Constructive Combinatorics of Dickson's Lemma

Iosif Petrakis
University of Munich
petrakis@math.lmu.de

April 3, 2018

Abstract

We study constructively the relations between the finite cases of Dickson's lemma. Although there are many constructive proofs of them, the novel aspect of our proofs is the extraction of a corresponding bound. We provide some new one-step unprovability results i.e., results of the form "a finite case of Dickson's lemma does not prove in one step a stronger case of it". Moreover, we study the infinite cases of Dickson's lemma from the point of view of constructive reverse mathematics. We work within Bishop's informal system of constructive mathematics BISH.

1 Introduction

1.1 The finite and infinite cases of a combinatorial theorem τ

According to [10], p.391, the basic propositions of (classical) combinatorics

> assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system.

The larger the subsystem is proven to be, the stronger the corresponding theorem is. Suppose that τ is a theorem of combinatorics asserting for a system S in a class of systems Σ the existence of a subsystem I of S that has property P, which generally S does not. In most cases property P is *hereditary*, i.e., if $I' \subseteq I$ and $P(I)$, then $P(I')$. If $|X|$ denotes the cardinality of a set X, $l \geq 1$ and ξ is a cardinal strictly larger than $\aleph_0$, usually the following finite and infinite cases of τ are considered.

1. The *finite* case $\tau(l)$: If $|S| \geq l$, there is $I \subseteq S$ such that $|I| = l$ and $P(I)$.
2. The *strong finite* case $\tau^*(l)$: There is $M(l) > 0$ such that if $l \leq |S| \leq M(l)$, there is $I \subseteq S$ such that $|I| = l$ and $P(I)$.
3. The *unbounded* case: If $|S| \geq \aleph_0$, then $\forall_{l \geq 1}(\tau(l))$.
4. The *infinite* case $\tau(\aleph_0)$: If $|S| \geq \aleph_0$, there is $I \subseteq S$ such that $|I| = \aleph_0$ and $P(I)$.
5. The *higher infinite* case $\tau(\xi)$: If $|S| \geq \xi$, there is $I \subseteq S$ such that $|I| = \xi$ and $P(I)$.

For the constructive study of such a combinatorial theorem τ a general pattern can be described.

a. The finite case $\tau()$ is constructively proved, although there are finite combinatoric propositions, like Friedman's Proposition B, which is provable only with the use of large cardinals (see [14] and [16]), or the proposition of Paris-Harrington, which is provable in second-order anlysis but not in Peano arithmetic, and also lacks a constructive proof[1].

b. In many cases a strong case $\tau^*(l)$ is also constructively proved. To find explicitly though, a bound for the strong case $\tau^*(l)$ is usually a difficult problem, and for many well-studied combinatorial theorem, like Higman's lemma, or Kruskal's theorem, the extraction of a bound $M(l)$ from a constructive proof of $\tau(l)$ is, to our knowledge, not yet known.

c.The unbounded case $\forall_{l \geq 1}(\tau(l))$ is generally constructively proved.

[1] On 2011, during a colloquium-talk at LMU, Veldman suggested to try to find such a proof.

d. The infinite case $\tau(\aleph_0)$ is not constructively provable, as one usually can provide a Brouwerian counterexample to it, or show that $\tau(\aleph_0)$ is constructively equivalent to some constructively unacceptable proposition, like the limited principle of omniscience LPO. It is possible though, to find a classically equivalent formulation of $\tau(\aleph_0)$, which admits a constructive proof (see e.g., the intuitionistic proof of the infinite Ramsey theorem in [6], or it's constructive proof in Type Theory in [25]). It is not uncommon that non-constructive proofs inspire, or have a constructive counterpart. E.g., minimal-bad-sequence-proofs of Higman's lemma, or of Dickson's lemma inspired corresponding constructive (inductive) proofs of them.

e. The higher infinite case $\tau(\xi)$ is generally beyond the scope of constructive combinatorics.

Often the proof of some case of τ is based on the use of a *repetitive argument*, that is on the repetition of the same proof-step for an appropriate number of times. In this way the power of repetition of a simple, single argument is revealed. Moreover, if a bound is extracted from the single proof-step, then a bound is extracted from the whole proof. Most of the proofs included in this paper are based on repetitive arguments. Although from such proofs we do not extract the best possible, or optimal bounds, we find them interesting because they are somehow "elementary".

1.2 The finite and infinite cases of Dickson's lemma

Dickson's lemma is the simplest theorem of the form "a certain quasi-order is a well-quasi-order", and it is connected to the the theory of Gröbner bases and the termination of Buchberger's algorithm for finding them (see [12] and [11]). This was one of the first examples of how a well-quasi-order can be used as a technique applied to program termination (for more on this see [25]). Here we present though, the finite and infinite cases of Dickson's lemma independently from the theory of well-quasi-orders. First we need a definition.

Definition 1.1. *If X, Y are sets, $\mathbb{F}(X, Y)$ denotes the set of functions from X to Y. Let $k \in \mathbb{N}$ such that $k \geq 1$, $\alpha_1, \ldots, \alpha_k \in \mathbb{F}(\mathbb{N}, \mathbb{N})$, $(i, j) \in \mathbb{N}^2$ such that $i < j$, and $I \subseteq \mathbb{N}$. The pair (i, j) is called a good pair of indices for $\alpha_1, \ldots, \alpha_k$, or $\alpha_1, \ldots, \alpha_k$ are called good on (i, j), if $\alpha_n(i) \leq \alpha_n(j)$, for every $n \in \{1, \ldots, k\}$. We say that $\alpha_1, \ldots, \alpha_k$ are good on I, or I is good for $\alpha_1, \ldots, \alpha_k$, if $\alpha_1, \ldots, \alpha_k$ are good on every pair of indices $(i, j) \in I^2$ such that $i < j$.*

If $k \geq 1$ and $l \geq 2$, the following finite and infinite cases of Dickson's lemma are usually considered.

1. DL(k, l): If $\alpha_1, \ldots, \alpha_k \in \mathbb{F}(\mathbb{N}, \mathbb{N})$, there exists $I_l = \{i_1 < i_2 < \ldots < i_l\} \subset \mathbb{N}$ such that $\alpha_1, \ldots, \alpha_k$ are good on I_l.
2. DL(k, ∞): If $\alpha_1, \ldots, \alpha_k \in \mathbb{F}(\mathbb{N}, \mathbb{N})$, there exists $I_\infty = \{i_1 < i_2 < \ldots < i_n < ß_{n+1} < \ldots\} \subseteq \mathbb{N}$ such that $\alpha_1, \ldots, \alpha_k$ are good on I_∞.
3. DL(k, U): If $\alpha_1, \ldots, \alpha_k \in \mathbb{F}(U, \mathbb{N})$, where U is an unbounded[2] subset of $\mathbb{N}$, there exists an unbounded subset I_U of U such that $\alpha_1, \ldots, \alpha_k$ are good on I_U.

If $\Sigma = \mathcal{P}(\mathbb{N})$, and $S = \mathbf{n}$, where $\mathbf{n} := \{0, \ldots, n-1\}$, or $S = \mathbb{N}$, and if $P(I)$, where $I \subseteq \mathbf{n}$, for some $n \in \mathbb{N}$, or $I \subseteq \mathbb{N}$, is the hereditary property defined as "the sequences $\alpha_1, \ldots, \alpha_k$ are good on I", then the cases DL(k, l) and DL(k, ∞) are special cases of a combinatorial theorem τ, for which no higher infinite case is meaningful.

Note that an infinite case DL$(\infty, 2)$ of DL$(k, 2)$ does not hold; if we consider the sequence of sequences $(\alpha_n)_{n=1}^{\infty}$, where for every $n \geq 1$ the sequence $\alpha_n \in \mathbb{F}(\mathbb{N}, \mathbb{N})$ is

$$\alpha_n = (n, n-1, \ldots, 1, n+1, n+2, n+3, \ldots),$$

we cannot find a pair of indices which is good for all α_n; If $n = \alpha_n(0)$, then $\alpha_n(n-1) = 1$, for every $n \geq 1$. Hence, if $i < j$, then $\alpha_{j+1}(i) > 1$, while $\alpha_{j+1}(j) = 1$, i.e., (i, j) cannot be a good pair for α_{j+1}.

