G}_\Sigma^{-1}(\mathbf{u})$ *is nonnegative, and* $G_{\Sigma,\alpha,\beta}(\mathbf{u}) \leq 0$ *for each* $(\alpha, \beta) \in I_\Sigma^2$, $\alpha \neq \beta$.

Proof. (a): Combining the definition of $\mathbf{G}_\Sigma$ with $\sum_{\beta \in I_\Sigma} \Lambda_{\alpha,\beta} = \lambda_2(\zeta_\alpha)$ yields

$$\begin{aligned} \sum_{\beta \in I_\Sigma \setminus \{\alpha\}} |G_{\Sigma,\alpha,\beta}(\mathbf{u})| &= \sum_{\beta \in I_\Sigma \setminus \{\alpha\}} \big(1 - \epsilon(u_\alpha)\big)\, \Lambda_{\alpha,\beta} \\ &= \big(1 - \epsilon(u_\alpha)\big) \big(\lambda_2(\zeta_\alpha) - \Lambda_{\alpha,\alpha}\big) \qquad (\alpha \in I_\Sigma), \end{aligned}$$

proving (a) since $\epsilon > 0$.

(b): $\Lambda_{\alpha,\beta} \geq 0$ implies $G_{\alpha,\beta}(\mathbf{u}) \leq 0$ for $\alpha \neq \beta$, whereas $G_{\alpha,\alpha}(\mathbf{u}) > 0$ by (a). Since $\mathbf{G}(\mathbf{u})$ is also strictly diagonally dominant, $\mathbf{G}(\mathbf{u})$ is an M-matrix (see Lem. 6.2 of Axelsson 1994: Iterative solution methods, Cambridge University Press). □

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (6)

Now, Lemma 5(b) allows to define $\mathbf{R}_\Sigma$ by

$$\mathbf{R}_\Sigma(\tilde{\mathbf{u}}, \mathbf{u}) := \mathbf{G}_\Sigma^{-1}(\tilde{\mathbf{u}})\, \mathbf{E}_\Sigma(\tilde{\mathbf{u}}, \mathbf{u}), \tag{20}$$

$$\mathbf{V}_\Sigma : (\mathbb{R}_0^+)^{I_\Sigma} \times (\mathbb{R}_0^+)^{I_\Sigma} \longrightarrow (\mathbb{R}_0^+)^{I_\Omega}, \qquad \mathbf{V}_\Sigma(\tilde{\mathbf{u}}, \mathbf{u}) = \big(V_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u})\big)_{\alpha \in I_\Sigma},$$

$$V_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u}) := \epsilon(\tilde{u}_\alpha) \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\tilde{\mathbf{u}}, \mathbf{u})\, \Lambda_{\alpha,\beta},$$

giving a precise meaning to the approximation (16) of $\int_{\zeta_\alpha} K(R(\theta_{\nu-1}, \theta_\nu))$:

$$\epsilon(\theta_{\nu-1}) \int_{\zeta_\alpha} K(R(\theta_{\nu-1}, \theta_\nu)) \approx \epsilon(\theta_{\nu-1,i(\alpha)}) \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu)\, \Lambda_{\alpha,\beta} = V_{\Sigma,\alpha}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu).$$

Note: $\mathbf{R}_\Sigma \geq 0$ and $\mathbf{V}_\Sigma \geq 0$ since $\mathbf{E}_\Sigma \geq 0$ and $\mathbf{G}_\Sigma^{-1} \geq 0$.

Finite Volume Scheme: Final Formulation

For $\mathbf{u} = (u_i)_{i \in I}$, let $\mathbf{u}\restriction_{I_\Omega} := (u_{i(\alpha)})_{\alpha \in I_\Omega}$, $\mathbf{u}\restriction_{I_\Sigma} := (u_{i(\alpha)})_{\alpha \in I_\Sigma}$.
Seek nonnegative $(\mathbf{u}_0, \dots, \mathbf{u}_N)$, $\mathbf{u}_\nu = (u_{\nu,i})_{i \in I}$, such that

