Limit Spaces with Approximations

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Sequential convergence in Topology

1. Intuitive, not enough, but still important.
2. Limit spaces and related notions capture the "sequential" part of topology.
3. A constructive theory of limit spaces is not elaborated so far.
4. How to add convergence in formal topology is still open.
5. Limit spaces are used in Computability at Higher Types (CHT).
6. Here we study limit spaces and their relation to CHT mostly within BISH.

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Limit Spaces

1. A **limit space**, or a Kuratowski limit space, or an $\mathcal{L}^*$-space, is a pair $\mathbb{L} = (X, \lim)$, where X is an inhabited set, and $\lim \subseteq X \times X^{\mathbb{N}}$ is a relation satisfying the following conditions:

   (L_1) If $x \in X$, then $\lim(x, x)$.

   (L_2) If $\mathcal{S}$ denotes the strictly monotone elements of the Baire space $\mathcal{N}$, then
   $$\forall_{\alpha \in \mathcal{S}}(\lim(x, x_n) \to \lim(x, x_{\alpha(n)})).$$

   (L_3) **Urysohn's axiom**: If $x \in X$ and $x_n \in X^{\mathbb{N}}$, then
   $$\forall_{\alpha \in \mathcal{S}} \exists_{\beta \in \mathcal{S}}(\lim(x, x_{\alpha(\beta(n))})) \to \lim(x, x_n).$$

2. $\mathbb{L}$ satisfies the **uniqueness property** (sequential Hausdorff), if

   (L_4) $$\forall_{x,y \in X} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \to \lim(y, x_n) \to x = y).$$

3. $\mathbb{L}$ satisfies the **weak uniqueness property**, if

   (L_5) $$\forall_{x,x' \in X}(\lim(x', x) \to x' = x).$$

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$\lim(R)$

Proposition (BISH)

Suppose that $R \subseteq X \times X^{\mathbb{N}}$. If $\lim(R) \subseteq X \times X^{\mathbb{N}}$ is defined inductively by the following clauses:

(i) $R \subseteq \lim(R)$ *and* $\{(x, x) \mid x \in X\} \subseteq \lim(R)$,
(ii) $\lim(R)(x, x_n) \to \lim(R)(x, x_{\alpha(n)})$, *for each* $\alpha \in \mathcal{S}$,
(iii) $\forall_{\alpha \in \mathcal{S}} \exists_{\beta \in \mathcal{S}} (\lim(R)(x, x_{\alpha(\beta(n))})) \to \lim(R)(x, x_n)$,

then $\lim(R)$ *is the smallest limit relation including* R.

Proposition (BISH)

There is a limit space satisfying the weak uniqueness but not the uniqueness property.

Proof.

Take $R = \{(x, x_n), (y, x_n)\}$, where $x, y \in X$ s.t. $x \neq y$ and x_n is a not eventually constant sequence in X. Take $\lim(R)$ to be the least limit relation including R defined as above satisfying the additional condition of the weak uniqueness property. □

$\mathcal{L}$-spaces

An $\mathcal{L}$-**space**, or a **pseudo-limit space** is a pair $(X, \lim)$ satisfying (L_1), (L_2) and (L_4).

Lemma (BISH)

Suppose that $(X, \lim)$ is a limit space, $x \in X$ and $x_n \in X^{\mathbb{N}}$ such that $\lim(x, x_n)$. If $\alpha \in \mathcal{N}$ such that $\alpha(n) > 0$ and ${x'}_n$ is the sequence defined by

$${x'}_{\alpha(k)} = \ldots = {x'}_{\alpha(k)+\alpha(k+1)-1} = x_k,$$

for each $k \geqslant 0$, then $\lim(x, {x'}_n)$.

An $\mathcal{L}$-space which is not a limit space: $(\mathbb{R}, \mathrm{rlim})$, where

$$\mathrm{rlim}(x, x_n) :\leftrightarrow \sum_{n \in \mathbb{N}} |x - x_n| < \infty.$$

Clearly, $\mathrm{rlim}(0, \frac{1}{2^n})$, while

$$\neg\mathrm{rlim}(0, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots).$$

If $(\mathbb{R}, \mathrm{rlim})$ was a limit space, we should have that the above sequence of finite repetitions converges to 0 too.

Filter Spaces

1. A **filter space**, or a Choquet space, is a pair $\mathbb{F} = (X, \mathrm{Lim})$, where X is an inhabited set, and $\mathrm{Lim} \subseteq X \times \mathcal{F}(X)$ is a relation satisfying the following conditions:

   (F_1) If $x \in X$ and $F_x = \{A \subseteq X \mid x \in A\}$, then $\mathrm{Lim}(x, F_x)$.

   (F_2) $F \subseteq G \to \mathrm{Lim}(x, F) \to \mathrm{Lim}(x, G)$.

   (F_3) $\forall_{G \supseteq F} \exists_{X \supsetneq H \supseteq G} (\mathrm{Lim}(x, H)) \to \mathrm{Lim}(x, F)$.

2. A **convergence space** is a pair $\mathbb{F} = (X, \mathrm{Lim})$ s.t. (F_1), (F_2) and

   (F_4) $\mathrm{Lim}(x, F) \to \mathrm{Lim}(x, G) \to \mathrm{Lim}(x, F \cap G)$.