[2] That is $\forall_{n \in \mathbb{N}} \exists_{m \in \mathbb{N}} (m > n \wedge m \in U)$.

This is a simple example of a finite combinatorial proposition the infinite case of which does not hold, even classically[3].

The original formulation of Dickson's lemma in [13] is equivalent to $\mathrm{DL}(k, U)$, which, as we show in section 4, is equivalent to $\mathrm{DL}(k, \infty)$ and cannot be constructively accepted. On the other hand, the finite case $\mathrm{DL}(k, l)$ has already a short constructive history. As Veldman and Bezem say in [6], p.210, it was John Burgess who, in a letter from 1983, asked for a constructive proof of $\mathrm{DL}(2, 2)$, which is shown to be a consequence of the intuitionistic Ramsey theorem in [6]. In [26] Veldman gave an elementary inductive, constructive proof of $\mathrm{DL}(k, 2)$, independently from the intuitionistic Ramsey theorem or some special intuitionistic principle. In [11] Coquand and Persson gave a constructive proof of an inductive version of $\mathrm{DL}(k, k)$. In [4] a program is extracted from a classical proof of $\mathrm{DL}(2, 2)$, by transforming the classical proof into a constructive one through a refined version of A-translation, and the proof is implemented in MINLOG (see also [5], [21], [24]). From the program extraction-point of view Dickson's lemma has been studied within systems like Mizar, Coq and ACL2 (see[23], [11], [19], respectively). In [17] Hertz proof-mined two classical proofs of $\mathrm{DL}(k, 2)$ using the Dialectica interpretation. We refer here only to direct constructive approaches to Dickson's lemma. Since the finite cases of Dickson's lemma follow easily from Higman's lemma, a constructive proof of the latter gives a constructive proof of the former (see [22]). In [2] it is shown that all finite cases of Dickson's lemma imply Higman's lemma for words of an alphabet with two letters.

The extraction of a bound for $\mathrm{DL}(k, l)$ i.e., the mining of a number $M_{\alpha_1,\ldots,\alpha_k}(l) > 0$ out of a proof of $\mathrm{DL}(k, l)$ such that $\{i_1 < \ldots < i_l\}$ is good for $\alpha_1, \ldots, \alpha_k$ and $i_l \leq M_{\alpha_1,\ldots,\alpha_k}(l)$ is, surprisingly, not well-studied (neither constructively nor classically). An exception to this is the work [3], where with the use of the finite pigeonhole principle a strong case of $\mathrm{DL}(2, 2)$ is shown. It doesn't seem possible though, to generalize this result to a method to prove strong cases of $\mathrm{DL}(k, 2)$, for $k > 2$.

The main results of this paper are the following.

1. Proposition 2.3, a strong case $\mathrm{DL}^*(1, l)$ of $\mathrm{DL}(1, l)$, for every $l \geq 3$.

2. Proposition 2.6, a strong case $\mathrm{DL}^*(2, 2)$ of $\mathrm{DL}(2, 2)$.

3. Proposition 2.7, a strong case $\mathrm{DL}^*(2, l)$ of $\mathrm{DL}(2, l)$, for every $l \geq 3$.

4. We explain how our proof of Proposition 2.7 generates a proof of a strong case $\mathrm{DL}^*(k, l)$ of $\mathrm{DL}(k, l)$, where $k > 2$ and $l \geq 3$, and how the latter together with the proof of Proposition 2.6 generate a proof of a strong case $\mathrm{DL}^*(k + 1, 2)$ of $\mathrm{DL}(k + 1, 2)$.

5. Theorem 3.2, a positive formulation of the non-existence of an one step-proof of $\mathrm{DL}(2, 2)$ from $\mathrm{DL}(1, l)$.

6. Theorem 3.4, a positive formulation of the non-existence of an one step-proof of $\mathrm{DL}(3, 2)$ from $\mathrm{DL}(2, 2)$.

7. Proposition 4.2 and Proposition 4.6, which express the (constructive) equivalence between $\mathrm{DL}(1, \infty)$ and LPO.

Results 5 and 6 are technically the more involved and are, as far as we know, together with result 7, new. They are motivated by Corollaries 3.3 and 3.5, respectively, which were conceived first.

We work within Bishop's informal system of constructive mathematics BISH (see [7], [8], [9]). A formal system that corresponds to BISH is CZF (see [1]) together with the principle of dependent choices (DC), or Myhill's system CST (see [20]).

2 Strong finite cases of Dickson's lemma

The strong form $\mathrm{DL}^*(1, 2)$ of $\mathrm{DL}(1, 2)$, although trivial, is essential to the description of a bound in all other strong cases $\mathrm{DL}^*(k, l)$ of $\mathrm{DL}(k, l)$ presented here.

[3] A deeper example is related to van der Waerden's theorem. According to it, if $\mathbb{N}$ is partitioned into two classes, then at least one of them contains arbitrarily long arithmetic progressions. But that does not imply that an infinite arithmetic progression in one of them exists (see [15], p.69).

Proposition 2.1 (DL$^*(1,2)$)**.** $\forall_{n\in\mathbb{N}}\forall_{\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})}(\alpha(0)\leq n\to\exists_{i<\alpha(0)+1}(\alpha(i)\leq\alpha(i+1)))$.

Proof. If $n=0$, then $\alpha(0)=0$, and $i=0$ is the required index. Next we suppose that $\forall_{\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})}(\alpha(0)\leq n\to\exists_{i<\alpha(0)+1}(\alpha(i)\leq\alpha(i+1)))$ and we show that $\forall_{\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})}(\alpha(0)\leq n+1\to\exists_{i<\alpha(0)+1}(\alpha(i)\leq\alpha(i+1)))$. Let $\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})$ such that $\alpha(0)\leq n+1$. If $\alpha(0)\leq n$, we use the inductive hypothesis. If $\alpha(0)=n+1$, then if $\alpha(0)\leq\alpha(1)$, we get $i=0$. If $\alpha(0)>\alpha(1)$, then $\alpha(1)\leq n$. By the inductive hypothesis on the sequence α^*, where $\alpha^*(n)=\alpha(n+1)$, for every $n\in\mathbb{N}$, there is $j<\alpha^*(0)+1=\alpha(1)+1$ such that $\alpha^*(j)\leq\alpha^*(j+1)$ i.e., $\alpha(j+1)\leq\alpha(j+2)$, and $i=j+1<(n+1)+1=\alpha(0)+1$. ☐

If $\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})$, we use the notation $M_\alpha(1,2):=\alpha(0)+1$ for the bound of DL$^*(1,2)$ that corresponds to α. It is immediate to see that $M_\alpha(1,2)$ is an optimal bound for DL$(1,2)$. The first part of the next simple corollary of DL$^*(1,2)$ expresses that for each sequence α we can find a good pair (i,j) for α such that $(j-i)$ is arbitrary large. For its last part recall that the lexicographic ordering $<_{\text{lex}}$ on $\mathbb{N}$ is defined by $(n_1,m_1)<_{\text{lex}}(n_2,m_2):\leftrightarrow(n_1<n_2)\vee(n_1=n_2\wedge m_1<m_2)$, for every $n_1,n_2,m_1,m_2\in\mathbb{N}$.