$$u_{0,i} = \theta_{\mathrm{init},i}, \qquad \mathcal{H}_{\nu,i}(\mathbf{u}_{\nu-1}, \mathbf{u}_\nu) = 0 \qquad (i \in I, \quad \nu \in \{1, \dots, n\}), \tag{21}$$

$$\mathcal{H}_{\nu,i} : (\mathbb{R}_0^+)^I \times (\mathbb{R}_0^+)^I \longrightarrow \mathbb{R},$$

$$\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}) = k_\nu^{-1} \sum_{m \in \{\mathrm{s},\mathrm{g}\}} \big(\varepsilon_m(u_i) - \varepsilon_m(\tilde{u}_i)\big)\, \lambda_3(\omega_{m,i}) \tag{22a}$$

$$- \sum_{m \in \{\mathrm{s},\mathrm{g}\}} \kappa_m \sum_{j \in \mathrm{nb}_m(i)} \frac{u_j - u_i}{\|x_i - x_j\|_2}\, \lambda_2\big(\partial\omega_{m,i} \cap \partial\omega_{m,j}\big) \tag{22b}$$

$$+ \sigma\, \epsilon(\tilde{u}_i)\, u_i^4\, \lambda_2\big(\partial\omega_{\mathrm{s},i} \cap \Gamma_\Omega\big) - \sum_{\alpha \in J_{\Omega,i}} V_{\Gamma,\alpha}(\tilde{\mathbf{u}}\restriction_{I_\Omega}, \mathbf{u}\restriction_{I_\Omega}) \tag{22c}$$

$$+ \sigma\, \epsilon(\tilde{u}_i)\, (u_i^4 - \theta_{\mathrm{ext}}^4)\, \lambda_2\big(\partial\omega_{\mathrm{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)\big) \tag{22d}$$

$$+\sigma\, \epsilon(\tilde{u}_i)\, u_i^4\, \lambda_2\big(\omega_i \cap \Sigma\big) - \sum_{\alpha \in J_{\Sigma,i}} V_{\Sigma,\alpha}(\tilde{\mathbf{u}}\restriction_{I_\Sigma}, \mathbf{u}\restriction_{I_\Sigma}) \tag{22e}$$

$$- \sum_{m \in \{\mathrm{s},\mathrm{g}\}} f_{m,\nu,i}\, \lambda_3(\omega_{m,i}). \tag{22f}$$

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (7): Maximum Principle

$$\min(\mathbf{u}) := \min\{u_i : i \in I\}, \quad \max(\mathbf{u}) := \max\{u_i : i \in I\}.$$

Lemma 6. *For each* $(\tilde{\mathbf{u}}, \mathbf{u}) \in (\mathbb{R}_0^+)^{I_\Sigma} \times (\mathbb{R}_0^+)^{I_\Sigma}$, $\alpha \in I_\Sigma$*:*

$$\sigma \min(\mathbf{u})^4 \leq R_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u}) \leq \sigma \max(\mathbf{u})^4, \tag{23a}$$

$$\sigma\,\epsilon(\tilde{u}_\alpha) \min(\mathbf{u})^4\; \lambda_2(\zeta_\alpha) \leq V_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u}) \leq \sigma\,\epsilon(\tilde{u}_\alpha) \max(\mathbf{u})^4\; \lambda_2(\zeta_\alpha). \tag{23b}$$

Proof. Since $\mathbf{R}_\Sigma(\tilde{\mathbf{u}}, \mathbf{u})$ satisfies (18), one has, for each $\alpha \in I_\Sigma$,

$$R_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u})\,\lambda_2(\zeta_\alpha) - \big(1 - \epsilon(\tilde{u}_\alpha)\big) \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\tilde{\mathbf{u}}, \mathbf{u})\,\Lambda_{\alpha,\beta} \leq \sigma \max(\mathbf{u})^4\; \epsilon(\tilde{u}_\alpha)\,\lambda_2(\zeta_\alpha)$$

$$= \sigma \max(\mathbf{u})^4 \left(\lambda_2(\zeta_\alpha) - \big(1 - \epsilon(\tilde{u}_\alpha)\big) \sum_{\beta \in I_\Sigma} \Lambda_{\alpha,\beta}\right),$$

