Category Theory and Universes in Explicit Mathematics.

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(work in progress)

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1. Categories in Explicit Mathematics
2. The category of “Sets” and some of its properties
3. A categorical universe translated back to a universe in Explicit Mathematics

Definition (Language)

The language used to write down the axioms for category theory is built from seven symbols

$$ob, mor, id, \circ, =_o, =_m, \xrightarrow{m}$$

and the following abbreviations.

$$x =_o y :\equiv \langle x, y \rangle \dot{\in} =_o$$
$$f =_m g :\equiv \langle f, g \rangle \dot{\in} =_m$$
$$f \circ g :\equiv \circ(f, g)$$

Definition (Category)

Let u be a universe. A Category (relative to u) is a six-tuple $\langle ob, mor, id, \circ, =_o, =_m\rangle$ *which satisfies the following properties (including (UNIV)):*

$$
\begin{array}{rl}
(CL) & \mathfrak{R}(ob) \wedge \mathfrak{R}(mor) \wedge \mathfrak{R}(=_o) \wedge \mathfrak{R}(=_m) \\
(UNIV) & ob \mathrel{\dot\in} u \wedge mor \mathrel{\dot\in} u \wedge =_o \dot\in u \wedge =_m \dot\in u \\
(MOR) & (\forall m \mathrel{\dot\in} mor)(\exists x, y \mathrel{\dot\in} ob)(m = \langle x, y, \pi_2 m\rangle) \\
(EQ_O1) & (=_o \dot\subset ob \times ob) \wedge (\forall x \mathrel{\dot\in} ob)(x =_o x) \\
(EQ_M1) & (=_m \dot\subset mor \times mor) \wedge (\forall f \mathrel{\dot\in} mor)(f =_m f) \\
(CMP1) & (\forall f, g \mathrel{\dot\in} mor) \\
 & \quad (\pi_1 g =_o \pi_0 f \rightarrow (f \circ g)\downarrow \wedge (f \circ g) \mathrel{\dot\in} mor) \\
(ID1) & (\forall x \mathrel{\dot\in} ob)(id(x) \mathrel{\dot\in} mor \\
 & \quad \wedge \pi_0 id(x) =_o x \wedge \pi_1 id(x) =_o x)
\end{array}
$$

Definition (Category (cont.))

$$
\begin{aligned}
(EQ_O2) &\quad (\forall x, y \mathrel{\dot{\in}} ob)(x =_o y \to y =_o x)\\
(EQ_O3) &\quad (\forall x, y, z \mathrel{\dot{\in}} ob)(x =_o y \wedge y =_o z \to x =_o z)\\
(EQ_M2) &\quad (\forall f, g \mathrel{\dot{\in}} mor)(f =_m g \to g =_m f)\\
(EQ_M3) &\quad (\forall f, g, h \mathrel{\dot{\in}} mor)(f =_m g \wedge g =_m h \to f =_m h)\\
(CMP2) &\quad (\forall f, g, h \mathrel{\dot{\in}} mor)(\pi_1 g =_o \pi_0 f \wedge \pi_1 h =_o \pi_0 g\\
&\qquad \to (f \circ g) \circ h =_m f \circ (g \circ h))\\
(ID2) &\quad (\forall x \in ob)(\forall f \mathrel{\dot{\in}} mor)(dom(f) =_o x \to f \circ id(x) =_m f)\\
(ID3) &\quad (\forall x \in ob)(\forall f \mathrel{\dot{\in}} mor)(cod(f) =_o x \to id(x) \circ f =_m f)
\end{aligned}
$$

Definition

The rest of the usual language of category theory (and other operations) can now be defined on top of this.

$$
\begin{aligned}
dom(f) &:\equiv \pi_0 f \\
cod(f) &:\equiv \pi_1 f \\
f_F &:\equiv \pi_2 f \\
f : a \overset{m}{\to} b &:\equiv f \mathrel{\dot{\in}} mor \wedge dom(f) =_o a \wedge cod(f) =_o b
\end{aligned}
$$

