Modelle der Mengenlehre - SS15

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May 28, 2015

The content of these notes includes the material discussed in the Exercises. Some extra, or optional exercises can also be found here. Please feel free to send me your comments, or your suggestions regarding these notes.

I. Sets, classes and proper classes

All begun when Cantor's **Full Comprehension Scheme** (FCS):

$$\exists_u (u = \{x \mid \phi(x)\}),$$

where ϕ is any formula of $L = (\in)$, was proved contradictory for $\phi(x) := x \notin x$. Zermelo's **Restricted Comprehension Scheme** (RCS), also known as Separation Scheme,

$$\exists_u (u = \{x \in v \mid \phi(x)\})$$

replaced the FCS and it implies that $V \notin V$: if $V \in V$, then $u = \{x \in V \mid x \notin x\} \in V$ and then $u \in u \leftrightarrow u \notin u$. If FCS was not contradictory, we wouldn't need so many axioms to describe our intuition about sets. E.g., the union of two sets would be defined as $u \cup v = \{x \mid x \in u \vee x \in v\}$.

I.1 The first-order non-logical axioms of ZF in the language $L = (\in)$

Extensionality: $\forall_{x,y}(\forall_z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

Empty set: $\exists_x \forall_y (y \notin x)$.

Pair: $\forall_{x,y}\exists_z \forall_w (w \in z \leftrightarrow w = x \vee w = y)$.

Union: $\forall_x \exists_y \forall_z (z \in y \leftrightarrow \exists_w (w \in x \wedge z \in w))$.

Replacement Scheme: If $\phi(x, y, \vec{w})$ is a function formula, then

$$\forall_x \exists_v \forall_y (y \in v \leftrightarrow \exists_z (z \in x \wedge \phi(z, y, \vec{w})).$$

Power-set: $\forall_x \exists_y \forall_z (z \in y \leftrightarrow \forall_w (w \in z \rightarrow w \in x))$.

Foundation: $\forall_x (x \neq \emptyset \rightarrow \exists_z (z \in x \wedge \neg \exists_w (w \in z \wedge w \in x)))$.

Infinity: $\exists_x (\emptyset \in x \wedge \forall_y (y \in x \rightarrow y \cup \{y\} \in x))$.

I.2 On the axioms of ZF

1. Unlike group-axioms (first the models, the groups, and then the axioms) the set-axioms are given first and then we study their models!!!

2. The axioms of ZF are generally "accepted". The system generated by the addition of the infamous **axiom of choice** AC

$$\forall_u \exists_{f:u \rightarrow V} (\forall_x (x \in u \rightarrow x \neq \emptyset \rightarrow f(x) \in x),$$

   i.e., form every non-empty element x of u the "choice-function" f selects an element of x, to ZF is called ZFC. The AC is not that "innocent". When a proof uses it, usually this is noted. It has many important consequences in standard mathematics, for example every ideal in a ring is contained in a maximal ideal, every vector space has a basis, and every product of compact spaces is compact. But it has also some counter-intuitive consequences like the **Banach-Tarski paradox**: it is possible to decompose the 3-dimensional solid unit ball into finitely many pieces and, using only rotations and translations, reassemble the pieces into two solid balls each with the same volume as the original.

3. There exist many set theories (e.g., in Bernays-Gödel ST we have two sorts (types) of objects: sets and classes).

4. There exist many constructive set theories e.g., CZF, based on intuitionistic logic (Aczel-Rathjen).

5. The axioms of ZF are due to **Zermelo**, except Replacement (**Fraenkel**, **Skolem**), and Foundation (**von Neumann**). Bolzano was maybe the first who considered infinite sets, he also introduced the term *Menge*, but it was Cantor who introduced cardinals-ordinals, who proved that $\mathbb{R}$ is uncountable, that the set of algebraic numbers $\mathbb{A}$ is countable, and he formulated the continuum hypothesis CH.

6. Note the difference between an axiom and an axiom-scheme.

7. The converse of Extensionality is provable by the equality axioms.

8. First we show by Extensionality the uniqueness of $\emptyset$, and *then* we write $\emptyset \in V$ or we use the symbol $\emptyset$ in the Foundation. The same pattern is followed in every similarly written non first-order axiom.

9. Note that the Restricted Comprehension Scheme (Aussonderung) is derivable from the Replacement Scheme.

10. Note that the existence of an inductive set is equivalent to the existence of an infinite set, but that requires the notion of a finite set. Of course, it is intuitively expected that an inductive set is not finite (see [3], p.26, Ex. 2.4).

