Remnants from the Bookshelf

cropped up again in jww Silvia Steila

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White spots within a basically known area

Γ_0

Bachmann-Howard ordinal

Π^1_1-CA + (BI)

Δ^1_2-CA + (BI)

Π^1_2-CA + (BI)

The general framework

The languages $\mathcal{L}_1$ and $\mathcal{L}^\star$

- $\mathcal{L}_1$: a standard language of first order arithmetic with a constant $\overline{m}$ for every natural number m and an n-ary relation symbol $R_{\mathcal{Z}}$ for every n-ary primitive recursive relation $\mathcal{Z}$; we write 0 for $\overline{0}$.
- $\mathcal{L}^\star \ := \ \mathcal{L}_1(N, S, Ad, \in)$
  - N a constant for the set of the natural numbers,
  - S a unary relation symbol to say that an object is a set.
  - Ad a unary relation symbol to say that an object is an admissible set.
- Δ_0, Σ_n, Π_n, Σ, and Π formulas of $\mathcal{L}^\star$ defined as usual.
- A^b is the relativization of formula A to set b.

Systems of basic set theory BS^0 and BS

Number-theoretic axioms of BS^0

(N.1) A^N for every closed axiom A of PA.

(N.2) $0 \in a \wedge (\forall x, y \in N)(x \in a \wedge R_{\mathcal{S}}(x, y) \rightarrow y \in a) \rightarrow N \subseteq a$.

Ontological axioms of BS^0**.**

(O.1) $S(a) \leftrightarrow a \notin N$.

(O.2) $a \in N \rightarrow b \notin a$.

(O.3) $0 \in N$.

(O.4) $R_{\mathcal{Z}}(a_1, \ldots, a_n) \rightarrow a_1, \ldots, a_n \in N$.

Set-theoretic axioms of BS^0.

(S.1) Pair: $\exists x(a \in x \wedge b \in x)$.

(S.2) Union: $\exists x(\forall y \in a)(\forall z \in y)(z \in x)$.

(S.3) Δ_0 separation (Δ_0-Sep): for all Δ_0 formulas $\varphi[y]$,

$$\exists x(x = \{y \in a : \varphi[y]\}).$$

BS $:=$ BS^0 plus full induction on N and full $\in$-induction, i.e.

$$\forall x((\forall y \in x)\varphi[y] \rightarrow \varphi[x]) \rightarrow \forall x \varphi[x]$$

for all formulas $\varphi[x]$ of $\mathcal{L}^\star$.

Theories for admissible sets

The schema of Δ_0 collection

For all Δ_0 formulas $\varphi[x,y]$ of $\mathcal{L}^\star$:

$$(\forall x \in a)\exists y \varphi[x,y] \;\rightarrow\; \exists z(\forall x \in a)(\exists y \in z)\varphi[x,y]. \qquad (\Delta_0\text{-Col})$$

KPu^0 and KPu

$$\mathsf{KPu}^0 \;:=\; \mathsf{BS}^0 + (\Delta_0\text{-Col}),$$

$$\mathsf{KPu} \;:=\; \mathsf{BS} + (\Delta_0\text{-Col}).$$

Theorem (Jä)

$$\mathsf{KPu}^0 \;\equiv\; \mathsf{PA} \qquad \textit{and} \qquad \mathsf{KPu} \;\equiv\; \mathsf{ID}_1.$$

Adding admisible sets

Ad-axioms

$(Ad.1)$ $Ad(d) \;\rightarrow\; N \in d \;\wedge\; Tran[d]$.

$(Ad.2)$ $Ad(d_1) \wedge Ad(d_2) \;\rightarrow\; d_1 \in d_2 \;\vee\; d_1 = d_2 \;\vee\; d_2 \in d_1$.