Definition

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$$dom(f) :\equiv \pi_0 f$$
$$cod(f) :\equiv \pi_1 f$$
$$f_F :\equiv \pi_2 f$$
$$f : a \xrightarrow{m} b :\equiv f \dot{\in} mor \wedge dom(f) =_o a \wedge cod(f) =_o b$$

$$\mathrm{hom}^{EC}(a, b) :\equiv \{f | f : a \xrightarrow{m} b\}$$

Definition (Functor)

In the following we will write f_o for $(\pi_0 f)$ and f_m for $(\pi_1 f)$. We say a term f is a functor between two categories $\mathcal{C}$ and $\mathcal{D}$ (Notation: $f \dot{\in} functor(\mathcal{C}, \mathcal{D})$) if the conjunction of the following properties holds:

$$
\begin{array}{ll}
(F1) & x, y \dot{\in} ob \ \wedge x =_o y \to f_o(x) =_o f_o(y) \\
(F2) & g, h \dot{\in} mor \ \wedge g =_m h \to f_m(g) =_m f_o(h) \\
(F3) & g \dot{\in} mor \ \to dom \ (f_m(g)) =_o f_o(dom(g)) \\
(F4) & g \dot{\in} mor \ \to cod \ (f_m(g)) =_o f_o(cod(g)) \\
(F5) & x \dot{\in} ob \ \to f_m(id(x)) =_m id \ (f_o(x)) \\
(F6) & g, h \dot{\in} mor \ \wedge dom(g) =_o cod(h) \\
 & \to f_m(g \circ h) =_m f_m(g) \circ f_m(h)
\end{array}
$$

All of this can be written as an elementary formula.

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\begin{array}{ll}
(F1) & x, y \dot{\in} ob_{\mathcal{C}} \wedge x =_o^{\mathcal{C}} y \rightarrow f_o(x) =_o^{\mathcal{D}} f_o(y) \\
(F2) & g, h \dot{\in} mor_{\mathcal{C}} \wedge g =_m^{\mathcal{C}} h \rightarrow f_m(g) =_m^{\mathcal{D}} f_o(h) \\
(F3) & g \dot{\in} mor_{\mathcal{C}} \rightarrow dom_{\mathcal{D}}(f_m(g)) =_o^{\mathcal{D}} f_o(dom(g)) \\
(F4) & g \dot{\in} mor_{\mathcal{C}} \rightarrow cod_{\mathcal{D}}(f_m(g)) =_o^{\mathcal{D}} f_o(cod(g)) \\
(F5) & x \dot{\in} ob_{\mathcal{C}} \rightarrow f_m(id(x)) =_m^{\mathcal{D}} id_{\mathcal{D}}(f_o(x)) \\
(F6) & g, h \dot{\in} mor_{\mathcal{C}} \wedge dom(g) =_o^{\mathcal{C}} cod(h) \\
 & \quad \rightarrow f_m(g \circ^{\mathcal{C}} h) =_m^{\mathcal{D}} f_m(g) \circ_{\mathcal{D}} f_m(h)
\end{array}
$$

All of this can be written as an elementary formula.

Definition (Natural transformation)

Given two functors $f, g \mathrel{\dot{\in}} functor(\mathcal{C}, \mathcal{D})$ *between fixed categories, we call a tuple* $\eta = \langle f, g, \eta_F \rangle$ *a natural transformation (Notation:* $\eta \mathrel{\dot{\in}} nat(\mathcal{C}, \mathcal{D}, f, g)$ *or* $\eta : f \Rightarrow g$*) if*

$$
\begin{aligned}
&(NAT1) \qquad (\forall x \mathrel{\dot{\in}} ob)(\eta_F(x) : f_o(x) \xrightarrow{m} g_o(x)) \\
&(NAT2) \qquad (\forall h \mathrel{\dot{\in}} mor)(g_m(h) \circ_{\mathcal{D}} \eta_F(dom(h)) \\
&\qquad\qquad\qquad =_m^{\mathcal{D}} \eta_F(cod(h)) \circ_{\mathcal{D}} f_m(h))
\end{aligned}
$$

Definition (Functor category)