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Spanier's quasi-topological spaces (1963)

A **quasi-topological space** is a structure $(X, (K \to X)_{K \in \mathbf{CHTop}})$, where **CHTop** is the category of compact Hausdorff spaces and for each $K, K', K_1, \ldots, K_n \in \mathbf{CHTop}$ the set of functions $K \to X \subseteq \mathbb{F}(K, X)$ satisfies the following conditions:

(QT_1) The constant function $\hat{x} \in K \to X$, for each $x \in X$.
(QT_2) $f \in K \to X \to g \in \mathbb{C}(K', K) \to g \circ f \in K' \to X$.
(QT_3) If $f_1 \in \mathbb{F}(K_1, K), \ldots, f_n \in \mathbb{F}(K_n, K)$ such that

$$\bigcup_{i=1}^{n} \mathrm{rng}(f_i) = K,$$

$$\forall_{g \in \mathbb{F}(K,X)} \forall_i (g \circ f_i \in K_i \to X),$$

then $g \in K \to X$.

Bishop's function spaces (1967)

1. A **function space** is a pair (X, F), where $F \subseteq \mathbb{F}(X, \mathbb{R})$, called the **topology**, satisfies the following clauses:

   (FS_1) The constant function $\hat{a} \in F$, for each $a \in \mathbb{R}$.
   (FS_2) $f, g \in F \to f + g, fg \in F$.
   (FS_3) $f \in F \to g \in \mathbb{C}(\mathbb{R}, \mathbb{R}) \to g \circ f \in F$.
   (FS_4) $f \in \mathbb{F}(X, \mathbb{R}) \to \forall_{\epsilon>0}\exists_{g \in F}\forall_{x \in X}(|f(x) - g(x)| \leqslant \epsilon) \to f \in F$.

2. $F(F_0)$ is the least topology including $F_0 \subseteq \mathbb{F}(X, \mathbb{R})$, defined like $\lim(R)$.

3. Bishop-Bridges 1985: This definition "should not be taken seriously. The purpose is merely to list a minimal number of properties that the set of all continuous functions in a topology should be expected to have. Other properties could be added; to find a complete list seems to be a nontrivial and interesting problem".

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The Birkhoff-Baer topology

1. A set $\mathcal{O} \subseteq X$ is lim-**open**, or $\mathcal{O} \in \mathcal{T}_{\mathrm{lim}}$, if

$$\forall_{x \in \mathcal{O}} \forall_{x_n \in X^{\mathbb{N}}} (\mathrm{lim}(x, x_n) \to \mathrm{ev}_{\mathcal{O}}(x_n)),$$

   where if $A \subseteq X$, we define

$$\mathrm{ev}_A(x_n) :\equiv \exists_{n_0} \forall_{n \geqslant n_0} (x_n \in A).$$

2. A set $F \subseteq X$ is called lim-**closed**, if it is the complement of a lim-open set, and in CLASS this is equivalent to

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3. A topological space $(X, \mathcal{T})$ induces a limit space $(X, \mathrm{lim}_{\mathcal{T}})$, where

$$\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \xrightarrow{\mathcal{T}} x.$$

4. A set $D \subseteq X$ is called lim-**dense**, if

$$\forall_{x \in X} \exists_{d_n \in D^{\mathbb{N}}} (\mathrm{lim}(x, d_n)).$$

5. (BISH) If D is lim-dense, then D is dense in $(X, \mathcal{T}_{\mathrm{lim}})$.
6. (CLASS) $\mathbb{I}$ is dense in $(\mathbb{R}, \mathcal{T}_{\mathrm{coc}})$, but it is not lim-dense in $(\mathbb{R}, \mathrm{lim}_{\mathcal{T}_{\mathrm{coc}}})$.

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Topological limit spaces and Sequential topological spaces

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$$\lim \supseteq \lim_{\mathcal{T}_{\lim}} .$$

2. Trivially, $\mathcal{T} \subseteq \mathcal{T}_{\lim_{\mathcal{T}}}$. A topological space is called **sequential**, if

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New limit spaces from old ones within BISH

Suppose that $(X, \lim)$ and $(Y, \lim)$ are limit spaces.

1. $(X \times Y, \lim)$ is the **product** limit space, where

$$\lim((x, y), (x_n, y_n)) :\leftrightarrow \lim(x, x_n) \ \wedge \ \lim(y, y_n).$$

2. $(X \to Y, \lim)$ is the **function** limit space, where

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3. If $A \subseteq X$, then $(A, \lim_A)$ is the **relative** limit space, where

$$\lim_A = \lim \cap (A \times A^{\mathbb{N}}).$$

4. If $f : (X, \lim) \to (Y, \lim)$ is lim-continuous, then $f : (X, \lim) \to (f(X), \lim_{f(X)})$ is lim-continuous.

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(BISH) If $(X, \mathcal{T}_{\lim})$ is a T_2-space, then $(X, \lim)$ has the uniqueness property.

The converse doesn't hold in general (Dudley 1964).