I.3 Classes and proper classes

In the metatheory of ZF *classes* i.e., formulas $\phi(x, \vec{c})$ in $L = (\in)$ with parameters are used. Usually we identify $\phi(x, \vec{c})$ with the informal object

$$A_\phi = \{x \mid \phi(x, \vec{c})\},$$

and we denote it by A, omitting the subscript in most cases.

A set a is a class: $a = \{x \mid x \in a\} = \{x \mid \psi(x, a)\}$, where $\psi(x, a) := x \in a$.

We use the following **definitions for classes**:

$$V := \{x \mid x = x\}$$

$$\emptyset := \{x \mid x \neq x\}$$

$$A_{\phi \vee \psi} := A_\phi \cup A_\psi := \{x \mid \phi(x) \vee \psi(x)\}$$

$$A_{\phi \wedge \psi} := A_\phi \cap A_\psi := \{x \mid \phi(x) \wedge \psi(x)\}$$

$$A_\phi^c := A_{\neg\phi} := \{x \mid \neg\phi(x)\}$$

$$A_\phi \setminus A_\psi := A_\phi \cap A_\psi^c.$$

By the principle of the excluded middle $A_{\phi \vee \neg\phi} = V$. We use the following **rules for classes**:

$$A_\phi \subseteq B_\psi :\leftrightarrow \text{ZF} \vdash \forall_x(\phi(x) \rightarrow \psi(x))$$

$$A_\phi = B_\psi :\leftrightarrow \text{ZF} \vdash \forall_x(\phi(x) \leftrightarrow \psi(x))$$

$$b \in A_\phi :\leftrightarrow \text{ZF} \vdash \phi(b)$$

$$B \in A_\phi :\leftrightarrow \text{ZF} \vdash \exists_b(B = b \wedge \phi(b)).$$

Note that the last equality is well-defined, since a set is a class. The same applies to the formulations $A \cap x$, or $A \subseteq x$. A class A is called *proper*, if

$$\text{ZF} \vdash \neg\exists_a(A = a),$$

and we denote this fact informally as $A \notin V$.

Basic facts on (proper) classes:

a. $R \notin V$.
b. $\forall_x (A \cap x \in V)$.
c. $\forall_x (A \subseteq x \to A \in V)$.
d. $A \subseteq B \to A \notin V \to B \notin V$.
e. $\bigcup A \notin V \to A \notin V$.
f. $F'' A \notin V \to A \notin V$.
g. $\mathrm{On} \notin V$, since an ordinal cannot be contained to itself.

I.4 On the axiom of Foundation

1. Verify the axiom of foundation on specific sets. There are sets x, like ω, for which there exists only one element not intersecting x, while there are sets, like $\mathbb{R}$, such that every element doesn't intersect x.

2. It is not used in actual mathematics but it is important for the formation of the set-theoretic universe. It is also very important in the construction of models of set theory.

3. There exist no infinite $\ni$-chains $x_0 \ni x_1 \ni x_2 \ni \ldots$.

4. There exist no cycles $x_0 \in x_1 \ldots x_n \in x_0$.

5. $\nexists_x (x \in x)$.

6. $\nexists_x (\mathcal{P}(x) \subseteq x)$.

7. $\forall_{x,y} (x \notin y \ \vee \ y \notin x)$.

II. Transitive classes/sets and well-founded relations

II.1 Transitive classes/sets

A class A is *transitive*, if it satisfies the rule

$$\frac{y \in x \in A}{y \in A} \ .$$

1. If u is transitive, then $u \cup \{u\}$ is transitive.
2. If $u \in V$, show that the following are equivalent:
(a) u is transitive.
(b) $\bigcup u \subseteq u$.
(c) $\bigcup u^+ = u$, where $u^+ = u \cup \{u\}$.
3. If u is a non-empty set each element of which is transitive, then $\bigcap u$ is also transitive.
4. If u is a non-empty set, then

$$u \text{ is transitive } \to \bigcap u = \emptyset.$$

5. For every set u there exists a transitive set v such that $u \subseteq v$. We can describe the least such transitive set $\overline{u}$, which is called the *transitive closure* of u. If we consider the following inductive rules to define $\overline{u}$

$$\frac{x \in u}{x \in \overline{u}} \ , \quad \frac{y \in x \in \overline{u}}{y \in \overline{u}} \ ,$$

which can also be rewritten as

$$\frac{x \in u}{x \in \overline{u}}\,, \quad \frac{x \in \bigcup \overline{u}}{x \in \overline{u}}\,,$$

they induce the corresponding induction principle:

$$\begin{aligned}
&\forall_{x \in u}(\phi(x)) \to \\
&\forall_{y \in \overline{u}} \forall_x (\phi(y) \to x \in y \to \phi(x)) \to \\
&\forall_{x \in \overline{u}}(\phi(x)),
\end{aligned}$$

where $\phi(x)$ is any formula. Show that $\overline{u}$ is a transitive set and if v is a transitive set such that $u \subseteq v$, then $\overline{u} \subseteq v$. We can also describe $\overline{u}$ as the result of the iteration of a certain operation on u for a specific "ordinal" number of times, as follows:

$$u_0 = u,$$

$$u_{n+1} = \bigcup u_n,$$

$$\overline{u} = \bigcup \{u_n \mid n \in \omega\}.$$

II.2 On the definition of addition in ω (Exercise 2, Blatt 2)