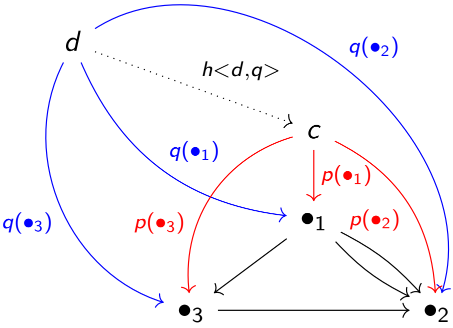
A (covariant) functor category from (fixed) categories $\mathcal{C}$ to $\mathcal{D}$ is a category $\mathcal{D}^{\mathcal{C}}$ defined as

$$
\begin{aligned}
ob &:\equiv functor(\mathcal{C}, \mathcal{D}) \\
mor &:\equiv \sum_{f \dot{\in} ob} \sum_{g \dot{\in} ob} nat(\mathcal{C}, \mathcal{D}, f, g) \\
=_o &:\equiv eqo \\
=_m &:\equiv eqm \\
id(f) &:\equiv \langle f, f, \lambda x.id_{\mathcal{D}}(f_o(x))\rangle \\
(\eta \circ \nu) &:\equiv \langle dom(\nu), cod(\eta), \lambda x.(\eta_F x \circ_{\mathcal{D}} \nu_F x)\rangle
\end{aligned}
$$

where

$$
\begin{aligned}
eqo :\equiv \{\langle f, g\rangle \mid f, g \dot{\in} ob \\
&\wedge (\forall x \dot{\in} ob_{\mathcal{C}})(f_o(x) =_o^{\mathcal{D}} g_o(x))) \\
&\wedge (\forall h \in mor_{\mathcal{C}})(f_m(h) =_m^{\mathcal{D}} g_m(h)))\} \\
eqm &:\equiv \{\langle \nu, \eta\rangle \mid \nu, \eta \dot{\in} mor \wedge (\forall x \dot{\in} ob_{\mathcal{C}})(\nu_F x =_m^{\mathcal{D}} \eta_F x)\}
\end{aligned}
$$

It is possible to encode (co)cones and (co)limits as pairs $\langle c, p \rangle$ and a term h which uniformly picks the unique map into (out of) the (co)limit.



The category of “Sets”

- We want something which is “close” to the usual category of sets.
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  2. Bishop Sets. (Sorry Ulrik, it's Setoid Hell)
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  1. Objects are class names, morphisms are operations.
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  3. One of the above with an added choice operator.
  4. Set theoretic maps with choice operator. But then why even use Explicit Mathematics?

Variant 2 vs. Variant 3

- Classes built from elementary comprehension restricted to *essentially* $\exists, \vee$-*free formulas*. and operations as morphisms.
  - built from $(x \in X)$, $x\downarrow$, $(x = y)$, and $\wedge, \to, \forall$
  - allows (AC) : $\quad (\forall x \in X)\exists y \varphi(x,y) \to \exists f (\forall x \in X)\varphi(x, fx)$

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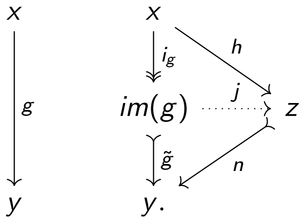
  - allows (AC) : $\quad (\forall x \in X)\exists y \varphi(x, y) \rightarrow \exists f (\forall x \in X)\varphi(x, fx)$
- Bishop Sets
  - Objects are pairs of names: A carrier and an ordinary equivalence relation (A class of pairs, giving a slight trivialization compared to the usual formulation in TT.)
  - Morphisms are operations which respect relations
  - No explicit transport operation necessary.
  - can use full elementary comprehension.

The category of "Sets"

- Example: Construction of the image of a morphism in Bishop Sets.



- This does not work for classes & operations without added choice.

Image construction

$$\begin{aligned} im(g) &:\equiv \langle dom(g), (x \sim_{im(g)} y) \leftrightarrow (g_{\mathsf{F}} x \sim_{cod(g)} g_{\mathsf{F}} y)\rangle \\ i_g &:\equiv \langle dom(g), im(g), \lambda x.x\rangle \\ \tilde{g} &:\equiv \langle im(g), cod(g), g_{\mathsf{F}}\rangle \\ j &:\equiv \langle im(g), z, h_{\mathsf{F}}\rangle \end{aligned}$$