Proposition

(CLASS) *Suppose that* $(X, \lim)$ *is a limit space,* $(Y, \lim)$ *is a limit space with the uniqueness property,* D *is a* lim*-dense subset of* X*, and* $f, g : X \to Y$ *are* lim*-continuous functions. Then the following hold:*

(i) If $f_{|D} = g_{|D}$*, then* $f = g$.
(ii) If $f : (D, \lim_D) \to (Y, \lim)$ *is* lim*-continuous, then it has at most one* lim*-continuous extension to* X.
(iii) The set $Z(f, g) = \{x \in X \mid f(x) = g(x)\}$ *is* lim*-closed.*
(iv) The graph $\mathbb{G}_f$ *of* f *is* lim*-closed in* $(X \times Y, \lim)$.
(v) If f *is* 1-1*, then* $(X, \lim)$ *has the uniqueness property.*

Useful continuity facts

1. $A \subseteq X$ is called a lim-**retract** of X, if there is a lim-continuous function $r : X \to A$ such that $r(a) = a$, for each $a \in A$.
2. (BISH) If $(X, \lim), (Y, \lim)$ are limit spaces, A is a lim-retract of X and $f : (A, \lim_A) \to (Y, \lim)$ is lim-continuous, then f has a lim-continuous extension $F : (X, \lim) \to (Y, \lim)$.
3. If $F = f \circ r$, then F is lim-continuous as a composition of lim-continuous functions, and $F(a) = f(r(a)) = f(a)$, for each $a \in A$.
4. (BISH) If $(X, \lim)$ and $(Y, \lim)$ are limit spaces and $f : X \to Y$ is lim-continuous, then $f : (X, \mathcal{T}_{\lim}) \to (Y, \mathcal{T}_{\lim})$ is continuous.
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1. **Scarpellini** 1971: introduced the category of limit spaces over $\mathbb{N}$ with the discrete limit relation $\lim_d$ as a model for bar recursion of finite types.
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3. **Motivation**:
    1. "the weaker tools we use to obtain a result, the more extra knowledge can be obtained from the process of obtaining the result"
    2. "proofs are simpler"
    3. "the internally computable functions are defined from elements, relations and functions present in the structure, using acceptable operators that form new functions. The problem will be to decide what the acceptable operators are"
    4. "on the one hand one does not have to translate everything to the set of representatives, and on the other hand an internally defined object will always be well defined"
4. **Soundness criterion**: an internally computable object is externally computable.
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Normann's Program of Internal Computability (IC)

1. **Normann 1982**: "the **internal** concepts [of computability] must grow out of the structure at hand, while **external** concepts maybe inherited from computability over superstructures via, for example, enumerations, domain representations, or in other ways"
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The core of IC

1. Mainly limit spaces are used.
2. It is "useful to see how far we can get towards constructing an effective infrastructure on such spaces without introducing superstructures and imposing external notions of computability on the given structures ... One way to create a useful part of an infrastructure will be to **isolate a dense subset** that in some way is **effectively dense**.".
3. Although Normann is working in CLASS, we work mostly in BISH.
4. **Normann 2008** presented Kleene's countable functionals over $\mathbb{N}$ using limit spaces and the corresponding density theorem using the notion of the ***n*th approximation** of a functional, for each $n \in \mathbb{N}$.
5. Here we generalize Normann's presentation by defining two new subcategories of limit spaces, **Appr** and **Gappr** and connect them to later work of Normann.
6. Dense sets are very direct to find in **Appr** and **Gappr**.
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Limit spaces with approximations

1. A **limit space with approximations** is a structure $\mathbb{A} = (X, \lim, (\mathrm{Appr}_n)_{n \in \mathbb{N}})$ such that $(X, \lim)$ is a limit space, and, for each $n \in \mathbb{N}$ the **approximation functions** $\mathrm{Appr}_n : X \to X$ satisfy the following properties:

   (A_1) Appr_n is lim-continuous.
   (A_2) $\mathrm{Appr}_n(\mathrm{Appr}_m(x)) = \mathrm{Appr}_{\min(n,m)}(x)$, for each $x \in X$.
   (A_3) $D_n = \mathrm{Appr}_n(X) = \{\mathrm{Appr}_n(x) \mid x \in X\}$ is an inhabited finite set.
   (A_4) $\lim(x, x_n) \to \lim(x, \mathrm{Appr}_n(x_n))$, for each $x \in X$ and $x_n \in X^{\mathbb{N}}$.

2. **Corollary 1**: (A_5) $\mathrm{Appr}_n(\mathrm{Appr}_n(x)) = \mathrm{Appr}_n(x)$.
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4. **Corollary 3**: $\mathcal{B} = \{D_n \mid n \in \mathbb{N}\}$ is a countable filter base on X.
5. A structure $\mathbb{A}$ satisfying (A_5), (A_3) and (A_4) is a **limit space with general approximations**.

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Density theorem

Proposition

(BISH) If $\mathbb{A}$ is a limit space with (general) approximations and $x \in X$, then $\lim(x, \mathrm{Appr}_n(x))$, *and the set*

$$D = \bigcup_{n \in \mathbb{N}} D_n$$

is an enumerable dense subset of $(X, \mathcal{T}_{\lim})$.

Proof.

By (A_4), considering the constant sequence x, we get $\lim(x, x) \to \lim(x, \mathrm{Appr}_n(x))$ i.e., D is a lim-dense subset of X. Therefore, D is a dense subset of $(X, \mathcal{T}_{\lim})$. D is enumerable. □

Extension theorem

For each function f defined on D there is a sequence of lim-continuous functions which extend uniformly arbitrary big "parts" of f.

Proposition

(BISH) If $\mathbb{A}$ is a limit space with approximations, then

(i) Each set D_n is a lim*-retract of X.*
(ii) If $(Y, \lim)$ is a limit space, any lim*-continuous function $f_n : (D_n, \lim_{D_n}) \to (Y, \lim)$ has a* lim*-continuous extension $F_n : (X, \lim) \to (Y, \lim)$.*
(iii) If $f : (D, \lim_D) \to (Y, \lim)$ is lim*-continuous, then there is a sequence $(F_n)_n$ of* lim*-continuous functions $F_n : X \to Y$ such that, for each n,*

$$F_{n|D_n} = f_{|D_n} \quad \textit{and} \quad F_{n+1|D_n} = F_{n|D_n}.$$

Proof.