1. Using the Rekursionssatz für ω, show the following special case of it:
If a is a non-empty set, x is a fixed element of a and $h : a \to a$, there exists a unique function $f : \omega \to a$ such that

$$f(0) = x,$$

$$f(n+1) = h(f(n)).$$

2. Show that if $m \in \omega$, there exists a function $A_m : \omega \to \omega$ such that

$$A_m(0) = m,$$

$$A_m(n+1) = A_m(n) + 1,$$

where $n + 1$ is the successor of n. Then we define the addition of natural numbers as the set

$$+ := \{((m, n), \sigma) \mid m, n \in \omega \wedge \sigma = A_m(n)\}.$$

II.3 Well-founded relations

If (u, r) is a structure, then r is called *well-founded* (w.f.r.), if

$$\forall_{v \subseteq u}(v \neq \emptyset \to \exists_{a \in v} \forall_{x \in v}(x \not r a))$$

i.e., if each non-empty subset v of u has an r-minimal element.

1. Show that a w.f.r. is irreflexive and asymmetric i.e.,

$$\forall_{x \in u}(x \not r x),$$

$$\forall_{x,y \in u}(x \; r \; y \to y \not r x).$$

2. Give an example of a w.f.r. which is not a transitive relation.

3. Show that if r is a well-ordering (see also next section), then r is a w.f.r., and find a w.f.r. which is not a well-ordering.

4. If (u, r) is a w.f.r., there exists no sequence $\alpha : \omega \to u$ such that

$$\alpha_1 r \alpha_0, \ \alpha_2 r \alpha_1, \ \alpha_3 r \alpha_2 \dots \ .$$

II.4 Well-orderings

A *partial ordering* (p.o) $(u, <)$ is an irreflexive and transitive relation $<$ on u. If $(u_1, <), (u_2, <)$ are p.o., a function $f : u_1 \to u_2$ is called *order-preserving*, if

$$\forall_{x,y \in u_1}(x < y \to f(x) < f(y)).$$

If $(u_1, <), (u_2, <)$ are linear p.o., an order-preserving f is also called *increasing.* If $f : u_1 \to u_2$ is $1{-}1$ and onto u_2, then f is called an *isomorphism*, if f, f^{-1} are o.p. (in this case we write $u_1 \cong u_2$). If $u_2 = u_1$ and f is an isomorphism, f is called an *automorphism.* A *well-ordering* (w.o.) is a p.o. $(w, <)$ such that

$$\forall_{u \subseteq w}(u \neq \emptyset \to u \text{ has a } < \text{-least element}).$$

Clearly, a w.o. is a linear p.o. (if $x \neq y$, then $\min\{x, y\}$ is in w).

1. If $(w, <)$ is a well-ordered set, then show that there exists no sequence $\alpha : \omega \to w$ such that

$$\alpha_0 > \alpha_1 > \alpha_2 > \dots \ .$$

   Show also that this property (the non-existence of infinitely decreasing chains) implies the existence of a minimum element for each non-empty subset of w (for that one uses actually the principle of dependent choices).

2. **(optional)** If $\alpha, \beta : \omega \to w$ show that there exist $i < j$ such that

$$\alpha(i) \leq \alpha(j) \ \wedge \ \beta(i) \leq \beta(j).$$

3. **(optional)** There exist quasi-orderings (q.o) $(u, \preceq)$ i.e., a reflexive and transitive relation on u, such that every sequence in u is *good*[1] but $(u, \preceq)$ is not a well-ordering.