Corollary 2.2. *(i) For every $\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})$ and $n>0$*

$$\exists_{i\in\mathbb{N}}\left(i\leq\sum_{j=0}^{n-1}\alpha(j)\ \wedge\ \alpha(i)\leq\alpha(i+n)\right).$$

Moreover, the bound $\sum_{j=0}^{n-1}\alpha(j)$ is the best possible i.e., there exists a sequence α such that $\alpha(i)>\alpha(i+n)$, for every $i\leq M<\sum_{j=0}^{n-1}\alpha(j)$.
(ii) If $n\in\mathbb{N}$, there is no sequence $\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})$ such that $\forall_{k\in\mathbb{N}}(\alpha(k)<\alpha(k+1)\wedge\alpha(k)<n)$.
(iii) There exists no function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that

$$f(n_1,m_1)<f(n_2,m_2)\leftrightarrow(n_1,m_1)<_{\text{lex}}(n_2,m_2),$$

for every $n_1,n_2,m_1,m_2\in\mathbb{N}$.

Proof. (i) If $\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})$ and $n>0$, we consider the sequence $\beta\in\mathbb{F}(\mathbb{N},\mathbb{N})$ defined by

$$\beta(m)=\sum_{j<n}\alpha(m+j),$$

for every $m\in\mathbb{N}$. By DL$^*(1,2)$ there exists $i\leq\beta(0)=\sum_{j=0}^{n-1}\alpha(j)$ such that

$$\begin{aligned}\beta(i)\leq\beta(i+1)&\leftrightarrow\sum_{j<n}\alpha(i+j)\leq\sum_{j<n}\alpha(i+1+j)\\&\leftrightarrow\alpha(i)\leq\alpha(i+n).\end{aligned}$$

In order to show the optimality of the specified bound[4] consider, for an arbitrary $n>0$, any infinite sequence α extending the finite sequence $1,\underbrace{0,\ldots,0}_{n}$. Clearly, $\sum_{j=0}^{n-1}\alpha(j)=1$ and $\alpha(0)>\alpha(0+n)$, while $\alpha(1)\leq\alpha(1+n)$.
(ii) Suppose that such a sequence α exists, and consider any infinite extension of the finite sequence $\beta(0)=n_0>\beta(1)=\alpha(n_0)>\beta(2)=\alpha(n_0-1)>\ldots>\beta(n_0+1)=\alpha(0)$. By DL$^*(1,2)$ there exists $i<\beta(0)+1=n_0+1$ such that $\beta(i)\leq\beta(i+1)$, which contradicts the supposed strict monotonicity of α.
(iii) Suppose that such a function f exists. By the definition of $<_{\text{lex}}$ we get $(0,0)<_{\text{lex}}(0,1)<_{\text{lex}}(0,2)<_{\text{lex}}\ldots<_{\text{lex}}(0,n)<_{\text{lex}}\ldots<_{\text{lex}}(1,0)$, while by the supposed property of f we have $f(0,0)<f(0,1)<f(0,2)<\ldots<f(0,n)<\ldots<f(1,0)$, which is impossible by (ii). ☐

Note that by the unbounded case $\forall_{l\geq 2}(\text{DL}(1,l))$ we get that $\forall_{\alpha\in\mathbb{F}(\mathbb{N},\mathbb{N})}\forall_{n>0}\exists_{i,j\in\mathbb{N}}(j-i\geq n\wedge\alpha(i)\leq\alpha(j))$, since by DL$(1,n+1)$ there exist $i_1<\ldots<i_{n+1}$, such that $\alpha(i_1)\leq\ldots\leq\alpha(i_{n+1})$, therefore $i_{n+1}-i_1\geq n$, and $\alpha(i_1)\leq\alpha(i_{n+1})$. By Corollary 2.2(i) though, we "strongly" know that the distance between the elements of the good pair is exactly n.

[4]For $n=1$ we get $\sum_{j=0}^{n-1}\alpha(j)=\alpha(0)$, the optimal bound of DL$^*(1,2)$.

Proposition 2.3 (DL$^*(1,l)$)**.** *If $l \geq 3$ and $\alpha \in \mathbb{F}(\mathbb{N},\mathbb{N})$, there exist $i_1, i_2, \ldots, i_l$, and $M_\alpha(1,l) \in \mathbb{N}$ such that*

$$i_1 < i_2 < \ldots < i_l \leq M_\alpha(1,l) \quad \textit{and}$$

$$\alpha(i_1) \leq \alpha(i_2) \leq \ldots \leq \alpha(i_l),$$

where

$$M_\alpha(1,l) = \sum_{j=1}^{N} M_j,$$

$$N = \alpha(i^{(1)}) + 2,$$

$$M_1 = M_\alpha(1, l-1),$$

$$M_{j+1} = M_{\alpha^{(j)}}(1, l-1),$$

for every $j \in \{1,...,N-1\}$, and $M_\alpha(1,l-1)$ is the bound according to DL$^*(1,l-1)$ *on α, $\alpha^{(j)}$ is the tail of α starting from the index M_j, $M_{\alpha^{(j)}}(1,l-1)$ is the bound according to* DL$^*(1,l-1)$ *on the sequence $\alpha^{(j)}$, and $i^{(1)}$ is the index determined by the application of* DL$^*(1,l-1)$ *on α.*

Proof. Suppose first that $l = 3$. If we apply DL$^*(1,2)$ on α, we get an index $i^{(1)} \leq \alpha(0)$, such that $\alpha(i^{(1)}) \leq \alpha(i^{(1)}+1)$. We write $M_1 = M_\alpha(1,2) = \alpha(0)+1$. If we apply DL$^*(1,2)$ on the tail $\alpha^{(1)}$ of α starting from M_1, i.e., $\alpha^{(1)}(n) = \alpha(M_1+n)$, for every $n \in \mathbb{N}$, then we get an index $i^{(2)} \leq \alpha^{(1)}(0) = \alpha(M_1)$, such that $\alpha^{(1)}(i^{(2)}) \leq \alpha^{(1)}(i^{(2)}+1)$. We write $M_2 = \alpha(M_1)+1$. Repeating these steps $N = \alpha(i^{(1)})+2$ number of times we get indices $i^{(1)} < i^{(2)} < \ldots < i^{(N)}$, such that the application of DL$^*(1,2)$ on $\alpha(i^{(1)}), \alpha(i^{(2)}), \ldots, \alpha(i^{(N)})$ gives the existence of an index $i^{(k)}$, where $k \leq \alpha(i^{(1)})$, such that $\alpha(i^{(k)}) \leq \alpha(i^{(k+1)})$. By the definition of the indices $i^{(1)} < i^{(2)} < \ldots < i^{(N)}$ we conclude that

$$\alpha(i^{(k)}) \leq \alpha(i^{(k+1)}) \leq \alpha(i^{(k+1)}+1).$$

The initial segment of α required to find the indices $i^{(k)}, i^{(k+1)}$, and $i^{(k+1)}+1$ is $M_\alpha(1,3) = \sum_{i=1}^{N} M_i$, where $M_1 = \alpha(0)+1$, and for every $i \in \{1,...,N-1\}$ we have that $M_{j+1} = \alpha(M_j)+1$.
If $l > 3$, we show that

$$\text{DL}^*(1,l) \to \text{DL}^*(1,l+1)$$

by repeating N number of times the application of DL$^*(1,l)$ on the corresponding tails of α, exactly as in the $l = 3$ case. In this way we get indices $i_1^{(1)} < i_1^{(2)} < \ldots < i_1^{(N)}$, such that the application of DL$^*(1,2)$ on $\alpha(i_1^{(1)}), \alpha(i_1^{(2)}), ..., \alpha(i_1^{(N)})$ gives the existence of an index $i_1^{(k)}$, such that $\alpha(i_1^{(k)}) \leq \alpha(i_1^{(k+1)})$. By the definition of the indices $i_1^{(1)} < i_1^{(2)} < \ldots < i_1^{(N)}$ we conclude that

$$\alpha(i_1^{(k)}) \leq \alpha(i_1^{(k+1)}) \leq \alpha(i_2^{(k+1)}) \leq \ldots \leq \alpha(i_l^{(k+1)}).$$

$\square$

Within the above proof the rightmost pair of the indices on which α weakly increases is a pair of consecutive numbers. Generally, these indices are not consecutive. E.g.,

$$\alpha(n) = \begin{cases} 0 & \text{, if } n = 2k \\ 1 & \text{, if } n = 2k+1. \end{cases}$$

doesn't weakly increase on any triad of consecutive numbers.

