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Blatt 10

Übungen zur Vorlesung "Modelle der Mengenlehre"

Carefully read each of the following statements, and decide whether it is true or false. Circle your answer accordingly. In case more than two answers (other than True or False) are given, circle the one you consider correct.

Each correct answer gives one point. Each false answer gives zero points. The optimal total sum is 40 points.

Exercise 1. a. The axiom of extensionality is the following formula

$$\forall_{x,y}(\forall_z(z \in x \leftrightarrow z \in y) \leftarrow x = y).$$

(True) (False)

b. If $A \subseteq B$ and $A \notin V$, then $B \notin V$.

(True)(False)

c. ZF $\vdash \forall x \forall y(x \notin y \ \vee \ y \notin x)$.

(True)(False)

d. Write down two proper classes with empty intersection.

Exercise 2. a. If u is transitive, then $u \cup \{u\}$ is transitive.

(True)(False)

b. If λ is a limit ordinal, then $\bigcup \lambda = \lambda$.

(True)(False)

c. Assume that A is a subset of the real numbers that is well ordered under the usual ordering on reals. Then A is countable.

(True)(False)

d. Define an ordering $<$ on the set of integers $\mathbb{Z}$ such that $(\mathbb{Z}, <)$ is isomorphic to $\omega + \omega$.

Exercise 3. **a.** Assume that $\forall \alpha \in \mathrm{On}\ (\forall \beta < \alpha(\phi(\beta)) \to \phi(\alpha))$. Then it holds $\forall \alpha \in \mathrm{On}\ (\phi(\alpha))$.

(True)(False)

b. If $F : \mathrm{On} \to \mathrm{On}$ is increasing i.e., $\forall_{\alpha,\beta\in\mathrm{On}}(\alpha < \beta \to F(\alpha) < F(\beta))$, then

$$\forall \alpha \in \mathrm{On}\ (F(\alpha) \geq \alpha).$$

(True)(False)

c. If C_1, C_2 are closed and unbounded classes of ordinals, their intersection is closed and unbounded.

(True)(False)

d. The function $F : \mathrm{On} \to \mathrm{On}$ defined by $\alpha \mapsto \alpha^2$ satisfies the condition $F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha)$, for every limit ordinal λ.

(True)(False)

Exercise 4. **a.** The set $\bigcup\{\{\omega\}\}$ is finite.

(True)(False)

b. If λ is a limit ordinal $> \omega$, then $V_\lambda \models \mathrm{ZF}$.

(True)(False)

c. $(\omega + 1) \cdot 2 = (\omega \cdot 2) + (1 \cdot 2)$.

(True)(False)

d. $2 \in 3$ and $2 + \omega \notin 3 + \omega$.

(True)(False)

Exercise 5. **a.** Let W be a class. If for every set x we have that $x \subseteq W \to x \in W$, then $W = V$.

(True)(False)

b. $V_{\omega\cdot 2} \models$ Replacement Scheme.

(True)(False)

c. "x is transitive" is equivalent to a Σ_0-formula.

(True)(False)

d. "x is a cardinal" is absolute.

(True)(False)

Exercise 6. **a.** $V = L$.

(i) True in ZF (ii) False in ZF (iii) Undecidable in ZF

b. Continuum Hypothesis.

(i) True in ZFC (ii) False in ZFC (iii) Undecidable in ZFC

c. There is a well-ordering on $(\bigcup \mathcal{P}(\omega))^{\mathrm{HOD}}$.

(i) True in ZF (ii) False in ZF (iii) Undecidable in ZF

d. $\forall\alpha \geq \omega\ (|L_\alpha| = |\alpha|)$.

(i) True in ZFC (ii) False in ZFC (iii) Undecidable in ZFC

Exercise 7. **a.** Con(ZF) $\to$ Con(ZF $+ V = L$).

(True)(False)

b. Con(ZF) $\to$ Con(ZF $+ V =$ HOD).

(True)(False)

c. $V = L \to \neg$AC.

(True)(False)

d. Con(ZF) $\to$ Con(ZFC + GCH).

(True)(False)

Exercise 8. Let $\mathcal{P}^\infty(y) = \{a \subseteq y \mid a \text{ is infinite}\}$. On $\mathcal{P}^\infty(\omega)$ we define the relation E by

$$xEy \leftrightarrow x \in \mathcal{P}^\infty(y).$$

The following are true in $< \mathcal{P}^\infty(\omega), E >$:

a. $x \in y \leftrightarrow x \subseteq y$.

(True)(False)

b. The Pairing Axiom.

(True)(False)

c. The Foundation Axiom.

(True)(False)

d. The Separation Axiom Scheme (Aussonderung).

(True)(False)

Exercise 9. Suppose that $<\mathbb{P}, \leq, \mathbb{1}>$ is a set of conditions included in a countable and transitive model M of ZFC.

a. If D is dense in $\mathbb{P}$, then $D \neq \emptyset$.

(True)(False)

b. If $p \in \mathbb{P}$ there is always some $G \subseteq \mathbb{P}$ such that G is M-generic and $p \in G$.

(True)(False)

c. For every $p \in \mathbb{P}$ and every M-generic G it holds $K_G(\{< \emptyset, p >\}) = \{\emptyset\}$.

(True)(False)

d. If $G \subseteq \mathbb{P}$ is M-generic, and $\sigma_1, \sigma_2, \sigma_3$ are $\mathbb{P}$-names in M, then

$$K_G(\{< \sigma_1, \mathbb{1} >, < \sigma_2, \mathbb{1} >, < \sigma_3, \mathbb{1} >\}) = \{K_G(\sigma_1), K_G(\sigma_2), K_G(\sigma_3)\}.$$

(True)(False)

Exercise 10. Suppose that M is a countable transitive model for ZFC, $\mathbb{P}$ is the set of all finite subsets of ω i.e.,

$$\mathbb{P} = \{p \subset \omega \mid p \text{ is finite}\},$$

while

$$p \leq q \leftrightarrow \exists_{n \in \omega} (q = p \cap n).$$

Let $G \subseteq \mathbb{P}$ be M-generic and Φ a name for $\bigcup G$.

a. $\emptyset \Vdash \Phi$ is infinite.

(True)(False)

b. $\emptyset \Vdash \Phi$ contains an odd number.

(True)(False)

c. $\{3, 9\} \Vdash 100 \notin \Phi$.

(True)(False)

d. $\{3, 9\} \Vdash 6 \notin \Phi$.

(True)(False)

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Besprechung. Donnerstag, 09. Juli 2015, in der Übung.