(i) Since each $\mathrm{Appr}_n : (X, \lim) \to (X, \lim)$ is lim-continuous, each $\mathrm{Appr}_n : (X, \lim) \to (D_n, \lim_{D_n})$ is lim-continuous too. Since any $a \in \mathrm{Appr}_n(X)$ has the form $\mathrm{Appr}_n(x)$, for some $x \in X$, we get that $\mathrm{Appr}_n(a) = \mathrm{Appr}_n(\mathrm{Appr}_n(x)) = a$.
(ii) Then a lim-continuous function $f_n : (D_n, \lim_{D_n}) \to (Y, \lim)$ has a lim-continuous extension F.
(iii) $f_n = f_{|D_n}$ is extended to a lim-continuous function $F_n : X \to Y$, and by $D_n \subseteq D_{n+1}$ we get that $F_{n+1|D_n} = F_{n|D_n}$, for each n. □

Product of limit spaces with approximations

Proposition

(BISH*) If* $(X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *and* $(Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *are limit spaces with (general) approximations, and if we define on* $X \times Y$

$$\mathrm{Appr}_n(x, y) := (\mathrm{Appr}_n(x), \mathrm{Appr}_n(y)),$$

for each n*, then* $(X \times Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *is a limit space with (general) approximations, where* lim *is the already defined* lim*-relation on* $X \times Y$*.*

Function space of limit spaces with approximations

Theorem

(BISH) If $(X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ and $(Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ are limit spaces with (general) approximations, and if we define, for each n and $f \in X \to Y$,

$$f \mapsto \mathrm{Appr}_n(f),$$

$$\mathrm{Appr}_n(f)(x) := \mathrm{Appr}_n(f(\mathrm{Appr}_n(x))),$$

for each $x \in X$, then $(X \to Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ is a limit space with (general) approximations, where lim *is the already defined* lim*-relation on $X \to Y$.*

On the proof of (A_3)

If $\mathrm{Appr}_n(Y)$ in inhabited, then $\mathrm{Appr}_n(X \to Y)$ is inhabited: If $y \in Y$, then for the constant function $\hat{y}$ we have that

$$\begin{aligned}\mathrm{Appr}_n(\hat{y})(x) &= \mathrm{Appr}_n(\hat{y}(\mathrm{Appr}_n(x))) \\ &= \mathrm{Appr}_n(y).\end{aligned}$$

Hence,

$$\mathrm{Appr}_n(\hat{y}) = \widehat{\mathrm{Appr}_n(y)}$$

If y inhabits $\mathrm{Appr}_n(Y)$, then the constant function $\mathrm{Appr}_n(\hat{y}) = \widehat{\mathrm{Appr}_n(y)} = \hat{y}$ inhabits $\mathrm{Appr}_n(X \to Y)$. I.e., in this case the nth approximation of $\hat{y}$ is identical to it. To prove the **finiteness of** $\mathrm{Appr}_n(X \to Y)$ we show that the nth-approximation of a function in the function limit space acts equally on its input and on the nth-approximation of it, since

$$\begin{aligned}\mathrm{Appr}_n(f)(\mathrm{Appr}_n(x)) &= \mathrm{Appr}_n(f(\mathrm{Appr}_n(\mathrm{Appr}_n(x)))) \\ &\overset{A_5}{=} \mathrm{Appr}_n(f(\mathrm{Appr}_n(x))) \\ &= \mathrm{Appr}_n(f)(x).\end{aligned}$$

Then $\mathrm{Appr}_n(f) : X \to Y$ is determined by its restriction

$$\mathrm{Appr}_n(f)_{|\mathrm{Appr}_n(X)} : \mathrm{Appr}_n(X) \to \mathrm{Appr}_n(Y)$$

$$\mathrm{Appr}_n(f)_{|\mathrm{Appr}_n(X)} = \mathrm{Appr}_n(g)_{|\mathrm{Appr}_n(X)} \to \mathrm{Appr}_n(f) = \mathrm{Appr}_n(g).$$

$$|\mathrm{Appr}_n(X \to Y)| \leqslant |\mathrm{Appr}_n(Y)^{\mathrm{Appr}_n(X)}|.$$

On the proof of (A$_4$)

$$\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}} (\lim(x, x_n) \to \lim(f(x), f_n(x_n)))$$

$$\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}} (\lim(x, x_n) \to \lim(f(x), \mathrm{Appr}_n(f_n)(x_n))).$$

We fix $x \in X$ and $x_n \in X^{\mathbb{N}}$ such that $\lim(x, x_n)$. By (A$_4$) on X we get that

$$\lim(x, x_n) \to \lim(x, \mathrm{Appr}_n(x_n)),$$

while by the definition of $\lim(f, f_n)$ on x and the sequence $\mathrm{Appr}_n(x_n)$ we have that $\lim(f(x), f_n(\mathrm{Appr}_n(x_n)))$. By (A$_4$) on Y we get that

$$\lim(f(x), \mathrm{Appr}_n(f_n(\mathrm{Appr}_n(x_n)))) \leftrightarrow \lim(f(x), \mathrm{Appr}_n(f_n)(x_n)).$$