$(Ad.3)$ For any closed instance of an axiom φ of KPu,

$$Ad(d) \;\rightarrow\; \varphi^d.$$

Remark. The *Ad*-axioms do not imply the existence of admissible sets. However, this is achieved by the limit axiom (Lim):

$$\forall x \exists y (x \in y \;\wedge\; Ad(y)).$$

KPl^0, KPl, KPi^0, KPi

$$\begin{aligned}
\mathsf{KPl}^0 &:= \mathsf{BS}^0 + \mathit{Ad}\text{-axioms} + (\mathsf{Lim}),\\
\mathsf{KPl} &:= \mathsf{BS} + \mathit{Ad}\text{-axioms} + (\mathsf{Lim}),\\
\mathsf{KPi}^0 &:= \mathsf{KPu}^0 + \mathit{Ad}\text{-axioms} + (\mathsf{Lim}),\\
\mathsf{KPi} &:= \mathsf{KPu} + \mathit{Ad}\text{-axioms} + (\mathsf{Lim}),
\end{aligned}$$

Theorem (Jä)

$$\begin{aligned}
\mathsf{KPl}^0 \;&\equiv\; \mathsf{KPi}^0 \;\equiv\; \mathsf{ATR}_0,\\
\mathsf{KPl} \;&\equiv\; \Pi^1_1\text{-}\mathsf{CA} + (\mathsf{BI}),\\
\mathsf{KPi} \;&\equiv\; \Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI}).
\end{aligned}$$

The role of Δ_1 separation

The schema of Δ_1 separation

For any Σ_1 formula $\varphi[x]$ and Π_1 formula $\psi[x]$ of $\mathcal{L}^\star$,

$$(\forall x \in a)(\varphi[x] \leftrightarrow \psi[x]) \rightarrow \exists y(y = \{x \in a : \varphi[x]\}). \qquad (\Delta_1\text{-Sep})$$

We know:

(i) (Δ_1-Sep) is provable in KPu^0.

(ii) Jensen, cf. Barwise: α is admissible if and only if α is a limit ordinal and L_α satisfies (Δ_1-Sep).

Conjecture/Theorem

$$\mathsf{BS}^0 + (\Delta_1\text{-Sep}) \equiv \Delta^1_1\text{-}\mathsf{CA}_0 \quad \textit{and} \quad \mathsf{BS} + (\Delta_1\text{-Sep}) \equiv \Delta^1_1\text{-}\mathsf{CA}.$$

Question

- Is there a natural way to formalize something like $(V{=}L)$ in the theory $\mathsf{BS} + (\Delta_1\text{-Sep})$?
- And if so, does it affect the proof-theoretic strength of this theory?

Σ and Π reduction

Definition

Let $\mathfrak{F}$ be a collection of formulas of $\mathcal{L}^\star$ and $\neg\mathfrak{F}$ the collection of its duals. Then the axioms schema ($\mathfrak{F}$-Red) of $\mathfrak{F}$ reduction consists of all formulas

$$(\forall x \in a)(\varphi[x] \to \psi[x]) \;\to\; \exists y(\{x \in a : \varphi[x]\} \subseteq y \subseteq \{x \in a : \psi[x]\}),$$

where $\varphi[x]$ is from $\neg\mathfrak{F}$ and $\psi[x]$ from $\mathfrak{F}$.

Conjecture/Theorem

1. $\mathsf{BS}^0 + (\Sigma\text{-Red}) \;\equiv\; \Sigma^1_1\text{-AC}_0$ *and* $\mathsf{BS} + (\Sigma\text{-Red}) \;\equiv\; \Sigma^1_1\text{-AC}$.
2. $\mathsf{BS}^0 + (\Pi\text{-Red}) \;\equiv\; \mathsf{ATR}_0$ *and* $\mathsf{BS} + (\Pi\text{-Red}) \;\equiv\; \mathsf{ATR}$.

Question

Clearly, KPu^0 proves (Σ-Red). But what can we say about

$$\mathsf{KPu}^0 + (\Pi\text{-Red}) \quad \text{and} \quad \mathsf{KPu} + (\Pi\text{-Red})?$$

It is clear that

$$\mathsf{KPu}^0 + (\Pi\text{-Red}) \subseteq \mathsf{KPu}^0 + (\Sigma_1\text{-Sep}),$$

$$\mathsf{KPu} + (\Pi\text{-Red}) \subseteq \mathsf{KPu} + (\Sigma_1\text{-Sep}).$$

Adding principles of second order arithmetic

Canonical embedding of $\mathcal{L}_2$ into $\mathcal{L}^\star$.