i.e. $\mathbf{G}_\Sigma(\tilde{\mathbf{u}})\,\mathbf{R}_\Sigma(\tilde{\mathbf{u}}, \mathbf{u}) \leq \mathbf{G}_\Sigma(\tilde{\mathbf{u}})\,\mathbf{U}_{\max}$,
where $\mathbf{U}_{\max} = (U_{\max,\alpha})_{\alpha \in I_\Sigma}$, $\quad U_{\max,\alpha} := \sigma \max(\mathbf{u})^4$,
implying $\mathbf{R}_\Sigma(\tilde{\mathbf{u}}, \mathbf{u}) \leq \mathbf{U}_{\max}$, as $\mathbf{G}_\Sigma^{-1}(\tilde{\mathbf{u}}) \geq 0$. Thus, $R_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u}) \leq \sigma \max(\mathbf{u})^4$.
Likewise, one shows $R_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u}) \geq \sigma \min(\mathbf{u})^4$, proving (23a). Now (23b) follows from
$V_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \mathbf{u}) := \epsilon(\tilde{u}_\alpha) \sum_{\beta \in I_\Sigma} R_{\Sigma,\beta}(\tilde{\mathbf{u}}, \mathbf{u})\,\Lambda_{\alpha,\beta}$ and $\sum_{\beta \in I_\Sigma} \Lambda_{\alpha,\beta} = \lambda_2(\zeta_\alpha)$. □

Finite Volume Scheme: Discretization of Nonlocal Radiation Terms (8): Lipschitz

Lemma 7. *For each* $r \in \mathbb{R}^+$ *and* $\tilde{\mathbf{u}} \in (\mathbb{R}_0^+)^{I_\Sigma}$, *with respect to the max-norm, the map* $V_{\Sigma,\alpha}(\tilde{\mathbf{u}}, \cdot)$ *is* $\big(4\,\sigma\,\epsilon(\tilde{u}_\alpha)\,\lambda_2(\zeta_\alpha)\,r^3\big)$*-Lipschitz on* $[0,r]^{I_\Sigma}$.

Proof. $\theta \mapsto \lambda\,\theta^4$ is $(4\lambda r^3)$-Lipschitz on $[0,r]$; $\mathbf{R}_\Sigma(\tilde{\mathbf{u}}, \cdot)$ is $(4\,\sigma\,r^3)$-Lipschitz on $[0,r]^{I_\Sigma}$:

$$\Big|\big(R_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{u}) - R_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{v})\big)\,\lambda_2(\zeta_\alpha) - \big(1-\epsilon(\tilde{u}_\alpha)\big)\sum_{\beta\in I_\Sigma}\big(R_{\Sigma,\beta}(\tilde{\mathbf{u}},\mathbf{u}) - R_{\Sigma,\beta}(\tilde{\mathbf{u}},\mathbf{v})\big)\,\Lambda_{\alpha,\beta}\Big|$$

$$= \sigma\,\epsilon(\tilde{u}_\alpha)\,\big|u_\alpha^4 - v_\alpha^4\big|\,\lambda_2(\zeta_\alpha) \le 4\,\sigma\,\epsilon(\tilde{u}_\alpha)\,|u_\alpha - v_\alpha|\,\lambda_2(\zeta_\alpha)\,r^3. \tag{24}$$

Fix $\alpha \in I_\Sigma$ with $N_{\max} := \|\mathbf{R}_\Sigma(\tilde{\mathbf{u}},\mathbf{u}) - \mathbf{R}_\Sigma(\tilde{\mathbf{u}},\mathbf{v})\|_{\max} = |R_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{u}) - R_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{v})|$. Then