The countable functionals over $\mathbb{N}$

$$\iota = \mathbb{N} \mid \rho \to \sigma,$$

$$\mathrm{Ct}(\iota) := (\mathbb{N}, \lim_{\mathcal{T}_{\mathrm{di}}}),$$

$$\mathrm{Ct}(\rho \to \sigma) := (\mathrm{Ct}(\rho) \to \mathrm{Ct}(\sigma), \lim_{\rho \to \sigma}),$$

To each limit space $(\mathrm{Ct}(\rho), \lim_{\rho})$ the following approximation functions are added:

$$\mathrm{Appr}_{n,\iota}(m) = \min(n, m),$$

while if $F \in \mathrm{Ct}(\rho \to \sigma)$ and $f \in \mathrm{Ct}(\rho)$ we define

$$F \mapsto \mathrm{Appr}_{n,\rho \to \sigma}(F),$$

$$\mathrm{Appr}_{n,\rho \to \sigma}(F)(f) = \mathrm{Appr}_{n,\sigma}(F(\mathrm{Appr}_{n,\rho}(f))).$$

Corollary 1

Corollary

(BISH) The structure $\mathbb{A}_\rho = (\mathrm{Ct}(\rho), \lim_\rho, (\mathrm{Appr}_{n,\rho})_{n\in\mathbb{N}})$ *is a limit space with approximations, for each* ρ. *Moreover, there exists an enumerable dense subset* D_ρ *in* $(\mathrm{Ct}(\rho), \mathcal{T}_{\lim_\rho})$, *for each* ρ.

Proof.

$\rho = \iota$: each Appr_n is $\lim_{\mathcal{T}_{\mathrm{di}}}$-continuous i.e., $\lim_{\mathcal{T}_{\mathrm{di}}}(m, m_l)$ implies that $\lim_{\mathcal{T}_{\mathrm{di}}}(\mathrm{Appr}_n(m), \mathrm{Appr}_n(m_l))$, since the hypothesis amounts to the sequence m_l being eventually the constant sequence m, therefore the sequence $\mathrm{Appr}_n(m_l)$ is eventually the constant sequence $\mathrm{Appr}_n(m)$.

$$\mathrm{Appr}_n(\mathbb{N}) = \{0, 1, \ldots, n\}.$$

Condition (iv) is written as $\lim_{\mathcal{T}_{\mathrm{di}}}(m, m_l) \to \lim_{\mathcal{T}_{\mathrm{di}}}(m, \mathrm{Appr}_l(m_l))$. Since the premiss says that the sequence m_l is after some index l_0 constantly m, then for $l \geqslant \max(l_0, m)$ we get that the sequence $\mathrm{Appr}_l(m_l)$ is constantly m.

The fact that $(\mathrm{Ct}(\rho \to \sigma), \lim_{\rho\to\sigma}, (\mathrm{Appr}_{n,\rho\to\sigma})_{n\in\mathbb{N}})$ is a limit space with approximations is a direct consequence of our Theorem. Moreover, by density theorem $D_\rho = \bigcup_{n\in\mathbb{N}} \mathrm{Appr}_n(\mathrm{Ct}(\rho))$ is an enumerable dense subset of $(\mathrm{Ct}(\rho), \mathcal{T}_{\lim_\rho})$, for each ρ. □

Approximation functions on the Cantor space $\mathcal{C}$

$$B(\overline{\alpha}(k)) = \{\beta \in \mathcal{C} \mid \overline{\alpha}(k) \prec \beta\}$$

is a countable base of a topology $\mathcal{T}$ on $\mathcal{C}$. The space $(\mathcal{C}, \mathcal{T})$ is a T_1, compact space with a countable base of clopen sets, and without isolated points. Consequently,

$$\begin{aligned}\lim_{\mathcal{T}}(\alpha, \alpha_n) &\leftrightarrow \forall_k \exists_{n_0} \forall_{n \geqslant n_0} (\alpha_n(k) = \alpha(k)) \\ &\leftrightarrow \forall_k \exists_{n_0} \forall_{n \geqslant n_0} (\overline{\alpha_n}(k) = \overline{\alpha}(k)),\end{aligned}$$

for each $\alpha \in \mathcal{C}$ and $\alpha_n \in \mathcal{C}^{\mathbb{N}}$. We define the approximation functions $\mathrm{Appr}_n : \mathcal{C} \to \mathcal{C}$ by

$$\alpha \mapsto \mathrm{Appr}_n(\alpha),$$

$$\mathrm{Appr}_n(\alpha) = \overline{\alpha}(n+1) * \overline{0}.$$

The functionals over the Cantor space $\mathcal{C}$

$$\iota = \mathcal{C} \mid \rho \to \sigma,$$

$$\mathcal{C}(\iota) := (\mathcal{C}, \lim_{\mathcal{T}}),$$

$$\mathcal{C}(\rho \to \sigma) := (\mathcal{C}(\rho) \to \mathcal{C}(\sigma), \lim_{\rho \to \sigma}),$$

and supply these spaces with the approximation functions $\mathrm{Appr}_{n,\iota}$ as defined above, and the arrow functions $\mathrm{Appr}_{n,\rho\to\sigma}$, we get the following corollary:

Corollary

(BISH) The structure $\mathbb{A}_\rho = (\mathcal{C}(\rho), \lim_\rho, (\mathrm{Appr}_{n,\rho})_{n\in\mathbb{N}})$ is a limit space with approximations, for each ρ. Moreover, there exists an enumerable dense subset D_ρ in $(\mathcal{C}(\rho), \mathcal{T}_{\lim_\rho})$, for each ρ.