Language $\mathcal{L}_2$ of second order arithmetic embedded into $\mathcal{L}^\star$ by translating

$$\begin{array}{llllll} \exists n(\ldots) & \text{into} & (\exists n \in \mathsf{N})(\ldots) & \text{and} & \forall n(\ldots) & \text{into} & (\forall n \in \mathsf{N})(\ldots), \\ \exists X(\ldots) & \text{into} & (\exists x \subseteq \mathsf{N})(\ldots) & \text{and} & \forall X(\ldots) & \text{into} & (\forall x \subseteq \mathsf{N})(\ldots). \end{array}$$

Theorem

1. $\mathsf{KPu} \;\equiv\; \mathsf{ID}_1 \;\equiv\; \mathsf{ACA} + (\Pi^1_1\text{-}\mathsf{CA})^-$.
2. $\mathsf{KPi} \;\equiv\; \Delta^1_2\text{-}\mathsf{CA} + (\mathsf{BI})$.

Question

What is the strength of $\mathsf{KPu} + (\Pi^1_1\text{-}\mathsf{CA})$?

Arithmetic operator forms

Arithmetic formulas (in their translation into $\mathcal{L}^\star$) of the form

$$\mathfrak{A}[X^+, n],$$

possibly with additional set and number parameters. We set

$$Fix_{\mathfrak{A}}[\mathsf{N}, x] \;:=\; x \subseteq \mathsf{N} \;\land\; (\forall n \in \mathsf{N})(n \in x \;\leftrightarrow\; \mathfrak{A}[x, n]).$$

Arithmetic fixed point axioms

Let $\mathfrak{A}[X^+, n]$ be an arithmetic operator form.

$$\exists x Fix_{\mathfrak{A}}[\mathsf{N}, x], \qquad (\Pi^0_\infty\text{-FP})$$

$$\exists x(Fix_{\mathfrak{A}}[\mathsf{N}, x] \;\land\; \forall y(Fix_{\mathfrak{A}}[\mathsf{N}, y] \to x \subseteq y)), \qquad (\Pi^0_\infty\text{-LFP})$$

Questions

1. Is $\mathsf{KPu}^0 + (\Pi^0_\infty\text{-FP})$ is proof-theoretically equivalent to ATR_0?
2. What is the proof-theoretic strength of $\mathsf{KPu} + (\Pi^0_\infty\text{-FP})$?
3. What is the proof-theoretic strength of $\mathsf{KPu} + (\Pi^0_\infty\text{-LFP})$?

Subset-bounded separations

$\Pi_1^{\mathcal{P}}$ separation ($\Pi_1^{\mathcal{P}}$-Sep)

For any Δ_0 formula $\varphi[x, y]$ and any a:

$$\exists z(z = \{x \in a : (\forall y \subseteq a)\varphi[x, y]\}).$$

$\Pi_1^{\mathcal{P}}(\Delta_1)$ separation ($\Pi_1^{\mathcal{P}}(\Delta_1)$-Sep)

For every Σ_1 formula $\varphi[x, y]$, every Π formula $\psi[x, y]$,and any a:

$$\forall x, y(\varphi[x, y] \leftrightarrow \psi[x, y]) \rightarrow \exists z(z = \{x \in a : (\forall y \subseteq a)\varphi[x, y]\}).$$

Questions

1. What is the exact relationship between $\mathsf{KPu} + (\Pi^1_1\text{-CA})$ and $\mathsf{KPu} + (\Pi^{\mathcal{P}}_1\text{-Sep})$?
2. What is the exact relationship between $\mathsf{KPu} + (\Pi^{\mathcal{P}}_1\text{-Sep})$ and $\mathsf{KPu} + (\Delta_0\text{-LFP})$?
3. Is $(\Delta^1_2\text{-CA})$ provable in $\mathsf{KPu} + (\Pi^{\mathcal{P}}_1(\Delta_1)\text{-Sep})$?

Thank you for your attention!