Definition 2.4. *Let A be an inhabited set and $n \geq 1$. A coloring of A with n colors, or an n-coloring of A, is a function $\chi : A \to \boldsymbol{n}$. If $a_1, a_2 \in A$, the set $\{a_1, a_2\}$ is called a monochromatic pair under χ, if $\chi(a_1) = \chi(a_2)$. A subset B of A is called monochromatic under χ, if every two elements of B form a monochromatic pair. The notation* PH(n,m,l)*, where $n \in \mathbb{N}$ and $m, l \in \mathbb{N} \cup \{\mathbb{N}\}$, expresses that if χ is an n-coloring of a sequence of A of length m, then this sequence contains a monochromatic subsequence B of length l.*

Consequently, the case $\mathrm{PH}(2,\mathbb{N},l)$ of the pigeonhole principle, where $l \in \mathbb{N}$ and $l \geq 2$, says that if χ is a 2-coloring of $\{\alpha_n : n \in \mathbb{N}\} \subseteq A$, then $\{\alpha_n : n \in \mathbb{N}\}$ has a monochromatic subsequence of length l.

Proposition 2.5. $\forall_{l\geq 2}(\mathrm{DL}(1,l) \to \mathrm{PH}(2,\mathbb{N},l))$.

Proof. Suppose that $l \geq 2$, $\alpha \in \mathbb{F}(\mathbb{N},\mathbb{N})$ and χ is a 2-coloring of $\{\alpha_n : n \in \mathbb{N}\}$. By $\mathrm{DL}(1,l)$ on $\chi \circ \alpha : \mathbb{N} \to 2$ there are indices $i_1 < i_2 < ... < i_l$, such that $\chi(\alpha_{i_1}) \leq \chi(\alpha_{i_2}) \leq \ldots \leq \chi(\alpha_{i_l})$. If $\chi(\alpha_{i_l}) = 0$, therefore $\chi(\alpha_{i_1}) = \chi(\alpha_{i_2}) = \ldots = \chi(\alpha_{i_l}) = 0$, the sequence $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_l}$ is a monochromatic subsequence of α of length l. If $\chi(\alpha_{i_l}) = 1$, then we repeat the previous step on the tail $\alpha_{i_l+1}, \alpha_{i_l+2}, \ldots,$ of α. By $\mathrm{DL}(1,l)$, there are indices $n_1 < n_2 < ... < n_l$, such that $\chi(\alpha_{i_l+n_1}) \leq \chi(\alpha_{i_l+n_2}) \leq \cdots \leq \chi(\alpha_{i_l+n_l})$. If $\chi(\alpha_{i_l+n_l}) = 0$, then we get a monochromatic subsequence of α of length l. If $\chi(\alpha_{i_l+n_l}) = 1$, we repeat the same procedure. It suffices to repeat the above steps at most l number of times to find a monochromatic subsequence of α of length l. $\square$

It is easy to provide a bound for $\mathrm{PH}(2,\mathbb{N},l)$ based on the bounds determined by $\mathrm{DL}^*(1,l)$ on the sequences considered in the previous proof.

Proposition 2.6 ($\mathrm{DL}^*(2,2)$)**.** *If $\alpha, \beta \in \mathbb{F}(\mathbb{N},\mathbb{N})$, there exist i, j and $M_{\alpha,\beta}(2,2) \in \mathbb{N}$ such that*

$$i < j < M_{\alpha,\beta}(2,2), \quad \textit{and}$$

$$\alpha(i) \leq \alpha(j) \ \wedge \ \beta(i) \leq \beta(j)),$$

where

$$M_{\alpha,\beta}(2,2) = \sum_{j=1}^{K} M_j,$$

$$1 \leq K \leq N,$$

$$N = \alpha(i_1^{(1)}) + 2,$$

$$M_1 = M_\alpha(1,3),$$

$$\beta(i_1^{(1)}) \leq 1 \to K = 1,$$

$$\beta(i_1^{(1)}) \geq 2 \to M_2 = M_{\alpha^{(1)}}(1, \beta(i_1^{(1)}) + 1),$$

$i_1^{(1)}$ is the first index of the application of $\mathrm{DL}^*(1,3)$ *on α and $\alpha^{(1)}$ is the tail of α starting from index M_1. If $1 \leq j \leq N-1$, then*

$$\beta(i_1^{(j+1)}) \leq \beta(i_1^{(j)}) - 1 \to K = j+1,$$

$$\beta(i_1^{(j+1)}) \geq \beta(i_1^{(j)}) \to M_{j+1} = M_{\alpha^{(j)}}(1, \beta(i_1^{(j+1)}) + 1),$$

and $i_1^{(j+1)}$ is the first index of the application of $\mathrm{DL}^*(1, \beta(i_1^{(j)} + 1))$ *on $\alpha^{(j)}$, where $\alpha^{(j)}$ is the tail of α starting from index M_j.*

Proof. We show that

$$\forall_{l\geq 2}(\mathrm{DL}^*(1,l)) \to \mathrm{DL}^*(2,2),$$

hence by Proposition 2.3 we get a proof of $\mathrm{DL}^*(2,2)$. Applying $\mathrm{DL}^*(1,3)$ on α we find indices $i_1^{(1)} < i_2^{(1)} < i_3^{(1)}$, for which $\alpha(i_1^{(1)}) \leq \alpha(i_2^{(1)}) \leq \alpha(i_3^{(1)})$, based on the initial segment of α of length $M_1 = M_\alpha(1,3)$. We also consider the finite sequence $\beta(i_1^{(1)}), \beta(i_2^{(1)}), \beta(i_3^{(1)})$.

Suppose that $\beta(i_1^{(1)}) \leq 1$. If we form the sequence $\gamma(0) = \beta(i_1^{(1)}), \gamma(1) = \beta(i_2^{(1)}), \gamma(2) = \beta(i_3^{(1)})$ and extend it in any way we like, then, by $\mathrm{DL}^*(1,2)$ there exists $j \leq \gamma(0) \leq 1$, such that $\gamma(j) \leq \gamma(j+1) \leftrightarrow \beta(i_{j+1}^{(1)}) \leq \beta(i_{j+2}^{(1)})$, while $\alpha(i_{j+1}^{(1)}) \leq \alpha(i_{j+2}^{(1)})$ also holds. Hence, in case $\beta(i_1^{(1)}) \leq 1$, we can find a pair of indices for which $\mathrm{DL}^*(2,2)$ is satisfied, and then trivially $K = 1$.

If $\beta(i_1^{(1)}) = \mu \geq 2$, we consider the tail $\alpha^{(1)}$ of α which starts from index M_1. By $\mathrm{DL}^*(1,\mu+1)$ on $\alpha^{(1)}$ we find a finite sequence of indices $i_1^{(2)} < i_2^{(2)} < \ldots < i_{\mu+1}^{(2)}$, for which $i_1^{(1)} < i_2^{(1)} < i_3^{(1)} <$

$i_1^{(2)} < i_2^{(2)} < \ldots < i_{\mu+1}^{(2)}$, such that $\alpha^{(1)}(i_1^{(2)}) \leq \alpha^{(1)}(i_2^{(2)}) \leq \ldots \leq \alpha^{(1)}(i_{\mu+1}^{(2)})$. Of course, α also weakly increases on these indices. Considering $\beta(i_1^{(2)})$ we work as follows:

If $\beta(i_1^{(2)}) \leq \mu - 1$, then we can find the required pair of indices using $\mathrm{DL}^*(1,2)$. If $\beta(i_1^{(2)}) \geq \mu = \beta(i_1^{(1)})$, we repeat the previous step working with the tail $\alpha^{(2)}$ of α which starts from index $M_2 = M_{\alpha^{(1)}}(1, \beta(i_1^{(1)}) + 1)$.