Compact metric spaces in BISH

If (X, d) is a metric space, a set $Y \subseteq X$ is called an ϵ-**approximation** to X, if

$$\forall_{x \in X} \exists_{y \in Y} (d(x, y) < \epsilon).$$

A metric space (X, d) is **totally bounded**, if for each $\epsilon > 0$ there exists some $Y \subseteq X$ s.t. Y is a finite ϵ-approximation to X, and it is **compact**, if it is complete and totally bounded.

Lemma

(BISH) If (X, d) is an inhabited compact metric space and $r \in (0, \frac{1}{2}]$, there exist sequences $(x_u)_{u \in 2^{<\mathbb{N}}}$ and $\gamma \in \mathcal{S}$ such that, for each $n \geqslant 1$, we have that

(i) $\{x_u \mid |u| = \gamma(n)\}$ is an r^n-approximation to X.

(ii) $|u| = \gamma(n) \rightarrow \forall_{w \in 2^{<\mathbb{N}}} (d(x_u, x_{u*w}) < \frac{r^{n-1}}{1-r})$.

(iii) $|u| = \gamma(n) \rightarrow d(x, x_u) < r^{n-1} - r^{n+1} \rightarrow$

$$\rightarrow \exists_{w \in 2^{<\mathbb{N}}} (|u * w| = \gamma(n+1) \ \wedge \ d(x, x_{u*w}) < r^{n+1}).$$

(iv) $|u| = \gamma(n) \rightarrow |u * w| < \gamma(n+1) \rightarrow x_{u*w} = x_u$.

Note that $u * w$ denotes the concatenation of the finite sequences u, w, and that the proof of the above lemma uses for the definition of γ the principle of dependent choices on $\mathbb{N}$.

Approximation functions on a compact metric space

Proposition

If (X, d) is an inhabited compact metric space, $\lim_d$ is the limit relation induced by its metric d, and $\mathrm{Appr}_n : X \to X$ is defined, for each n, by

$$\mathrm{Appr}_n(x) = \begin{cases} x_{\min_{\prec}\{u \in 2^{<\mathbb{N}} \mid x_u \in \mathrm{Appr}_n(X) \ \wedge \ d(x, x_u) < r^n\}} & , \text{ if } x \notin \mathrm{Appr}_n(X) \\ x & , \text{ if } x \in \mathrm{Appr}_n(X), \end{cases}$$

where $\prec$ is any fixed total ordering on $2^{<\mathbb{N}}$, and

$$\mathrm{Appr}_n(X) = \{x_u \mid |u| = \gamma(n)\}$$

and the sequences $(x_u)_{u \in 2^{<\mathbb{N}}}$ and $\gamma \in \mathcal{S}$ are determined in the Lemma, then the structure $\mathbb{A} = (X, \lim_d, (\mathrm{Appr}_n)_{n \in \mathbb{N}})$ is a limit space with **general** *approximations.*

Proof

Proof.

The property $\mathrm{Appr}_n(\mathrm{Appr}_n(x)) = \mathrm{Appr}_n(x)$ follows automatically by the definition of $\mathrm{Appr}_n(x)$. The fact that $\mathrm{Appr}_n(X)$ is finite follows by the finiteness of the set of nodes in $2^{<\mathbb{N}}$ of fixed length $\gamma(n)$. Finally we show that $\lim_d(x, x_n) \to \lim_d(x, \mathrm{Appr}_n(x_n))$. The premiss is

$$\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geqslant n_0}(d(x, x_n) < \epsilon),$$

while the conclusion amounts to

$$\forall_{\epsilon>0}\exists_{n_0}\forall_{n\geqslant n_0}(d(x, \mathrm{Appr}_n(x_n)) < \epsilon).$$

We fix some $\epsilon > 0$, and by the unfolding of the premiss we find $n_0(\frac{\epsilon}{2})$ such that $d(x, x_n) < \frac{\epsilon}{2}$, for each $n \geqslant n_0(\frac{\epsilon}{2})$. Also, there is some n_1 such that $r^n < \frac{\epsilon}{2}$, for each $n \geqslant n_1$. For each $n \geqslant \max(n_0(\frac{\epsilon}{2}), n_1)$ we have that

$$\begin{aligned} d(x, \mathrm{Appr}_n(x_n)) &\leqslant d(x, x_n) + d(x_n, \mathrm{Appr}_n(x_n)) \\ &< \frac{\epsilon}{2} + r^n \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

□

The functionals over a compact metric space X

$$\iota = X \mid \rho \to \sigma,$$

$$X(\iota) := (X, \lim_d),$$

$$X(\rho \to \sigma) := (X(\rho) \to X(\sigma), \lim_{\rho \to \sigma}),$$

and add to these spaces the approximation functions $\mathrm{Appr}_{n,\iota}$ as defined above, and the arrow functions $\mathrm{Appr}_{n,\rho \to \sigma}$, we get directly by the fact that **Gappr** is cartesian closed the following corollary.

Corollary

The structure $\mathbb{A}_\rho = (X(\rho), \lim_\rho, (\mathrm{Appr}_{n,\rho})_{n \in \mathbb{N}})$ is a limit space with general approximations, for each ρ. Moreover, there exists an enumerable dense subset D_ρ in $(X(\rho), \mathcal{T}_{\lim_\rho})$, for each ρ.

Of course, we could use a type system where the base types are determined by more than one compact metric spaces and have a similar result similar.