$$\begin{aligned}
&4\,\sigma\,\epsilon(\tilde{u}_\alpha)\,\|\mathbf{u}-\mathbf{v}\|_{\max}\,\lambda_2(\zeta_\alpha)\,r^3 \\
&\overset{(24)}{\ge} \Big|N_{\max}\,\lambda_2(\zeta_\alpha) - \big(1-\epsilon(\tilde{u}_\alpha)\big)\Big|\sum_{\beta\in I_\Sigma}\big(R_{\Sigma,\beta}(\tilde{\mathbf{u}},\mathbf{u}) - R_{\Sigma,\beta}(\tilde{\mathbf{u}},\mathbf{v})\big)\,\Lambda_{\alpha,\beta}\Big|\Big| \\
&\overset{\text{Lem. 5(a)}}{\ge} N_{\max}\,\lambda_2(\zeta_\alpha) - \big(1-\epsilon(\tilde{u}_\alpha)\big)\Big|\sum_{\beta\in I_\Sigma}\big(R_{\Sigma,\beta}(\tilde{\mathbf{u}},\mathbf{u}) - R_{\Sigma,\beta}(\tilde{\mathbf{u}},\mathbf{v})\big)\,\Lambda_{\alpha,\beta}\Big| \\
&\ge N_{\max}\Big(\lambda_2(\zeta_\alpha) - \big(1-\epsilon(\tilde{u}_\alpha)\big)\sum_{\beta\in I_\Sigma}\Lambda_{\alpha,\beta}\Big) \ge N_{\max}\,\epsilon(\tilde{u}_\alpha)\,\lambda_2(\zeta_\alpha). \quad \square
\end{aligned}$$

Outline

Part 1:

- The Model: Domains, mathematical assumptions, transient nonlinear heat equations, nonlocal interface and boundary conditions
- Formulation of Finite Volume Scheme: Focus on discretization of nonlocal radiation terms, maximum principle
- Discrete Maximum Principle, Discrete Existence and Uniqueness

Part 2:

- Piecewise Constant Interpolation, Existence of Convergent Subsequence
  - A Priori Estimates I: Discrete norms, discrete continuity in time
  - Riesz-Fréchet-Kolmogorov Compactness Theorem, A Priori Estimates II: Space and time translate estimates
- Weak Solution: Formulation, discrete analogue, convergence of the discrete analogue to a weak solution

Discrete Maximum Principle, Discrete Existence (Local in Time)

Theorem 8. *Assume* (A-1) – (A-7), (DA-1) – (DA-7), (AA-1) *and* (AA-2). *Moreover, assume* $\nu \in \{1, \ldots, N\}$ *and* $\tilde{\mathbf{u}} = (\tilde{u}_i)_{i \in I} \in \left(\mathbb{R}_0^+\right)^I$. *Let*

$$B_{f,\nu} := \max\left\{ \sum_{m \in \{\mathrm{s},\mathrm{g}\}} f_{m,\nu,i} \, \frac{\lambda_3(\omega_{m,i})}{\lambda_3(\omega_i)} : i \in I \right\}, \tag{25a}$$

$$L_{\mathbf{V}} := 4\,\sigma \,\max\left\{ \frac{\lambda_2(\omega_i \cap \Sigma)}{\lambda_3(\omega_i)} + \sum_{\alpha \in J_{\Omega,i}} \frac{\lambda_2(\zeta_\alpha) - \Lambda_{\alpha,\mathrm{ph}}}{\lambda_3(\omega_i)} : i \in I \right\}, \tag{25b}$$

$$m(\tilde{\mathbf{u}}) := \min\left\{ \theta_{\mathrm{ext}}, \, \min(\tilde{\mathbf{u}}) \right\}, \quad M_\nu(\tilde{\mathbf{u}}) := \max\left\{ \theta_{\mathrm{ext}}, \, \max(\tilde{\mathbf{u}}) + \frac{k_\nu}{C_\varepsilon} B_{f,\nu} \right\}. \tag{25c}$$

Then each solution $\mathbf{u}_\nu = (u_{\nu,i})_{i \in I} \in \left(\mathbb{R}_0^+\right)^I$ *to*

$$\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}_\nu) = 0 \tag{26}$$

must lie in $[m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$. *Furthermore, if* k_ν *is such that*

$$k_\nu \left(M_\nu(\tilde{\mathbf{u}})^3 - m(\tilde{\mathbf{u}})^3 \right) L_{\mathbf{V}} < C_\varepsilon, \tag{27}$$

then there is a unique $\mathbf{u}_\nu \in [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$ *satisfying* (26).