If we are at step j, where $1 \leq j \leq N-1$, we find index $i_1^{(j+1)}$, which is the first index of the application of $\mathrm{DL}^*(1, \beta(i_1^{(j)} + 1))$ on $\alpha^{(j)}$, the tail of α starting from the index M_j.

If $\beta(i_1^{(j+1)}) \leq \beta(i_1^{(j)}) - 1$, then, by $\mathrm{DL}^*(1,2)$, the required pair of indices is found, and $K = j+1$.

If $\beta(i_1^{(j+1)}) \geq \beta(i_1^{(j)})$, we repeat the procedure at most $N = \alpha(i_1^{(1)}) + 2$ number of times. Then indices $i_1^{(1)} < i_1^{(2)} < \ldots < i_1^{(N)}$ will have been constructed for which, by the previous constructions, we have that

$$\beta(i_1^{(1)}) \leq \beta(i_1^{(2)}) \leq \ldots \leq \beta(i_1^{(N)}).$$

Applying $\mathrm{DL}^*(1,2)$ on any extension of the finite sequence $\alpha(i_1^{(1)}), \alpha(i_1^{(2)}), \ldots, \alpha(i_1^{(N)})$, we find a pair of indices on which α weakly increases. Since β already weakly increases on them, we have found the required pair based on an initial segment of α, β of length at most $M = \sum_{j=1}^{K} M_j$. $\square$

Proposition 2.7 ($\mathrm{DL}^*(2,l), l \geq 3$)**.** *If $l \geq 3$ and $\alpha, \beta \in \mathbb{F}(\mathbb{N}, \mathbb{N})$, there exist $i_1, i_2, \ldots, i_l$, and $M_{\alpha,\beta}(2,l) \in \mathbb{N}$ such that*

$$i_1 < i_2 < \ldots < i_l \leq M_{\alpha,\beta}(2,l) \quad \textit{and}$$
$$\alpha(i_1) \leq \alpha(i_2) \leq \ldots \leq \alpha(i_l),$$
$$\beta(i_1) \leq \beta(i_2) \leq \ldots \leq \beta(i_l),$$

where

$$M_{\alpha,\beta}(2,l) = \sum_{k=1}^{K} M^{(k)},$$
$$1 \leq K \leq \Lambda,$$
$$\Lambda = \alpha(i^{(1)}) + 1,$$

and $i^{(1)}$ is the first index determined by the application of $\mathrm{DL}^*(1,l)$ on the sequence $\alpha^*(n) = \alpha(i_n)$, where the indices i_n are formed as follows: i_1 is the first component of the common good pair resulting from the application of $\mathrm{DL}^*(2,2)$ on α, β requiring the initial segment of α, β of length $M_1 = M_{\alpha,\beta}(2,2)$, and i_{n+1} is the first component of the common good pair resulting from the application of $\mathrm{DL}^*(2,2)$ on $\alpha^{(n)}, \beta^{(n)}$, which are the tails of α, β starting from index M_n. Moreover,

$$M^{(1)} = \sum_{j=1}^{N_1} M_j^{(1)},$$

$$M_1^{(1)} = M_{\alpha,\beta}(2,l),$$
$$M_{j+1}^{(1)} = M_{\alpha^{(j)},\beta^{(j)}}(2,l),$$

while

$$\beta(i_1^{(1)}) \leq 1 \rightarrow K = 1,$$
$$\beta(i_1^{(1)}) \geq 2 \rightarrow M^{(2)} = \sum_{j=1}^{N_2} M_j^{(2)},$$

where $N_2 = M_{\alpha^{*(1)}}(1, \beta(i^{(1)}) + 1)$, $\alpha^{*(1)}$ is the tail of α^* starting from index i_{N_1}, $M_1^{(2)} = M_{\alpha_2^{(1)},\beta_2^{(1)}}(2,l)$, $M_{j+1}^{(2)} = M_{\alpha_2^{(j)},\beta_2^{(j)}}(2,l)$, where $\alpha_2^{(1)}, \beta_2^{(1)}$ are the tails of α, β starting from index $M^{(1)}$ and $\alpha_2^{(j)}, \beta_2^{(j)}$ are the tails of α, β, respectively, starting from index $M_j^{(2)}$. If $2 \leq k \leq \Lambda - 1$, then $M^{(k+1)}$ is defined through $M_j^{(k)}$'s, $1 \leq j \leq N_k$, in a similar way.

Proof. For simplicity we show here only the case $l = 3$. Applying $\mathrm{DL}^*(2, 2)$ on α, β using their initial segment of length $M_1^{(1)} = M_{\alpha,\beta}(2, 2)$ we find a common good pair of indices (i_1, j_1) for them. Then we apply $\mathrm{DL}^*(2, 2)$ on $\alpha^{(1)}, \beta^{(1)}$, the tails of α, β starting from index M_1, using the initial segment of them of length $M_2^{(1)} = M_{\alpha^{(1)},\beta^{(1)}}(2, 2)$, and we find a common good pair of indices (i_2, j_2) for them. We repeat this procedure enough number of times so that the sequences $\alpha^*(n) = \alpha(i_n)$, $\beta^*(n) = \beta(i_n)$ reach a common good pair of indices (i_s, i_t) for them. Then (i_s, i_t, j_t) is the required good triplet for α, β. In order to find the pair (i_s, i_t) we need to repeat the initial procedure so many times so that for the sequences α^*, β^* we can find a common good pair of indices. It is clear that M, Λ and $i^{(1)}$ as defined above for the case $l = 3$ determine the bound which corresponds to this proof. □

The formulation of $\mathrm{DL}^*(3, 2)$ has a complexity similar to that of the formulation of $\mathrm{DL}^*(2, 3)$, while its proof follows the pattern of the proof of $\mathrm{DL}^*(2, 2)$. If α, β, γ are given sequences, then applying $\mathrm{DL}^*(2, 3)$ on α, β using their initial segment of length $M_1 = M_{\alpha,\beta}(2, 3)$ we find indices $i_1^{(1)} < i_2^{(1)} < i_3^{(1)} \leq M_1$, such that both α and β weakly increase on them. If $\gamma(i_1^{(1)}) \leq 1$, we are done, while if not, we apply $\mathrm{DL}^*(2, \mu + 1)$ on $\alpha^{(1)}, \beta^{(1)}$, the tails of α, β starting from the index M_1, where $\mu = \gamma(i_1^{(1)})$. Let $i_1^{(2)}$ be the first index of this application. If $\gamma(i_1^{(2)}) \leq \mu - 1$ we stop, while if $\gamma(i_1^{(1)}) \geq \mu$ we repeat the procedure. At any step, either we have found the required pair, or the sequence $\gamma(i_1^{(1)}) \leq \gamma(i_1^{(2)}) \leq \dots$, is formed. Our algorithm of finding the required pair terminates with bound $M = M_{\alpha,\beta,\gamma}(3, 2)$, where M is the bound within which sequences $\alpha(i_1^{(1)}), \alpha(i_1^{(2)}), \dots$, and $\beta(i_1^{(1)}), \beta(i_1^{(2)}), \dots$, have a common good pair of indices. Consequently, this is a good pair for γ too. To determine M we work in a completely similar way to the determination of $M_{\alpha,\beta}(2, 3)$.