Remark

(CLASS) *A metric space (X, d) is a sequential space.*

A corollary of Kisyński's theorem

Kisyński's theorem suffices to prove classically that all limit spaces in the above hierarchies are topological, since all of them satisfy the uniqueness property.

Corollary

(CLASS) (i) If $f : (X, \mathcal{T}_{\lim_X}) \to (Y, \mathcal{T}_{\lim_Y})$ is continuous and $(Y, \lim_Y)$ has the uniqueness property, then $f : (X, \lim_X) \to (Y, \lim_Y)$ is lim*-continuous.*
(ii) If $(X, \lim)$ is a limit space and $(Y, \lim_Y)$ has the uniqueness property, then

$$\mathbb{C}(X, Y) = X \to Y,$$

where $\mathbb{C}(X, Y)$ denotes the set of continuous functions from X to Y w.r.t. the topologies induced by the corresponding limits.

Functionals over separable and non-compact metric spaces

1. The use of probability distributions first in the study of hierarchies of functionals over $\mathbb{R}$ in Normann 2008 following the work of DeJaeger 2003.
2. We study the notion of a positive probabilistic projection adding the property of positivity to Normann's notion of probabilistic projection.
3. The probabilistic projections proved to exist by Normann are actually positive ones.
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Probabilistic projections

Suppose that $(Y, \mathcal{T})$ is a **sequential** topological space, A_n is an inhabited finite subset of Y, for each $n \in \mathbb{N}$, and

$$A = \bigcup_{n \in \mathbb{N}} A_n \subseteq X \subseteq Y.$$

A **probabilistic projection** from Y to X is a sequence of functions

$$\mu_n : Y \to \mathbb{F}(A_n, [0,1]) \quad y \mapsto \mu_n(y),$$

(P_1) $\mu_n(y) : A_n \to [0,1]$ is a probability distribution on A_n, for each $n \in \mathbb{N}$ i.e., it satisfies the condition

$$\sum_{a \in A_n} \mu_n(y)(a) = 1.$$

(P_2) The function $\hat{a} : Y \to [0,1]$ defined by

$$y \mapsto \mu_n(y)(a)$$

is **continuous**, for each $a \in A_n$ and for each $n \in \mathbb{N}$.

(iii) For each $x \in X$, $x_n \subseteq X$ such that $\lim(x, x_n)$ and for each $a_n \subseteq A$ such that $a_n \in A_n$, for each $n \in \mathbb{N}$, we have that

$$\forall_n (\mu_n(x_n)(a_n) > 0) \to \lim_{\mathcal{T}}(x, a_n),$$

where $\lim_{\mathcal{T}}$ is the limit relation on X induced by the limit relation $\lim_{\mathcal{T}}$ on Y.

Positive probabilistic projections

1. We denote a probability projection by $\mathcal{P} = (Y, \mathcal{T}, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$, while the sequence of sets $(A_n)_{n\in\mathbb{N}}$ is called the **support** of $\mathcal{P}$.
2. We call a probabilistic projection from Y to X **general**, if conditions (P_1) and (P_3) are satisfied but not necessarily the continuity condition (P_2).
3. We call a (general) probabilistic projection from Y to X **positive**, if

$$\mu_n(a)(a) > 0,$$

for each $a \in A_n$ and for each $n \in \mathbb{N}$.

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for each $a \in A_n$ and for each $n \in \mathbb{N}$.

Next natural density theorem explains why Y is considered sequential. Without this hypothesis we can only conclude that A is lim-dense in $(X, \lim_{\mathcal{T}})$.

Proposition

Suppose that Y is a sequential space, X is a closed (or open) subspace of Y and $\mathcal{P} = (Y, \mathcal{T}, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a (general) probability projection from Y to X. Then A is dense in X with the relative topology.

Proof.

Since $\lim(x, x)$, by (P_3) we get $\lim(x, a_n)$, for some a_n such that $\mu_n(x)(a_n) > 0$, for each $n \in \mathbb{N}$. There is always such an a_n, since $\mu_n(x)$ is a probability distribution on A_n. Thus, A is $\lim_{\mathcal{T}'}$-dense in X, where $\mathcal{T}'$ is the relative topology of Y on X. Since a closed (or open) subspace of a sequential space is also sequential, A is $\mathcal{T}'$-dense. □

lim-probabilistic projections

A lim-**probabilistic projection** $\mathcal{P} = (Y, \lim, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ where $(Y, \lim)$ is a limit space, X, A are as above and the functions $(\mu_n)_{n\in\mathbb{N}}$ satisfy (P_1), (P_3) and

(P_4) The function $\hat{a} : Y \to [0,1]$ defined by $y \mapsto \mu_n(y)(a)$ is lim-continuous, for each $a \in A_n$ and for each $n \in \mathbb{N}$ i.e.,

$$\lim(y, y_m) \to \lim(\mu_n(y)(a), \mu_n(y_m)(a)),$$

Proposition

(BISH) (i) If $(Y, \mathcal{T}, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a (positive) probabilistic projection, then $(Y, \lim_{\mathcal{T}}, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a (positive) lim*-probabilistic projection.*

(BISH) (ii) If $(Y, \lim)$ is a topological limit space and $(Y, \lim, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a (positive) lim*-probabilistic projection, then $(Y, \mathcal{T}_{\lim}, X, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a (positive) probabilistic*

Probabilistic selections

1. A **probabilistic selection** on a sequential space Y is a probabilistic projection from Y to Y, and we denote it by $(Y, \mathcal{T}, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$.
2. Normann: "a probabilistic selection from a dense subset may replace the use of a continuous or even effective selection of a sequence from a dense subset converging to a given point, when such topological selections are impossible".
3. A lim-**probabilistic selection** on a limit space Y is a lim-probabilistic projection from Y to Y, and we denote it by $(Y, \lim, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$.
4. Next proposition shows the connection of positive probabilistic selections to limit spaces with general approximations.