Discrete Maximum Principle $(\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}_\nu) = 0) \Rightarrow \mathbf{u}_\nu \in [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$

Sketch of Proof: Proof abstract max principle for class of nonlinear systems.

Lemma 9. *Consider a continuous operator* $\mathcal{H} : [a,b]^I \longrightarrow \mathbb{R}^I, \quad \mathcal{H}(\mathbf{u}) = \big(\mathcal{H}_i(\mathbf{u})\big)_{i\in I}$.
Assume there are continuous functions
$b_i \in C([a,b],\mathbb{R})$, $\tilde{h}_i \in C([a,b],\mathbb{R})$, $\tilde{g}_i \in C([a,b]^I,\mathbb{R})$, $i \in I$, *satisfying*

(i) *There is* $\tilde{\mathbf{u}} \in [a,b]^I$ *such that, for each* $i \in I$, $\mathbf{u} \in [a,b]^I$*:*

$$\mathcal{H}_i(\mathbf{u}) = b_i(u_i) + \tilde{h}_i(u_i) - b_i(\tilde{u}_i) - \tilde{g}_i(\mathbf{u}).$$

(ii) *There are* $\theta_{\text{ext}} \in [a,b]$ *and families of numbers* $\beta_i \leq 0$, $B_i \geq 0$, *such that, for each* $i \in I$, $\mathbf{u} \in [a,b]^I$, $\theta \in [a,b]$*:*

$$\max\left\{\max(\mathbf{u}),\, \theta_{\text{ext}}\right\} \leq \theta \quad \Rightarrow \quad \tilde{g}_i(\mathbf{u}) - \tilde{h}_i(\theta) \leq B_i, \tag{28a}$$

$$\theta \leq \min\left\{\theta_{\text{ext}},\, \min(\mathbf{u})\right\} \quad \Rightarrow \quad \tilde{g}_i(\mathbf{u}) - \tilde{h}_i(\theta) \geq \beta_i. \tag{28b}$$

(iii) *There is a family of numbers* $C_{b,i} > 0$ *such that, for each* $i \in I$ *and* $\theta_1, \theta_2 \in \tau$*:*
$\theta_2 \geq \theta_1 \quad \Rightarrow \quad b_i(\theta_2) \geq (\theta_2 - \theta_1)\, C_{b,i} + b_i(\theta_1)$.

Letting $\beta := \min\{\beta_i/C_{b,i} : i \in I\}, \quad B := \max\{B_i/C_{b,i} : i \in I\}$,
$m(\tilde{\mathbf{u}}) := \min\left\{\theta_{\text{ext}},\, \min(\tilde{\mathbf{u}}) + \beta\right\}, \quad M(\tilde{\mathbf{u}}) := \max\left\{\theta_{\text{ext}},\, \max(\tilde{\mathbf{u}}) + B\right\}$ *one has:*

$\mathbf{u}_0 \in [a,b]^I$ *and* $\mathcal{H}(\mathbf{u}_0) = \mathbf{0} := (0,\ldots,0)$ *imply* $\mathbf{u}_0 \in [m(\tilde{\mathbf{u}}), M(\tilde{\mathbf{u}})]^I$.

Discrete Maximum Principle, Sketch of Proof: Choose Functions and Constants for Lemma (1)

$$b_{\nu,i} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \qquad b_{\nu,i}(\theta) := k_\nu^{-1} \sum_{m\in\{\mathrm{s},\mathrm{g}\}} \varepsilon_m(\theta)\, \lambda_3(\omega_{m,i}), \tag{29a}$$

$$L_{\kappa,i} := \sum_{m\in\{\mathrm{s},\mathrm{g}\}} \kappa_m \sum_{j\in \mathrm{nb}_m(i)} \frac{\lambda_2\big(\partial\omega_{m,i} \cap \partial\omega_{m,j}\big)}{\|x_i - x_j\|_2} \geq 0, \tag{29b}$$