If $k > 3$, the formulations of $\mathrm{DL}^*(k, 2)$, and of $\mathrm{DL}^*(k, l)$, for every $l \geq 3$, are similar to the formulations of $\mathrm{DL}^*(3, 2)$ and of $\mathrm{DL}^*(3, k)$, respectively. The general proofs

$$\mathrm{DL}^*(k, 2) \to \mathrm{DL}^*(k, l),$$

and

$$\forall_{l\geq 2}(\mathrm{DL}^*(k, l)) \to \mathrm{DL}^*(k + 1, 2)$$

are similar to the proofs of Propositions 2.7 and the proof of $\mathrm{DL}^*(3, 2)$, respectively. Although we avoid here the cumbersome details of the general case, we may conclude the following regarding our proof of $\mathrm{DL}^*(k, l)$:

1. It is based on two simple repetitive arguments, a "horizontal" one, found in the proof of the implication $\mathrm{DL}^*(k, l) \to \mathrm{DL}^*(k, l + 1)$, and a "vertical" one, found in the proof of the implication $\forall_{l\geq 2}(\mathrm{DL}^*(k, l)) \to \mathrm{DL}^*(k + 1, 2)$. Both arguments depend on the simplest case $\mathrm{DL}^*(1, 2)$, something which is not the case in other constructive proofs of the finite cases of Dickson's lemma (e.g., like the ones in [26], [3]).

2. It provides a method to extract a bound $M_{\alpha_1,\dots,\alpha_k}(k, l)$ for $\mathrm{DL}(k, l)$.

3. Our proof of $\forall_{l\geq 2}(\mathrm{DL}^*(k, l)) \to \mathrm{DL}^*(k + 1, 2)$ is the constructive analogue of the constructively non-accepted proof
$$\mathrm{DL}(k, \infty) \to \mathrm{DL}(k + 1, l),$$
according to which one first applies the case $\mathrm{DL}(k, \infty)$ on $\alpha_1, \dots, \alpha_k$ to determine some $I_\infty \subseteq \mathbb{N}$, which is good for $\alpha_1, \dots, \alpha_k$, and then applies $\mathrm{DL}(1, l)$ on the subsequence of α_{k+1} determined by I_∞. Here we replaced $\mathrm{DL}(k, \infty)$ by $\forall_{l\geq 2}(\mathrm{DL}^*(k, l))$.

3 One-step unprovability results

The results included in this section are, as far as we know new, and they are motivated by our intuition that it is not possible to prove $\mathrm{DL}(k + 1, 2)$ from a finite number of cases $\mathrm{DL}(k, l)$ i.e., from "less information" than $\forall_{l\geq 2}(\mathrm{DL}(k, l))$. First we show that no single case $\mathrm{DL}(1, l)$ proves $\mathrm{DL}(2, 2)$ "directly in one step". We give a simple example to explain what we mean: if we define

$$(n_1, n_2) \leq (m_1, m_2)) :\leftrightarrow n_1 \leq m_1 \ \wedge \ n_2 \leq m_2,$$

for every $n_1, n_2, m_1, m_2 \in \mathbb{N}$, then

$$\nexists_{f \in \mathbb{F}(\mathbb{N}^2, \mathbb{N})} \forall_{n_1,n_2,m_1,m_2 \in \mathbb{N}} (f(n_1, n_2) \leq f(m_1, m_2) \rightarrow$$

$$(n_1, n_2) \leq (m_1, m_2)),$$

since, if there was such a function f, then $f(0,1) > f(1,0) > f(0,1)$. From this we conclude that $\mathrm{DL}(1,2)$ doesn't prove $\mathrm{DL}(2,2)$ in one step, since if there was such a function f and α, β are given sequences, by $\mathrm{DL}(1,2)$ on $(f(\alpha(n), \beta(n)))_n$ there are indices $i < j$ such that

$$f(\alpha(i), \beta(i)) \leq f(\alpha(j), \beta(j))$$

hence

$$(\alpha(i), \beta(i)) \leq (\alpha(j), \beta(j)).$$

A positive version of the above negation is the following, constructively stronger, formula:

$$\forall_{f \in \mathbb{F}(\mathbb{N}^2, \mathbb{N})} \exists_{n_1,n_2,m_1,m_2 \in \mathbb{N}} (f(n_1, n_2) \leq f(m_1, m_2) \wedge$$

$$(n_1, n_2) \not\leq (m_1, m_2)).$$

Next we prove constructively a strong form of this positive version, for arbitrary $l > 1$, concluding that no single case $\mathrm{DL}(1, l)$ can prove $\mathrm{DL}(2,2)$ in one step. In this way a "meta-mathematical" question leads to a positive mathematical fact. First we show the following lemma.

Lemma 3.1. *Let $M \in \mathbb{N}$, $l > 1$ and $\alpha, \beta \in \mathbb{F}(\mathbb{N}, \mathbb{N})$.*

$$\exists_{n_1 < n_2 < \ldots < n_l} (\alpha(n_1) = \ldots = \alpha(n_l) < M \vee$$

$$\beta(n_1) = \ldots = \beta(n_l) < M) \vee$$

$$\exists_{n,m \in \mathbb{N}} (M \leq \alpha(n) \leq \beta(m) \ \vee \ M \leq \beta(n) \leq \alpha(m)).$$

Proof. The number $K = (l-1)M+1$ is the bound on the length of a sequence colored with the M colors of $\{0, \ldots, M-1\}$ in order to have a monochromatic subsequence of length l (this simple case of the finite pigeonhole principle has an immediate inductive proof within BISH). If all the first K-terms of α are strictly smaller than M, or all the first K-terms of β are strictly smaller than M, then the conclusion follows immediately. Suppose that not all the first K-terms of α and not all the first K-terms of β are strictly smaller than M. The use of the principle of the excluded middle here is unproblematic as the related property is decidable. Hence, there are $n_1, m_1 < K$ such that $\alpha(n_1), \beta(m_1) \geq M$. We repeat the previous step on the tails $\alpha^{(1)}, \beta^{(1)}$ of α, β starting from $\alpha(K+1), \beta(K+1)$, respectively. Then again either the first K-terms of $\alpha^{(1)}$ are strictly smaller than M, or the first K-terms of $\beta^{(1)}$ are strictly smaller than M. If not there are numbers n_2, m_2 such that $K < n_2, m_2 < 2K$ and $\alpha(n_2), \beta(m_2) \geq M$. We repeat this procedure at most $\Lambda = (\alpha(n_1) + 1)$-number of times. If the first disjunct has not been proved, applying $\mathrm{DL}^*(1,2)$ on the sequence

$$\gamma(0) = \alpha(n_1), \ \gamma(1) = \beta(m_1),$$

$$\gamma(2) = \alpha(n_2), \ \gamma(3) = \beta(m_2), \ldots \ ,$$

we get an index $i < \Lambda$ such that $M \leq \alpha(n_i) \leq \beta(m_i) \ \vee \ M \leq \beta(m_i) \leq \alpha(n_{i+1})$. □

It is clear that the proof also works if $M = 0$, and that $n_1, n_2, \ldots, n_l, n, m \leq B = K(\alpha(n_1)+1)$ i.e., B is an extracted bound. If $\phi_1, \ldots, \phi_n$ are formulas, then $\bigwedge_{i=1}^n \phi_i$ ($\bigvee_{i=1}^n \phi_i$) denotes the conjunction (disjunction) of $\phi_1, \ldots, \phi_n$.

Theorem 3.2. *If $l > 1$ and $m \in \mathbb{N}$, then*

$$\forall_{f \in \mathbb{F}(\mathbb{N}^2, \mathbb{N})} \exists_{i_1, j_1, \ldots, i_l, j_l \in \mathbb{N}} \Bigg(\bigwedge_{s=1}^{l} (m \leq i_s) \wedge \bigwedge_{s=1}^{l} (m \leq j_s) \wedge$$

$$\bigwedge_{r=1}^{l-1}[f(i_r,j_r)\leq f(i_{r+1},j_{r+1})]\ \wedge$$

$$\bigwedge_{1\leq r<s\leq l}(i_r,j_r)\nleq(i_s,j_s)\Bigg).$$

Proof. First we show this for the cases $l=2,3$ and then we prove that the case $l-2$ implies the case l, for every $l>3$.