4. **(optional)** We define

$$\omega^{(\infty)} = ({}^{\omega}\omega, \preceq_p),$$

   where

$$\alpha \preceq_p \beta \leftrightarrow \forall_n(\alpha(n) \leq \beta(n)),$$

   for each $\alpha, \beta \in {}^{\omega}\omega$. The $\omega^{(\infty)}$ has a bad sequence: Clearly, the pointwise ordering $\preceq_p$ on $\mathbb{N}^{\mathbb{N}}$ is a q.o., but the following sequence of elements of $\mathbb{N}^{\mathbb{N}}$

$$\alpha_1 = (1, 2, 3, 4, 5, ...)$$

$$\alpha_2 = (2, 1, 3, 4, 5, ...)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\alpha_n = (n, n-1, ..., 1, n+1, n+2, ...),$$

   is bad. If $n = \alpha_n(0)$, then $\alpha_n(n-1) = 1$, for each $i \in \mathbb{N}$. If $i < j$, then $\alpha_j(j-1) = 1$, while $\alpha_i(j-1) > 1$. Therefore, it can never be the case that $\alpha_i \preceq_p \alpha_j$, for some $i < j$. The interesting thing about this example is that the corresponding qo $\mathbb{N}^{(k)} = (\mathbb{N}^k, \preceq_p)$, where $u \preceq_p v \leftrightarrow (\forall i \in k) u_i \leq v_i$, for each $u, v \in \mathbb{N}^k$, is proved to be a wqo, for each $k \in \mathbb{N}$.

[1] A sequence $\alpha : \omega \to w$ is called *good*, if there exist $i < j$ such that $\alpha(i) \preceq \alpha(j)$; otherwise it is called *bad*.

5. If $(w,<)$ is a w.o. and $f:w\to w$ is increasing, then

$$\forall_{x\in w}(f(x)\geq x).$$

6. If $(w,<)$ is a w.o. and $f:w\to w$ is an automorphism, then $f=\mathrm{id}_w$.
7. If $(w_1,<)$, $(w_2,<)$ are w.o. and $w_1\cong w_2$, then the isomorphism between them is unique.
8. If $(w,<)$ is a w.o. and for each $x\in w$ we define

$$\hat{x}=\{y\in w\mid y<x\},$$

then there exists no isomorphism between $w,\hat{x}$.

9. If $(w_1,<)$, $(w_2,<)$ are w.o., then

$$w_1\cong w_2\ \vee\ \exists_{y\in w_2}(w_1\cong\hat{y})\ \vee\ \exists_{x\in w_1}(w_2\cong\hat{x}).$$

(The proof of this proposition is of the same style to the first Satz of the Vorlesung notes.)

III. Ordinals

1. They were invented by Cantor to solve a problem in Fourier series.
2. As it is noted by T. Forster in [2], "ordinals are the kind of numbers that measures the length of precisely this sort of process: transfinite and discrete". E.g., as it is asked in the exercise 4 of Blatt 3, if (u,r) is a structure and r is founded, we define

$$\begin{aligned}
&u_0:=\emptyset,\\
&u_{\alpha+1}:=\{x\in u\mid\forall_y(y\ r\ x\to y\in u_\alpha\},\\
&u_\lambda:=\bigcup_{\alpha<\lambda}u_\alpha.
\end{aligned}$$

By Replacement one shows that there exists an ordinal ξ such that $u_{\xi+1}=u_\xi$. Also,

$$u_0\subseteq u_1\subseteq\ldots u_\zeta=u,$$

where ζ is the minimum ordinal satisfying $u_{\zeta+1}=u_\zeta$.
3. **The intuition behind addition and multiplication of ordinals**:

$\alpha+\beta$ is the order structure generated by considering each element of β greater than element of α and the orderings of α,β are kept as they are.

Using this intuition verify that $1+\omega\neq\omega+1$, and $\omega+\omega^2\neq\omega^2+\omega$.

$\alpha\cdot\beta$ is the order structure generated by considering each element of β equal to α.

Using this intuition explain why $2\cdot\omega\neq\omega\cdot 2$.
4. **Some countable ordinals**:

$$0,1,2,\ldots,\omega,$$

$$\omega+1,\omega+2,\ldots,\omega+\omega=\omega\cdot 2,$$

$$\omega\cdot 2+1,\omega\cdot 2+2,\ldots,\omega\cdot 2+\omega=\omega\cdot 3,$$

$$\omega\cdot 3+1,\omega\cdot 3+2,\ldots,\omega\cdot 3+\omega=\omega\cdot 4,\ldots$$

$$\ldots,\omega\cdot\omega=\omega^2,\ldots,\omega^2+\omega,\ldots,\omega^2+\omega\cdot 2,\ldots$$

$$\ldots,\omega^2+\omega^2=\omega^2\cdot 2,\ldots,\omega^2\cdot 3,\ldots,\omega^2\cdot\omega=\omega^3,\ldots$$

$$\ldots, \omega^4, \ldots, \omega^5, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \epsilon_0.$$

5. Show that

$$\lim(\lambda) \leftrightarrow \forall_\alpha(\alpha < \lambda \to \alpha + 1 < \lambda).$$

