How large are proper classes?

Silvia Steila
joint work with Gerhard Jäger

Universität Bern

ABM

München
December 14-15, 2017

NGB

The theory NGB is formulated in a two-sorted language and consists of the following axioms:

- extensionality, pair, union,powerset, infinity for sets,
- Extensionality, Foundation for classes,
- Class Comprehension Schema: i.e, for every formula φ containing no quantifiers over classes there exists a class C such that

$$\forall x(\varphi[x] \leftrightarrow x \in C)$$

- Limitation of Size: i.e, for every proper class C there is a bijection between C and the class V of all sets.

$\mathbf{KP}^c$

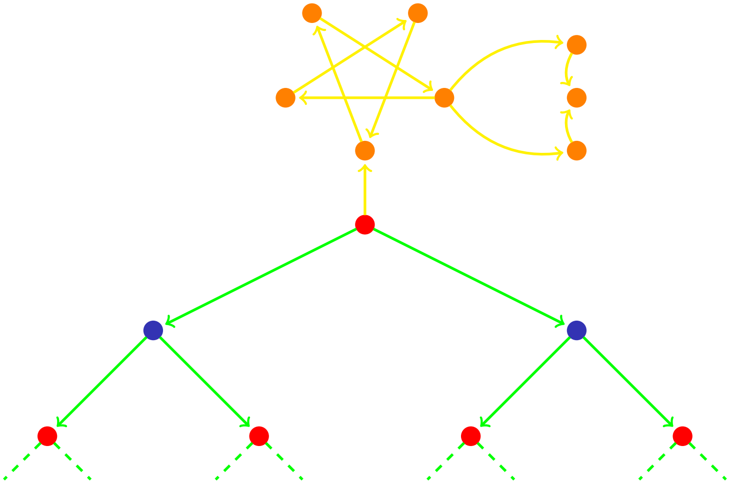
- Let $\mathcal{L}^c$ be the extension of $\mathcal{L}$ with countably many class variables.
- The atomic formulas comprise the ones of $\mathcal{L}$ and all expression of the form "$a \in C$".
- An $\mathcal{L}^c$ formula is elementary if it contains no class quantifiers.
- Δ_n^c, Σ_n^c and Π_n^c are defined as usual, but permitting subformulas of the form "$a \in C$".

$\mathbf{KP}^c$

The theory KP^c is formulated in $\mathcal{L}^c$ and consists of the following axioms:

- extensionality, pair, union, infinity,
- Δ_0^c-Separation: i.e, for every Δ_0^c formula φ in which x is not free and any set a,
$$\exists x(x = \{y \in a : \varphi[y]\})$$
- Δ_0^c-Collection: i.e, for every Δ_0^c formula φ and any set a,
$$\forall x \in a \exists y \varphi[x, y] \to \exists b \forall x \in a \exists y \in b \varphi[x, y]$$
- Δ_1^c-Comprehension: i.e, for every Σ_1^c formula φ and every Π_1^c formula ψ,
$$\forall x(\varphi[x] \leftrightarrow \psi[x]) \to \exists X \forall x(x \in X \leftrightarrow \varphi[x])$$
- Elementary $\in$-induction: i.e, for every elementary formula φ,
$$\forall x((\forall y \in x \varphi[y]) \to \varphi[x]) \to \forall x \varphi[x]$$

Motivations: ... last ABM



Operators

- We call a class an operator if all its elements are ordered pairs and it is right-unique (i.e. functional).
- We use F to denote operators.
- Given an operator F and a set a we write $\mathrm{Mon}[F, a]$ for:

$$\forall x(F(x) \subseteq a) \wedge \forall x, y(x \subseteq y \rightarrow F(x) \subseteq F(y)).$$

Least fixed point statements

FP

$$\mathrm{Mon}[F, a] \to \exists x(F(x) = x)$$

LFP

$$\mathrm{Mon}[F, a] \to \exists x(F(x) = x \land \forall y(F(y) = y \to x \subseteq y))$$

Separation

Σ_1^c-separation

For every Σ_1^c formula φ in which x is not free and any set a,

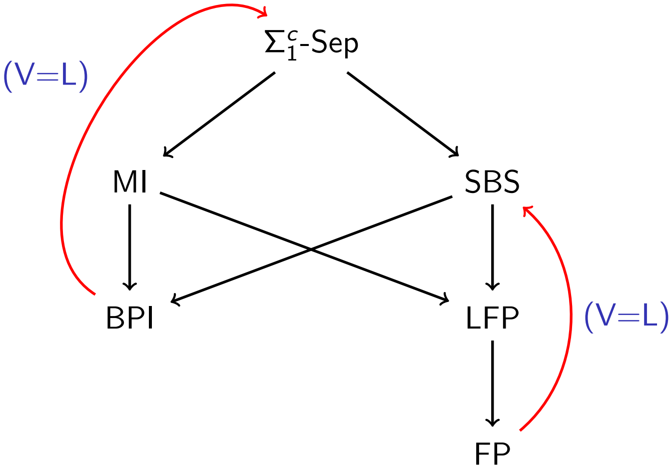
$$\exists x(x = \{y \in a : \varphi[y]\}).$$

SBS ($\sim \Pi_1^{\mathcal{P}}(\Delta_1^c)$-Sep)

For every Δ_1^c formula φ and sets a and b,

$$\exists z(z = \{x \in a : \exists y \subseteq b(\varphi[x, y])\})$$

Fixed point principles in KP^c + (V=L)



If we add to our theory the Axiom of Limitation of Size:

- we have a global well-ordering of V,
- all our principles are equivalent,
- But... I am not able to prove the consistency of:
$$\mathsf{KP}^c + \mathsf{FP} + \text{Limitation of size},$$
from the consistency of $\mathsf{KP}^c + \mathsf{FP}$.