If $l=2$, then fixing m and applying $\mathrm{DL}^*(1,2)$ on the sequence

$$\alpha(0)=f(m+1,m),\ \alpha(1)=f(m,m+1),$$

$$\alpha(2)=f(m+2,m),\ \alpha(3)=f(m,m+2),\ldots\ ,$$

i.e.,

$$\alpha(2n)=f(m+(n+1),m),$$

$$\alpha(2n+1)=f(m,m+(n+1)),$$

we get $i<\alpha(0)+1$ such that $\alpha(i)\leq\alpha(i+1)$. If $i=2k$, for some $k\in\mathbb{N}$, then

$$f(m+k+1,m)\leq f(m,m+k+1),$$

and if $i=2k+1$, for some $k\in\mathbb{N}$, then

$$f(m,m+k+1)\leq f(m+k+2,m)$$

while

$$(m+k+1,m)\nleq(m,m+k+1),$$

$$(m,m+k+1)\nleq(m+k+2,m).$$

If $l=3$, we apply Lemma 3.1 on

$$M=f(m+1,m+1),$$

$$l=3,$$

$$\alpha(n)=f(m,m+n+1),$$

$$\beta(n)=f(m+n+1,m).$$

If there are $n_1<n_2<n_3$ such that $f(m,m+n_3+1)=f(m,m+n_2+1)=f(m,m+n_1+1)<M$, then

$$(m,m+n_3+1)\nleq(m,m+n_2+1)\ \wedge$$

$$(m,m+n_3+1)\nleq(m,m+n_1+1)\ \wedge$$

$$(m,m+n_2+1)\nleq(m,m+n_1+1).$$

If there are $n_1<n_2<n_3$ such that $\beta(n_3)=\beta(n_2)=\beta(n_1)<M$, we work similarly. Next we suppose that there exist indices i,j such that

$$f(m+1,m+1)\leq f(m,m+i+1)\leq f(m+j+1,m).$$

Again we conclude that

$$(m+1,m+1)\nleq(m,m+i+1)\ \wedge$$

$$(m+1,m+1)\nleq(m+j+1,m)\ \wedge$$

$$(m,m+i+1)\nleq(m+j+1,m).$$

If there exist indices i,j such that $f(m+1,m+1)\leq f(m+i+1,m)\leq f(m,m+j+1)$, we work similarly.

For the inductive step we fix f and we suppose that there exist $i_1, j_1, i_2, j_2, \ldots, i_{l-2}, j_{l-2}$ such that

$$m+1 \leq i_1, j_1, i_2, j_2, \ldots, i_{l-2}, j_{l-2} \ \wedge$$
$$\bigwedge_{r=1}^{l-3} [f(i_r, j_r) \leq f(i_{r+1}, j_{r+1})] \ \wedge$$
$$\bigwedge_{1 \leq r < s \leq l-2} (i_r, j_r) \nleq (i_s, j_s)).$$

Applying Lemma 3.1 on

$$M = f(i_{l-2}, j_{l-2}),$$
$$l,$$
$$\alpha(n) = f(m, m+n+1),$$
$$\beta(n) = f(m+n+1, m),$$

and working as in case $l = 3$, we reach the required conclusion for f. Note that if $1 \leq r \leq l-2$, then $(i_r, j_r) \nleq (m, m+i+1)$ and $(i_r, j_r) \nleq (m+j+1, m)$, since by our hypothesis $m+1 \leq i_r, j_r$. $\square$

The next corollary is an immediate consequence of Theorem 3.2 (the condition $m \leq i_1, j_1, i_2, j_2, \ldots, i_l, j_l$ in Theorem 3.2, which shows that many such l-tuples of natural numbers can be found, is not necessary to its proof).

Corollary 3.3. *If $l > 2$, then*

$$\nexists_{f \in \mathbb{F}(\mathbb{N}^2, \mathbb{N})} \forall_{i_1, j_1, \ldots, i_l, j_l \in \mathbb{N}} \Bigg(\bigwedge_{r=1}^{l-1} [f(i_r, j_r) \leq f(i_{r+1}, j_{r+1})] \rightarrow$$
$$\bigvee_{1 \leq r < s \leq l} (i_r, j_r) \leq (i_s, j_s) \Bigg).$$

Corollary 3.3 can be interpreted as the mathematical formulation of the expression "DL$(1, l)$ doesn't prove DL$(2, 2)$ in one step". If there was such a function f, and α, β are given sequences, applying DL$(1, l)$ on the sequence$(f(\alpha(n), \beta(n)))_n$ we would get indices $i_1 < \ldots < i_l$ such that

$$f(\alpha(i_1), \beta(i_1)) \leq f(\alpha(i_2), \beta(i_2)) \leq \ldots \leq f(\alpha(i_l), \beta(i_l)).$$

Then we would have

$$\bigvee_{1 \leq r < s \leq l} (\alpha(i_r), \beta(i_r)) \leq (\alpha(i_s), \beta(i_s)),$$

which by the constructive interpretation of disjunction implies DL$(2, 2)$. The inequality $(n_1, n_2, n_3) \leq (m_1, m_2, m_3)$ on $\mathbb{N}^3$ is defined, as in the case of $\mathbb{N}^2$, pointwisely.

Theorem 3.4.

$$\forall_{f_1, f_2 \in \mathbb{F}(\mathbb{N}^3, \mathbb{N})} \exists_{n_1, n_2, n_3, m_1, m_2, m_3 \in \mathbb{N}} \Bigg($$
$$f_1(n_1, n_2, n_3) \leq f_1(m_1, m_2, m_3) \ \wedge$$
$$f_2(n_1, n_2, n_3) \leq f_2(m_1, m_2, m_3) \ \wedge$$
$$(n_1, n_2, n_3) \nleq (m_1, m_2, m_3) \Bigg).$$

Proof. We suppose first that $f_1(1,0,0) = f_2(1,0,0) = 0$. Then $f_1(1,0,0) \leq f_1(0,m_2,m_3)$, $f_2(1,0,0) \leq f_2(0,m_2,m_3)$ and $(1,0,0) \nleq (0,m_2,m_3)$, for every $m_2, m_3 \in \mathbb{N}$.

Next we suppose that $f_1(1,0,0) = 0$ and $f_2(1,0,0) = l_2 > 0$. Clearly, if there are $m_2, m_3 \in \mathbb{N}$ such that $f_2(0,m_2,m_3) \geq l_2$, then $(1,0,0)$ and $(0,m_2,m_3)$ are the required triplets. Taking $L = l_2 + 1$ and $m > 0$ and applying Theorem 3.2 on L, m and the function $f(i,j) = f_1(0,i,j)$ we find indices $i_1, j_1, \ldots, i_L, j_L \geq m$ such that

$$\bigwedge_{r=1}^{l_2} [f_1(0,i_r,j_r) \leq f_1(0,i_{r+1},j_{r+1})] \wedge$$

$$\bigwedge_{1 \leq r < s \leq L} (i_r, j_r) \nleq (i_s, j_s)).$$

Next we consider the sequence $f_2(0,i_1,j_1), \ldots, f_2(0,i_L,j_L)$. Either there is a term $f_2(0,i_t,j_t)$, where $1 \leq t \leq L$, such that $f_2(0,i_t,j_t) \geq l_2$, which gives directly what we want to show, or all these L terms are numbers strictly smaller than l_2. But then there are two of them which are equal i.e., there exist $r < s$ such that

$$f_2(0,i_r,j_r) = f_2(0,i_s,j_s).$$

Clearly $(0,i_r,j_r)$ and $(0,i_s,j_s)$ are the required triplets. Note that both of them are non-zero triplets, since the indices determined by Theorem 3.2 were larger than m, and $m > 0$.

We call the previous two cases the *basic proof-step*, and the arguments used for them work for any fixed non-zero triplet (k_1,k_2,k_3) for which $f_1(k_1,k_2,k_3) = f_2(k_1,k_2,k_3) = 0$, or $f_1(k_1,k_2,k_3) = 0$ and $f_2(k_1,k_2,k_3) = l_2 > 0$. If, for example, $k_2 > 0$, we consider the function $f(n,m) = f_1(n,0,m)$.