6. Find a non-transitive subset of some ordinal α which does not belong to α.
7. Show that for all ordinals α, β

$$\alpha < s(\beta) \leftrightarrow \alpha \leq \beta.$$

8. If $\alpha, \beta, \gamma, \delta \in \mathrm{On}$, show the following properties:
(i) $\alpha < \beta \to \gamma + \alpha < \gamma + \beta$. Especially, $0 < \beta \to \gamma < \gamma + \beta$.
(ii) $\alpha \leq \beta \to \alpha + \gamma \leq \beta + \gamma$. Find α, β, γ such that $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$.
(iii) $\alpha < \beta \to \gamma \cdot \alpha < \gamma \cdot \beta$. Especially, $1 < \beta \to \gamma < \gamma \cdot \beta$.
(iv) $\alpha \leq \beta \to \alpha \cdot \gamma \leq \beta \cdot \gamma$. Find α, β, γ such that $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$.
(v) $\alpha + \gamma < \beta + \gamma \to \alpha < \beta$.
(vi) $\alpha \cdot \gamma = \beta \cdot \gamma \to \gamma$ is successor ordinal $\to \alpha = \beta$.
(vii) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. Is it true that $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$?
(ix) If $\alpha \in \mathrm{On}$ and $x \in \alpha$, then $x \in \mathrm{On}$.
(x) $\forall_{\alpha, \beta \in \mathrm{On}}(\alpha \leq \beta \leftrightarrow \alpha \subseteq \beta)$.
(xi) If $a \subseteq \mathrm{On}$ and $a \neq \emptyset$, then $\min a = \bigcap a$ is the least ordinal in a.

9. $\forall_{\alpha, \beta \in \mathrm{On}}(\beta \leq \alpha \to \exists!_{\gamma \in \mathrm{On}}(\alpha = \beta + \gamma))$.
10. If $\alpha < \beta$ and $\gamma > 1$, then $\gamma^\alpha < \gamma^\beta$. Check that this doesn't hold for the corresponding operations on cardinals. 11. If ε_0 is the ordinal defined by[2]

$$\varepsilon_0 := \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$$

show that $\omega^{\varepsilon_0} = \varepsilon_0$ and also that ε_0 is the least ordinal α satisfying $\omega^\alpha = \alpha$. Therefore, ε_0 is the least ordinal bigger than ω which is closed under addition, multiplication and exponentiation of ordinals.
12. $\forall_{\alpha > 0, \gamma} \exists!_{\beta, \rho}(\rho < \alpha \ \wedge \ \gamma = \alpha \cdot \beta + \rho)$.
13. **Cantor's normal form theorem**: Every ordinal $\alpha > 0$ can be written uniquely as

$$\alpha = \omega^{\beta_m} \cdot k_m + \ldots + \omega^{\beta_0} \cdot k_0,$$

where $m \geq 1$, $\alpha \geq \beta_m > \ldots > \beta_0$, and $k_0, \ldots, k_m \in \mathbb{N} \setminus \{0\}$. Thus the ordinal ε_0 has the following convenient representation

$$\varepsilon_0 = \omega^{\varepsilon_0} \cdot 1,$$

and that's why in general $\alpha \geq \beta_m$. The above convenience is one of the reasons we use ω as a base in the representation of ordinals in a normal form.
14. Show that for every ordinal α there is a limit ordinal λ such that $\alpha < \lambda$.

IV. The cumulative hierarchy

1. Show that $\forall_\alpha(V_\alpha \in V)$.

2. Find a transitive set u having a non-transitive element x.

3. If the rank of a set x is defined as in the Vorlesung by[3]

$$\mathrm{rn}(x) := \min\{\alpha \in \mathrm{On} \mid x \in V_\alpha\},$$

   show that $y \in x \to \mathrm{rn}(y) < \mathrm{rn}(x)$. Does the converse hold?

[2]This is a very important ordinal in proof theory as the infamous theorem of Gentzen on the consistency of arithmetic shows. For that see [6], Chapter 10.
[3]If one defines the rank of x as the least α such that $x \in V_{\alpha+1}$, one gets $\mathrm{rn}(\alpha) = \alpha$.

4. Show that the following are equivalent:

   (i) Foundation Axiom.
   (ii) $V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$.

5. Show the following:

   (i) $\forall_{\alpha \in \mathrm{On}}(\alpha \in V_{\alpha+1} \setminus V_\alpha)$.
   (ii) $\forall_{\alpha \in \mathrm{On}}(\mathrm{rn}(\alpha) = \alpha + 1)$.