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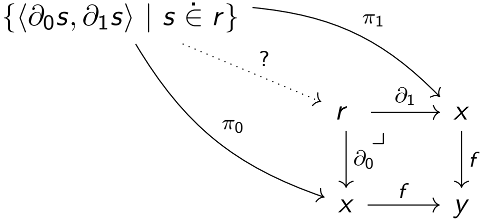
j &:\equiv \langle im(g), z, h_{\mathsf{F}}\rangle
\end{aligned}
$$

- i_g is a regular epi.
- the pullback $k^* i_g$ along any $k : z \xrightarrow{m} cod(g)$ is the coequalizer of its kernel pair.
- Coequalizers of kernel pairs are stable under pullback.
- So in Bishop Sets every morphism has a factorization into some (regular-)epi and a monomorphism.
- coequalizers in general are *not* stable under pullback.
- Bishop Sets are a regular category.

Other properties:

- For all variants we have: LCCC
- Exactness (= Every congruence is a kernel pair) is only possible with choice: Let $r \overset{(\partial_0,\partial_1)}{\rightarrowtail} x \times x$ be a congruence



Congruences are, upto renaming, equivalence relations, but we can't construct an inverse morphism to $r_0 \mapsto \langle \partial_0 r_0, \partial_1 r_0 \rangle$ w/o some way to choose an element from the preimage.

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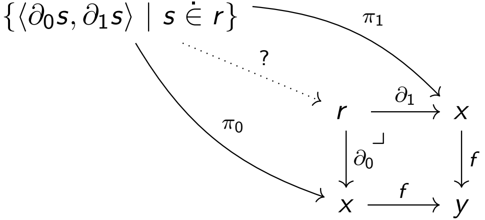

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  - The equivalent notion in MLTT does not suffer from this: Moerdijk, Palmgren (2002). Theorem 12.7: The category **Sets** [Bishop Sets] is a stratified pseudotopos [...]

Categorical Universes

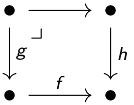
There have been several definitions of universes in categories. (Without any claim to completeness)

1. Joyal, Moerdijk. Algebraic Set Theory. (1995)
2. Moerdijk, Palmgren. Type theories, toposes and constructive set theory: Predicative aspects of ast. (2002)
3. Streicher. Universes in Toposes. (2004)
4. Awodey, Warren. Predicative algebraic set theory. (2005)

Categorical Universes translated back into Explicit Math.

Let $\mathcal{C}$ be a locally cartesian closed category, *el* be some morphism in $\mathcal{C}$ and $\mathcal{S}[x]$ be a formula. We call $\mathcal{S}$ a universe in $\mathcal{C}$ if the following axioms hold.

$$(U1) \qquad h \,\dot{\in}\, mor \wedge \mathcal{S}[h] \wedge f \,\dot{\in}\, mor \rightarrow (PB[h, f, g, q] \rightarrow \mathcal{S}[g])$$



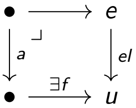
$$\mathcal{S}[h] \Rightarrow \mathcal{S}[g]$$

$$(U2) \qquad a \,\dot{\in}\, mor \wedge MONO[a] \rightarrow \mathcal{S}[a]$$

$$(U3) \qquad f : b \overset{m}{\rightarrow} c \wedge g : a \overset{m}{\rightarrow} b \wedge \mathcal{S}[f] \wedge \mathcal{S}[g] \rightarrow \mathcal{S}[\Sigma_f g]$$

$$(U4) \qquad f : a \overset{m}{\rightarrow} i \wedge g : b \overset{m}{\rightarrow} a \wedge \mathcal{S}[f] \wedge \mathcal{S}[g] \rightarrow \mathcal{S}[\Pi_f g]$$

$$(U5) \qquad a \,\dot{\in}\, mor \wedge \mathcal{S}[a] \rightarrow \exists f, pr_1(f : cod(a) \overset{m}{\rightarrow} cod(el) \wedge PB[f, el, a, pr_1])$$



- Taking $\mathcal{S}[x] :\equiv x \in \mathcal{S}$, and assuming the existence of $\mathcal{S}$ is inconsistent with elementary comprehension.