$$\begin{aligned} C_{\mathbf{V},i}(\tilde{\mathbf{u}}) := {} & \sigma\, \epsilon(\tilde{u}_i)\, \lambda_2\big(\partial\omega_{\mathrm{s},i} \cap \Gamma_\Omega\big) \\ & + \sigma\, \epsilon(\tilde{u}_i)\, \lambda_2\big(\partial\omega_{\mathrm{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)\big) + \sigma\, \epsilon(\tilde{u}_i)\, \lambda_2(\omega_i \cap \Sigma) \geq 0, \end{aligned} \tag{29c}$$

$$\tilde{h}_i : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+, \qquad \tilde{h}_i(\theta) := \theta\, L_{\kappa,i} + \theta^4\, C_{\mathbf{V},i}(\tilde{\mathbf{u}}), \tag{29d}$$

Discrete Maximum Principle, Sketch of Proof: Choose Functions and Constants for Lemma (2)

$$\tilde{g}_{\nu,i} : (\mathbb{R}_0^+)^I \longrightarrow \mathbb{R}_0^+,$$

$$\tilde{g}_{\nu,i}(\mathbf{u}) := \sum_{m\in\{\mathrm{s},\mathrm{g}\}} \kappa_m \sum_{j\in \mathrm{nb}_m(i)} \frac{u_j}{\|x_i - x_j\|_2} \lambda_2(\partial\omega_{m,i} \cap \partial\omega_{m,j})$$
$$+ \sum_{\alpha\in J_{\Omega,i}} V_{\Gamma,\alpha}(\tilde{\mathbf{u}}\restriction_{I_\Omega}, \mathbf{u}\restriction_{I_\Omega}) + \sigma\,\epsilon(\tilde{u}_i)\,\theta_{\mathrm{ext}}^4\,\lambda_2\big(\partial\omega_{\mathrm{s},i} \cap (\partial\Omega \setminus \Gamma_\Omega)\big)$$
$$+ \sum_{\alpha\in J_{\Sigma,i}} V_{\Sigma,\alpha}(\tilde{\mathbf{u}}\restriction_{I_\Sigma}, \mathbf{u}\restriction_{I_\Sigma}) + \sum_{m\in\{\mathrm{s},\mathrm{g}\}} f_{m,\nu,i}\,\lambda_3(\omega_{m,i}), \qquad (30a)$$

$$\beta_i := 0, \qquad B_{\nu,i} := \sum_{m\in\{\mathrm{s},\mathrm{g}\}} f_{m,\nu,i}\,\lambda_3(\omega_{m,i}), \qquad C_{b,\nu,i} := k_\nu^{-1}\,C_\varepsilon\,\lambda_3(\omega_i) > 0. \qquad (30b)$$

Use $\sigma\,\epsilon(\tilde{u}_\alpha)\,\min(\mathbf{u})^4\,\lambda_2(\zeta_\alpha) \le V_{\Sigma,\alpha}(\tilde{\mathbf{u}},\mathbf{u}) \le \sigma\,\epsilon(\tilde{u}_\alpha)\,\max(\mathbf{u})^4\,\lambda_2(\zeta_\alpha)$ to prove estimates (28).

Discrete Maximum Principle, Sketch of Proof: Proof of Lemma

Consider $\mathbf{u} \in [a,b]^I$ satisfying $\max(\mathbf{u}) > M(\tilde{\mathbf{u}})$. Let $i \in I$ be such that $u_i = \max(\mathbf{u})$. Then, since $u_i > M(\tilde{\mathbf{u}}) \geq \theta_{\mathrm{ext}}$, (28a) applies with $\theta = u_i$, yielding

$$\tilde{g}_i(\mathbf{u}) - \tilde{h}_i(u_i) \leq B_i. \tag{31}$$

Moreover, since $u_i > M(\tilde{\mathbf{u}}) \geq \max(\tilde{\mathbf{u}}) + B \geq \tilde{u}_i$, one can apply (iii) with $\theta_2 = u_i$ and $\theta_1 = \tilde{u}_i$ to get