6. Show that

$$A \in V \leftrightarrow \{\mathrm{rn}(x) \mid x \in A\} \text{ is bounded in On.}$$

7. The rank of a set, as defined above, is always a successor ordinal i.e.,

$$\forall_{x \in V} \exists_{\alpha \in \mathrm{On}}(\mathrm{rn}(x) = \alpha + 1),$$

   where

$$\alpha = \sup\{\mathrm{rn}(y) \mid y \in x\}.$$

8. Using the standard set-theoretic constructions of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ (see for example [1], Chapter 5), show that $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, {}^{\mathbb{R}}\mathbb{R} \in V_{\omega+\omega}$.

Suppose that $u \in V$ and ϕ is a formula of the language of ZFC. The *relativization* ϕ^u of ϕ to u is generated by replacing all occurrences $x \in V$ in ϕ by $x \in u$. Then we define

$$u \models \phi := \mathrm{ZFC} \vdash \phi^u.$$

Then show the following:

1. $\forall_{\alpha \in \mathrm{On}}(V_\alpha \models$ Extensionality Axiom$)$.
2. $\forall_{\alpha \in \mathrm{On}}(\alpha \neq 0 \to V_\alpha \models$ Empty set Axiom$)$.
3. $\forall_{\alpha \in \mathrm{On}}(\mathrm{limitordinal}(\alpha) \to V_\alpha \models$ Pair Axiom$)$.
4. $\forall_{\alpha \in \mathrm{On}}(V_\alpha \models$ Union axiom$)$.
5. $\forall_{\alpha \in \mathrm{On}}(\mathrm{limitordinal}(\alpha) \to V_\alpha \models$ Power set Axiom$)$.
6. $\forall_{\alpha \in \mathrm{On}}(V_\alpha \models$ Separation Scheme$)$.
7. $\forall_{\alpha \in \mathrm{On}}(\alpha > \omega \to V_\alpha \models$ Infinity Axiom$)$.
8. What about the converse to 2, 3, 5, 7?
9. Consider the following formulation of the Axiom of Choice: if R is a relation, then there exists a function F such that $\mathrm{dom}(F) = \mathrm{dom}(R)$, in order to show:

$$\forall_{\alpha \in \mathrm{On}}(V_\alpha \models \text{ Axiom of Choice}).$$

In summary we conclude that if λ is any limit ordinal number $> \omega$, then

$$V_\lambda \models \mathrm{ZF}_0,$$

where

$$\mathrm{ZF}_0 = \mathrm{ZF} \setminus \{\text{Replacement Axiom}\},$$

and ZF is considered here to contain the Separation Scheme[4]. One could ask if the Replacement Scheme is derivable by the rest axioms. Next results show the *independence* of the Replacement

[4] As you already know the Separation scheme is proved by the Replacement scheme, but in many textbooks it is included in the list of axioms of ZF and it is noted or proved later that it is derivable (see e.g., [3] or [4]). In that way it is clear that ZF_0 contains the required Separation Scheme.

Scheme.

(i) There exists a well-ordering structure $(w, <)$ such that $<\in V_{\omega+\omega}$ and $\mathrm{otp}(w, <) \notin V_{\omega+\omega}$.
(ii) Conclude by (i) that

$$V_{\omega+\omega} \not\models \text{ Replacement Axiom.}$$

The above result is expected from Gödel's second incompleteness theorem. If there was a limit ordinal λ such that $V_\lambda \models$ Replacement Axiom too, then we would have that $V_\lambda \models \mathrm{ZF}$. But then ZF could prove its own consistency, and that contradicts Gödel's second incompleteness theorem.

V. Normal functions

A function $F : \mathrm{On} \to \mathrm{On}$ is called *normal*, if it satisfies the following:

(i) $\forall_{\alpha,\beta}(\alpha < \beta \to F(\alpha) < F(\beta))$, i.e., F is (strictly) increasing.
(ii) $\forall_{\lambda\in\mathrm{LOn}}(F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha))$, where LOn is the class of limit ordinals. This condition is the continuity condition on F.

1. Show that $F_1(\alpha) = \beta + \alpha,\ F_2(\alpha) = \beta \cdot \alpha$ and $F_3(\alpha) = \beta^\alpha$ are normal functions.

2. Give an example of a monotone function which is not continuous.

3. Give an example of a continuous function which is not monotone.

4. Show that the function $F : \mathrm{On} \to \mathrm{On}$ defined by

$$\alpha \mapsto \alpha^2$$

is not continuous.

5. Suppose that F is continuous. Then the following are equivalent:

(i) F is normal.
(ii) $\forall_\alpha(F(\alpha) < F(\alpha + 1))$.