What does it happen if we consider something weaker than a bijection?

Injections from ordinals to reals

Proposition

Assume that there are no injections from Ord to $\mathcal{P}(\omega)$. Then MI hold!

Injections from ordinals to reals

Proposition

Assume that there are no injections from Ord to $\mathcal{P}(\omega)$. Then MI hold!

Question

And if there is an injection from Ord to $\mathcal{P}(\omega)$?

Injections from reals to ordinals

Proposition

Assume that there is an injection from $\mathcal{P}(\omega)$ to Ord. Then BPI implies MI.

Injections from reals to ordinals

Proposition

Assume that there is an injection from $\mathcal{P}(\omega)$ to Ord. Then BPI implies MI.

Question

Assume that there are no injections from $\mathcal{P}(\omega)$ to Ord... BPI holds.

Surjections from ordinals to reals

Proposition

Assume that there is a surjection from Ord to $\mathcal{P}(\omega)$. Then there exists a strong well ordering of $\mathcal{P}(\omega)$.

Surjections from ordinals to reals

Proposition

Assume that there is a surjection from Ord to $\mathcal{P}(\omega)$. Then there exists a strong well ordering of $\mathcal{P}(\omega)$.

Question

Which is the strength of the statement: "For every class C, there exists either an injection from C to the ordinals or a surjection from the ordinals to C"?

Cofinal maps from reals to ordinals

Theorem

Assume that there exists a cofinal map $F : \mathcal{P}(\omega) \to \mathrm{Ord}$. Then SBS implies Σ_1^c-Separation for ordinals.

- Given φ we want to show that $\{x \in \omega : \exists \alpha \varphi[\alpha, x]\}$ is a set.
- By using F:

$$\exists \alpha \varphi[x, \alpha] \iff \exists y \subseteq \omega (\exists \alpha < F(y)(\varphi[x, \alpha])).$$

- The formula "$\exists \alpha < F(y)(\varphi[x, \alpha])$" is Δ^c.
- By applying SBS we get the thesis.

Cofinal maps from reals to ordinals

Let CM be the statement: there exists a cofinal map $F : \mathcal{P}(\omega) \to \mathrm{Ord}$.

- $L \models (\mathrm{CM} \vee (\mathcal{P}(\omega) \text{ is a set}))$.
- Axiom Beta does not imply CM.
- CM does not imply Axiom Beta.
- CM does not imply that every the least fixed point of any arithmetical operator is Δ^c-definable.

Cofinal maps from reals to ordinals

What about the negation of CM?

Cofinal maps from reals to ordinals

Theorem

Assume that there are no cofinal maps from the reals to the ordinals. Then Π_1-Reduction for ordinals holds.

Π_1-Reduction for ordinals

Let φ and ψ be two Δ_0 formulas such that

$$\forall x \in \omega(\exists \alpha \varphi[x, \alpha] \implies \forall \alpha \psi[x, \alpha]).$$

there exists a set z such that

$$\{x \in \omega : \exists \alpha \varphi[x, \alpha]\} \subseteq z \subseteq \{x \in \omega : \forall \alpha \psi[x, \alpha]\}.$$

Cofinal maps from reals to ordinals

- Assume that we have a set ω and two Δ formulas φ and ψ such that

$$\forall x \in \omega(\exists\alpha\varphi[x,\alpha] \implies \forall\alpha\psi[x,\alpha])$$

  and Π_1-Reduction for them does not hold.
- We derive

$$\forall z \subseteq \omega\exists x \in \omega\exists\alpha((\varphi[x,\alpha] \wedge x \notin z) \vee (x \in z \wedge \neg\psi[x,\alpha]))$$

- Define the following operator $F : \mathcal{P}(\omega) \to \mathrm{Ord}$.

$$F(z) = \mu\alpha(\exists x(\varphi[x,\alpha] \wedge x \notin z) \vee (x \in z \wedge \neg\psi[x,\alpha])).$$

- There exists β such that

$$\forall z \subseteq \omega\exists x \in \omega\exists\alpha \in \beta\ ((\varphi[x,\alpha] \wedge x \notin z) \vee (x \in z \wedge \neg\psi[x,\alpha]))$$

- Define the set

$$\{x \in \omega : \exists\alpha < \beta\varphi[x,\alpha]\}.$$

  and derive a contradiction.

Cofinal maps from reals to ordinals

Moreover:

- SBS implies Π_1-Reduction for ordinals.
- The Axiom of Powerset implies $\neg$CM.
- $\neg$CM does not imply Axiom Beta.

Question

- Which is the strength of Π_1-Reduction for ordinals?
- Does Axiom Beta imply $\neg$CM?

Cofinal maps from reals to ordinals

Moreover:

- SBS implies Π_1-Reduction for ordinals.
- The Axiom of Powerset implies $\neg$CM.
- $\neg$CM does not imply Axiom Beta.

Question

- Which is the strength of Π_1-Reduction for ordinals?
- Does Axiom Beta imply $\neg$CM?

Thank you!