- Taking $\mathcal{S}[x] :\equiv x \in \mathcal{S}$, and assuming the existence of $\mathcal{S}$ is inconsistent with elementary comprehension.
- The problem is $(U2)$. Closure under all monos is a very strong condition.
- Weakening:

$$(U2\text{-W}) \qquad\qquad a \mathrel{\dot{\in}} ob \rightarrow \mathcal{S}[\Delta(a)]$$

We only require diagonals $\Delta(a) : a \overset{m}{\rightarrow} a \times a$ to be small.

Definition (Universe Condition)

Given some universe of Explicit Mathematics u and a morphism f, we say that f is in $\mathfrak{CU}$ iff there exist h, h^{-1} and g such that For all $x \dot{\in} cod(f)$

- $g(x) \dot{\in} u$
- $h(x) : f^{-1}\{x\} \stackrel{m}{\rightarrow} gx$
- $h^{-1}(x) : gx \stackrel{m}{\rightarrow} f^{-1}\{x\}$
- $iso(h(x), h^{-1}(x))$

"For all preimages of f there is some isomorphic class in u."

Universe Condition

Theorem

$\mathfrak{CU}$ is closed under $(U1), (U2\text{-}W), (U3), (U4), (U5)$.

Where we have for arbitrary $\mathfrak{CU}[f : a \overset{m}{\to} b, u]$

$$
\begin{array}{ccc}
a & \xrightarrow{\langle cod(h(f_{\mathsf{F}} x)), h(f_{\mathsf{F}} x)\rangle} & \sum\limits_{x \dot{\in} u} x \\
\big\downarrow f & \lrcorner & \big\downarrow el :\equiv pr_0 \\
b & \xrightarrow{g(x)} & \langle u, _{\exists}\cong\rangle
\end{array}
$$

Strength

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- This definition has a defect.
- Closure under the join axiom on the EM side is *not* required.
- Possible Fix: (CA) as defined by Joyal, Moerdijk (1995), & Moerdijk, Palmgren (2002)

A commutative square

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A & \longrightarrow & X \end{array}$$

is called a *quasi-pullback* whenever the canonical map $C \to A \times_X B$ is an epi.

Collection Axiom (CA)

Definition (CA)

For any small morphism $a \overset{m}{\to} x$ and any epi $c \overset{m}{\to} a$ there exists a quasi-pullback of the form*

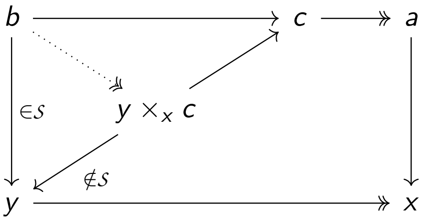
$$\begin{array}{ccccc} b & \longrightarrow & c & \twoheadrightarrow & a \\ \downarrow & & & & \downarrow \\ y & & \twoheadrightarrow & & x \end{array}$$

where $y \overset{m}{\twoheadrightarrow} x$ is epi and $b \overset{m}{\to} y$ is small.

$(*)$: part of $\mathcal{S}$

Collection Axiom (CA)

Some Intuition. Let $c \notin \mathcal{S}$



Existence of small subcovers.

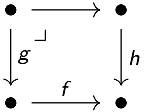
- May or may not hold for $\mathfrak{CU}$
- Might give closure under join only indirectly by interpreting some other system in the internal logic.

Thank You

Type theories, toposes and constructive set theory

- Pullback stability

$$h \,\dot{\in}\, mor \wedge \mathcal{S}[h] \wedge f \,\dot{\in}\, mor \rightarrow (PB[h, f, g, q] \rightarrow \mathcal{S}[g])$$



$$\mathcal{S}[h] \Rightarrow \mathcal{S}[g]$$

- Descent: In the pullback square above, if f is epi then $\mathcal{S}[h] \Leftrightarrow \mathcal{S}[g]$
- $\mathcal{S}[f : a \xrightarrow{m} b] \wedge \mathcal{S}[g : a' \xrightarrow{m} b'] \rightarrow \mathcal{S}[f + g : a + a' \xrightarrow{m} b + b']$
- $\mathcal{S}[f : b \xrightarrow{m} c] \wedge g : a \xrightarrow{m} b \rightarrow (\mathcal{S}[f \circ g] \leftrightarrow \mathcal{S}[g])$