6. Suppose that $F : \mathrm{On} \to \mathrm{On}$ is increasing. Then the following hold:

(i) $\forall_\alpha(F(\alpha) \geq \alpha)$.
(ii) $\forall_{\alpha,\beta}(F(\alpha) < F(\beta) \to \alpha < \beta)$.

7. If F is normal, then F has an unbounded set of *fixed points*. Actually, show that

$$\forall_\alpha\exists_\beta(\alpha \leq \beta\ \wedge\ F(\beta) = \beta\ \wedge\ \forall_{\alpha<\gamma<\beta}(F(\gamma) > \gamma)),$$

i.e., β is the least fixed point of F above α.
Hint: Define

$$\beta := \bigcup_{n\in\omega} \beta_n,$$

where

$$\beta_0 := \alpha,$$
$$\beta_{n+1} := F(\beta_n).$$

8. If F is normal, then the *derivative* F' of F is the function $F' : \mathrm{On} \to \mathrm{On}$ defined by:

$$F'(0) := \mu\beta(F(\beta) = \beta)$$
$$F'(\alpha + 1) := \mu\beta(F'(\alpha) < \beta\ \wedge\ F(\beta) = \beta)$$
$$F'(\lambda) := \mu\beta(\forall_{\alpha<\lambda}(F'(\alpha) < \beta)\ \wedge\ F(\beta) = \beta),$$

where λ is a limit ordinal and $\mu\beta(\phi(\beta))$ denotes the minimum ordinal satisfying the formula ϕ. Clearly, F' enumerates the fixed points of F, in other words the $\operatorname{rng}(F')$ is the class of the fixed points of F. Show that F' is also normal.

9. A closed and unbounded class of ordinals is called a *club*. Show that if $F : \mathrm{On} \to \mathrm{On}$ the following are equivalent:

   (i) F is normal.
   (ii) The $\operatorname{rng}(F)$ is a club.

   Using the normality of the derivative F' of some normal function F, we conclude that the class of fixed points of a normal function is a club.

10. Show that a club is a proper class.

11. If C_1, C_2 are clubs, then their intersection $C_1 \cap C_2$ is also a club. Hence, if F_1, F_2 are normal functions, the class of their common fixed points is a club. This, of course, applies to the normal functions F, F'.

12. Suppose that $F, G : \mathrm{On} \to \mathrm{On}$ are normal functions. Then the following hold:

   (i) Their composition $F \circ G$ is a normal function.
   (ii) If $\mathrm{Fix}(F)$ denotes the club of the fixed points of a normal function F, then

   $$\mathrm{Fix}(F \circ G) = \mathrm{Fix}(F) \cap \mathrm{Fix}(G).$$

13. If F is normal, then

   $$\lambda \text{ is a limit ordinal} \to F(\lambda) \text{ is a limit ordinal.}$$

14. If F is normal, then

   $$\forall_\beta \exists_\alpha (F(\alpha) \leq \beta \ \wedge \ \forall_\gamma (F(\gamma) \leq \beta \to \gamma \leq \alpha)),$$

   i.e., given an ordinal β and a normal function F, then $F(\alpha)$ is the best approximation to β below that one can give using F.

15. Using 7 and 14, try to sketch a graph modeling the graph of a normal function.

16. Show that the function $\omega : \mathrm{On} \to \mathrm{On}$ defined by

   $$\alpha \mapsto \omega_\alpha$$

   is normal and also that $\operatorname{rng}(\omega) = \mathrm{Kn} \setminus \omega$, where Kn denotes the class of cardinals.

17. Show that the class of limit ordinals LOn is a club.

VI. Cofinality

If λ is a limit ordinal its *cofinality* is defined by

$$\mathrm{cf}(\lambda) := \min\{|c| \mid c \subseteq \lambda \wedge \bigcup c = \lambda\}.$$

If α is an ordinal and X is a set, a *sequence* $(x_\xi)_{\xi<\alpha}$ *in* X *indexed by* α, or an α*-sequence of* X, is a function $x : \alpha \to X$. An ω-sequence of X is a called simply a sequence. We can describe the cofinality of λ by

$$\mathrm{cf}(\lambda) = \min\{\alpha \in \mathrm{On} \mid \exists x : \alpha \to \mathrm{On} \text{ strictly monotone } \wedge \lim_{\xi \to \alpha} x_\xi = \lambda\},$$

where $\lim_{\xi\to\alpha} x_\xi = \sup_{\xi<\alpha} x_\xi$. It is clear then why

a. $\mathrm{cf}(\omega+\omega)=\omega$.
b. $\mathrm{cf}(\omega^2)=\omega$.
c. $\mathrm{cf}(\omega_\omega)=\omega$.
d. $\mathrm{cf}(\omega_{\alpha+\omega})=\omega$, for every $\alpha\in\mathrm{On}$.