$$b_i(u_i) \geq (u_i - \tilde{u}_i)\, C_{b,i} + b_i(\tilde{u}_i). \tag{32}$$

Combining (31) and (32) with (i), we compute

$$\begin{aligned} \mathcal{H}_i(\mathbf{u}) &= b_i(u_i) - b_i(\tilde{u}_i) + \tilde{h}_i(u_i) - \tilde{g}_i(\mathbf{u}) \geq (u_i - \tilde{u}_i)\, C_{b,i} - B_i \\ &> (\tilde{u}_i + B - \tilde{u}_i)\, C_{b,i} - B_i \geq 0, \end{aligned}$$

i.e. $\mathbf{u}$ is not a root of $\mathcal{H}$. An analogous argument shows that, if $\mathbf{u} \in [a,b]^I$ and $\min(\mathbf{u}) < m(\tilde{\mathbf{u}})$, then $\mathbf{u}$ is not a root of $\mathcal{H}$, showing that each root of $\mathcal{H}$ must lie in $[m(\tilde{\mathbf{u}}), M(\tilde{\mathbf{u}})]^I$.

Discrete Existence: Basic Idea of the Proof

The hypothesis

$$k_\nu \left(M_\nu(\tilde{\mathbf{u}})^3 - m(\tilde{\mathbf{u}})^3\right) L_{\mathbf{V}} < C_\varepsilon,$$

allows the construction of a contracting map

$$f : [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I \mapsto [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$$

such that $\mathbf{u}_\nu \in [m(\tilde{\mathbf{u}}), M_\nu(\tilde{\mathbf{u}})]^I$ is a fixed point of f if, and only if,

$$\mathcal{H}_{\nu,i}(\tilde{\mathbf{u}}, \mathbf{u}_\nu) = 0.$$

Discrete Maximum Principle, Discrete Existence (Global in Time)

Theorem 10. *Assume* (A-1) – (A-7), (DA-1) – (DA-7), (AA-1) *and* (AA-2). *Let*

$$m := \min\left\{\theta_{\text{ext}},\ \operatorname{ess\,inf}(\theta_{\text{init}})\right\}, \tag{33}$$

$$M_\nu := \max\left\{\theta_{\text{ext}},\ \|\theta_{\text{init}}\|_{L^\infty(\Omega,\mathbb{R}_0^+)}\right\} + \frac{t_\nu}{C_\varepsilon} \sum_{m\in\{\text{s},\text{g}\}} \|f_m\|_{L^\infty(0,t_\nu,L^\infty(\Omega_m))} \tag{34}$$

for each $\nu \in \{0,\dots,N\}$.

If $(\mathbf{u}_0,\dots,\mathbf{u}_N) = (u_{\nu,i})_{(\nu,i)\in\{0,\dots,N\}\times I} \in (\mathbb{R}_0^+)^{I\times\{0,\dots,N\}}$ *is a solution to* $\mathcal{H}_{\nu,i}(\mathbf{u}_{\nu-1},\mathbf{u}_\nu) = 0$ *for each* $\nu \in \{1,\dots,N\}$, *then* $\mathbf{u}_\nu \in [m, M_\nu]^I$ *for each* $\nu \in \{0,\dots,N\}$. *Furthermore, if*

$$k_\nu\left(M_\nu^3 - m^3\right) L_{\mathbf{V}} < C_\varepsilon \qquad \left(\nu \in \{1,\dots,n\}\right), \tag{35}$$

where $L_{\mathbf{V}}$ *is defined according to* (25b), *then the finite volume scheme has a unique solution* $(\mathbf{u}_0,\dots,\mathbf{u}_N) \in (\mathbb{R}_0^+)^{I\times\{0,\dots,N\}}$. *A sufficient condition for* (35) *to be satisfied is*

$$\max\left\{k_\nu : \nu \in \{1,\dots,n\}\right\}\left(M_N^3 - m^3\right) L_{\mathbf{V}} < C_\varepsilon. \tag{36}$$

Proof: Induction on $n \in \{0,\dots,N\}$.

Thank You for Your Attention !