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4. Next proposition shows the connection of positive probabilistic selections to limit spaces with general approximations.

Positive probabilistic selections and limit spaces with general approximations

Proposition

If $(Y, \lim_{\mathcal{T}}, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ *is a positive* lim*-probabilistic selection on* Y*, there are approximation functions* Appr_n *such that* $(Y, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *is a limit space with general approximations and* $\mathrm{Appr}_n(X) = A_n$*, for each* n*.*

Proof.

Suppose that each A_n is given with a fixed modulus of finiteness e_n. We define

$$\mathrm{Appr}_n(x) = \begin{cases} a_{i_0} & \text{, if } x \notin A_n \\ x & \text{, if } x \in A_n, \end{cases}$$

where

$$i_0 = \min\{j \in \mathbb{N} \mid e_n(a) = j \wedge \mu_n(x)(a) > 0\}.$$

Clearly, $\mathrm{Appr}_n(X) = A_n$ by the second case of the above definition. The condition $\mathrm{Appr}_n(\mathrm{Appr}_n(x)) = \mathrm{Appr}_n(x)$ is also satisfied by definition. Suppose next that $x \in X, x_n \subseteq X$ and $\lim(x, x_n)$. We also have that

$$\mu_n(x)(\mathrm{Appr}_n(x)) > 0,$$

since, if $x \notin A_n$, then by definition $\mu_n(x)(a_{i_0}) > 0$, while if $x \in A_n$, we have $\mu_n(x)(x) > 0$ by the positivity condition. Hence, $\mu_n(x_n)(\mathrm{Appr}_n(x_n)) > 0$ for each n. By (P_3) we conclude $\lim(x, \mathrm{Appr}_n(x_n))$. □

Conversely

Proposition

(i) (BISH) *A limit space with general approximations* $(X, \lim, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *induces a positive, general* lim*-probabilistic selection* $(X, \lim, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ *on* X.

(ii) (CLASS) *A limit space with approximations* $(X, \lim!, (\mathrm{Appr}_n)_{n\in\mathbb{N}})$ *that satisfies the uniqueness property induces a positive* lim*-probabilistic selection* $(X, \lim, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ *on* X.

Proof.

(i) We define $A_n = D_n = \mathrm{Appr}_n(X)$ and $x \mapsto \mu_n(x)$, where

$$\mu_n(x)(a) = \begin{cases} 1 & , \text{if } a = \mathrm{Appr}_n(x) \\ 0 & , \text{ow.} \end{cases}$$

Clearly $\mu_n(x)$ is a probability distribution on A_n, and, since $\mu_n(x_n)(a_n) > 0 \leftrightarrow a_n = \mathrm{Appr}_n(x_n)$, we get $\lim(x, x_n) \rightarrow \lim(x, a_n)$. Also, $\mu_n(a)(a) = 1 > 0$, since $a = \mathrm{Appr}_n(x)$, for some $x \in X$, therefore, $\mathrm{Appr}_n(a) = \mathrm{Appr}_n(\mathrm{Appr}_n(x)) = a$.

(ii) Suppose that $\lim(x, x_m)$ and $\mu_n(x)(a) = 1 \leftrightarrow a = \mathrm{Appr}_n(x)$. The sequence $(\mathrm{Appr}_n(x_m))_m$ is eventually constant a. Thus, $\mu_n(x_m)(a)$ is eventually constant 1. The case $a \neq \mathrm{Appr}_n(x)$ is treated similarly. $\square$

Existence of a probabilistic projection

Proposition

(CLASS) Suppose that (X, d) is a separable metric space where $A = \{a_1, a_2, \ldots,\}$ is a countable dense subset of X. If we define $A_n = \{a_1, \ldots, a_n\}$ and, for each $1 \leqslant j \leqslant n$,

$$\mu_n(x)(a_j) := \frac{(d(x, A_n) + 2^{-n}) \dot{-} d(x, a_j)}{\sum_{i=1}^{n}[(d(x, A_n) + 2^{-n}) \dot{-} d(x, a_i)]},$$

where $d(x, A_n) = \min\{d(x, a_i) \mid 1 \leqslant i \leqslant n\}$ is the distance of x from A_n and $a \dot{-} b := \max(a - b, 0)$, then $(X, \mathcal{T}_d, (A_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}})$ is a positive probabilistic selection on X.

Final comments

1. The product of lim-probabilistic selections does not preserve the continuity condition.
2. Normann 2009 defined $\mathcal{Q}$-**spaces**: sequential Hausdorff spaces with a countable pseudo-base of closed sets, to show that for semi-convex Y, and X, Y are $\mathcal{Q}$-spaces with probabilistic selection, then $X \to Y$ is a $\mathcal{Q}$-space with a probabilistic selection.
3. The existence of dense subsets in the product $X \times Y$ and the function spaces $X \to Y$ is direct by the fact that X, Y are limit spaces with general approximations. It suffices that X, Y are sequential spaces admitting positive lim-probabilistic selections.
4. Of course, the results of Normann on $\mathcal{Q}$-spaces are of independent interest and value.
5. The limit spaces with approximations are useful in the separable non-compact case too.
6. There are more related results but even more open questions.

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