Some basic properties are the following:

1. A countable limit ordinal has cofinality ω.
2. $\omega\leq\mathrm{cf}(\lambda)$.
3. $\omega_{\alpha+1}$ is regular, for every $\alpha\in\mathrm{On}$.
4. $\kappa\leq|V_k|$, for every $\kappa\in\mathrm{Kn}$.
5. $\omega_\alpha\leq|V_\alpha|$, for every $\alpha\in\mathrm{On}$ such that $\alpha>\omega$.
6. If $\kappa=\omega_\alpha$, then
a. $c\subseteq\kappa\to|c|<\mathrm{cf}(\kappa)\to c$ is bounded in k.
b. $\lambda<\mathrm{cf}(\kappa)\to f:\lambda\to\kappa\to\mathrm{rng}(f)$ is bounded in k.

VII. Inner models

1. A class W is an *inner* model of ZF, if

(IM_1) W is transitive.
(IM_2) $\mathrm{On}\subseteq W$.
(IM_3) $\forall_{\phi\in\mathrm{Ax(ZF)}}(\mathrm{ZF}\vdash\phi^W)$, where ϕ^W has been defined inductively.

The word "inner" derives from (IM_3), since we need to show that the relativisations ϕ^W of the axioms of ZF are provable in ZF itself.

2. V is an inner model of ZF.
3. In the following W denotes an inner model of ZF.
4. $\mathrm{ZF}\vdash\theta\to\mathrm{ZF}\vdash\theta^W$.
5. $\mathrm{ZF}\vdash\theta^W\to$ ZF is consistent $\to\mathrm{ZF}+\theta$ is consistent.

This is the main use of the inner models; in order to show the consistency of $\mathrm{ZF}+\theta$ it suffices, given the consistency of ZF, to show that $\mathrm{ZF}\vdash\theta^W$, for some appropriate inner model of ZF. For example, take $\theta=\mathrm{AC}$ and $W=\mathrm{HOD}$.

6. $\phi(\vec{x})$ is called *absolute* for W, if $\forall_{\vec{x}\in W}(\phi(\vec{x})\leftrightarrow\phi^W(\vec{x}))$.
7. If $\phi(\vec{x})$ is Σ_0, then $\phi(\vec{x})$ is absolute for W.
8. If $\phi(\vec{x})$ is Σ_1, then $\forall_{\vec{x}\in W}(\phi(\vec{x})\leftarrow\phi^W(\vec{x}))$.
8. If $\phi(\vec{x})$ is Π_1, then $\forall_{\vec{x}\in W}(\phi(\vec{x})\rightarrow\phi^W(\vec{x}))$.
9. The following relations and functions are equivalent to Σ_0-formulas:

a. $x\subseteq y$.
b. $z=\{x,y\}$.
c. $z=\{x\}$.
d. $z=(x,y)$.
e. $z=\emptyset$.
f. $z=x\cup y$.
g. $z=x\cap y$.
h. $z=x\setminus y$.
i. $z=S(x)$.
k. z is transitive.
l. $z=\bigcup x$.
m. $z=\bigcap x$, where $\bigcap\emptyset=\emptyset$.
n. z is an ordinal.
o. z is a limit ordinal.
p. z is a successor ordinal.

10. Absolute notions are closed under composition: if $\theta(\vec{x}), F(\vec{x})$ and $G_i(\vec{y})$, where $1\leq i\leq|\vec{x})|$,

are absolute for W, then so is the formula

$$\phi(G_1(\vec{y}), \ldots, G_n(\vec{y})),$$

and the function

$$F(G_1(\vec{y}), \ldots, G_n(\vec{y})).$$

11. The following relations and functions are absolute for W:

a. z is an ordered pair.
b. $u \times v$.
c. r is a relation.
d. $z = \mathrm{dom}(r)$.
e. $z = \mathrm{rng}(r)$.
f. r is a function.
g. $z = f(x)$.
h. r is an injection.
i. r is a surection.
j. r is a bijection.

12. $\mathrm{ZF} \vdash A^W = \delta \leftrightarrow \mathrm{ZF} \vdash (A = \delta)^W$.
13. $\forall_{x \in W}(|x| \leq |x|^W)$.
14. $\forall_{\kappa \in \mathrm{Kn}}(\kappa^W \leq \kappa)$.
15. $\forall_{\kappa, \lambda \in \mathrm{Kn}} \forall_{x \in W}(|x| = \kappa \to |x|^W = \lambda^W \to \kappa \leq \lambda)$.
16. $\mathrm{cf}(\lambda) \leq \mathrm{cf}^W(\lambda)$.

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