Finally, we treat[5] the case $f_1(1,0,0) = l_1 > 0$ and $f_2(1,0,0) = l_2 > 0$. Without loss of generality we assume that $l_1 \leq l_2$. We consider the functions

$$g_i(k_1,k_2,k_3) = f_i(k_1,k_2,k_3) \dot{-} l_1,$$

where $\dot{-}$ is the modified subtraction and $i \in \{1,2\}$. Clearly, $g_1(1,0,0) = 0$ and $g_2(1,0,0) = l_2 - l_1 \geq 0$, hence by the previous basic proof-step there exist

$$(n_1,n_2,n_3), (m_1,m_2,m_3) \neq (0,0,0)$$

such that

$$\bigwedge_{i=1}^{2} (f_i(n_1,n_2,n_3) \dot{-} l_1) \leq (f_i(m_1,m_2,m_3) \dot{-} l_1) \wedge$$

$$(n_1,n_2,n_3) \nleq (m_1,m_2,m_3).$$

First we suppose that $f_i(n_1,n_2,n_3) \geq l_1$, for every $i \in \{1,2\}$, and we consider the following cases:

If $f_i(n_1,n_2,n_3) > l_1$, for every $i \in \{1,2\}$, then we get $f_i(m_1,m_2,m_3) > l_1$, and we conclude

$$\bigwedge_{i=1}^{2} (f_i(n_1,n_2,n_3) \leq f_i(m_1,m_2,m_3)) \wedge$$

$$(n_1,n_2,n_3) \nleq (m_1,m_2,m_3).$$

If $f_1(n_1,n_2,n_3) = l_1$ and $f_1(m_1,m_2,m_3) < l_1$, then we repeat the previous basic proof-step starting from the two values $f_2(m_1,m_2,m_3)$ and $f_1(m_1,m_2,m_3) < l_1$. If $f_1(n_1,n_2,n_3) = l_1$ and $f_1(m_1,m_2,m_3) \geq l_1$, then if $f_2(n_1,n_2,n_3) > l_1$, then $(n_1,n_2,n_3), (m_1,m_2,m_3)$ is the required pair of triplets, while if $f_2(n_1,n_2,n_3) = l_1$, we consider two cases: If $f_2(m_1,m_2,m_3) < l_1$, then we repeat the basic proof-step starting from the inequality $f_1(m_1,m_2,m_3)$ and $f_2(m_1,m_2,m_3) < l_1$. If $f_2(m_1,m_2,m_3) \geq l_1$, then $(n_1,n_2,n_3), (m_1,m_2,m_3)$ is the required pair of triplets.

[5]Classically this case has a simpler proof. Given functions f_1, f_2 either one of them is 0 on some non-zero triplet, or not. In the latter case let $\Lambda_i = \min\{f_i(n_1,n_2,n_3 \mid (n_1,n_2,n_3) \neq (0,0,0)\}$ and $\Lambda = \min\{\Lambda_1, \Lambda_2\}$. If we consider the functions $g_i(n_1,n_2,n_3) = f_i(n_1,n_2,n_3) - \Lambda$, there is a triplet on which one of them takes the value 0.

If $f_1(n_1,n_2,n_3) < l_1$ or $f_2(n_1,n_2,n_3) < l_1$, we repeat the basic proof-step starting from $f_1(n_1,n_2,n_3)$ and $f_2(n_1,n_2,n_3)$.

In each case either we find the required pair of triplets, or we find a starting triplet on which f_1 or f_2 has less value than at the starting triplet of the previous step. If we repeat the above steps at most l_1 number of times[6], we reach a basic proof-step, where f_1 or f_2 has on some non-zero triplet the value 0. $\square$

Corollary 3.5.

$$\nexists_{f_1,f_2\in\mathbb{F}(\mathbb{N}^3,\mathbb{N})}\forall_{n_1,n_2,n_3,m_1,m_2,m_3\in\mathbb{N}}\Bigg(f_1(n_1,n_2,n_3) \leq f_1(m_1,m_2,m_3) \wedge f_2(n_1,n_2,n_3) \leq f_2(m_1,m_2,m_3) \to (n_1,n_2,n_3) \leq (m_1,m_2,m_3)\Bigg).$$

The above immediate consequence of Theorem 3.4 can be interpreted as a mathematical formulation of the expression "DL$(2,2)$ doesn't prove DL$(3,2)$ in one step". If there were such functions f_1, f_2 and $\alpha_1,\alpha_2,\alpha_3 \in \mathbb{F}(\mathbb{N}^3,\mathbb{N})$ are given, then applying DL$(2,2)$ on

$$(f_1(\beta(n)))_n, \quad (f_2(\beta(n)))_n,$$

where, for each $n \in \mathbb{N}$,

$$\beta(n) = (\alpha_1(n), \alpha_2(n), \alpha_3(n)),$$

we would get indices $i < j$ such that

$$f_1(\alpha_1(i),\alpha_2(i),\alpha_3(i)) \leq f_1(\alpha_1(j),\alpha_2(j),\alpha_3(j)) \wedge$$

$$f_2(\alpha_1(i),\alpha_2(i),\alpha_3(i)) \leq f_2(\alpha_1(j),\alpha_2(j),\alpha_3(j))$$

which would imply

$$(\alpha_1(i),\alpha_2(i),\alpha_3(i)) \leq (\alpha_1(j),\alpha_2(j),\alpha_3(j)).$$

4 On the infinite cases of Dickson's lemma

In this section we study the infinite cases of Dickson's lemma from the point of view of constructive reverse mathematics (for more information on this subject see [18]). First we show the equivalence between the various infinite cases of Dickson's lemma.

Proposition 4.1. *If $k > 1$, the following are equivalent.*
(i) DL$(1,\infty)$.
(ii) DL(k,∞).
(iii) DL$(1,U)$.
(iv) DL(k,U).

Proof. (i) $\to$ (ii) DL$(1,\infty)$ is the first step in the inductive proof of DL(k,∞). It is also used in the proof of the inductive step $DL(k,\infty) \leftrightarrow DL(k+1,\infty)$. If $\alpha_1,\alpha_2,\ldots,\alpha_{k+1} \in \mathbb{F}(\mathbb{N},\mathbb{N})$, by DL$(k,\infty)$, there is a sequence $i_1 < i_2 < i_3 < \ldots$, such that $\alpha_m(i_1) \leq \alpha_m(i_2) \leq \alpha_m(i_3) \leq \ldots$, for evey $m \in \{1,2,\ldots,k\}$. If we apply DL$(1,\infty)$ on the sequence $\alpha_{m+1}(i_1), \alpha_{m+1}(i_2), \alpha_{m+1}(i_3), \ldots$, we get a weakly increasing subsequence of it. By hypothesis, the sequences $\alpha_1,\alpha_2,\ldots,\alpha_k$ weakly increase on its indices too.
The implication (ii) $\to$ (i) is trivial.
Next we show that (i) $\to$ (iii). With the use of the principle of dependent choices DC a sequence $s_0 < s_1 < \ldots < s_n < s_{n+1} < \ldots$, of elements of U is constructed. By DL$(1,\infty)$ on the sequence α^*,

[6]It is easy to extract a bound from this proof considering the bound of Theorem 3.2. Note also that the whole argument can be rephrased as an inductive one over the minimum of the values of f_1, f_2 on a non-zero triplet.

where $\alpha^*(n) = \alpha(s_n)$, for every $n \in \mathbb{N}$, a subsequence $(k(n))_{n\in\mathbb{N}}$ is formed on which α is good. But then α is also good on $M = \{s_{k(n)} : n \in \mathbb{N}\}$, and M is an unbounded subset of $\mathbb{N}$.
The equivalence (iii) $\leftrightarrow$ (iv) is shown as the equivalence (i) $\leftrightarrow$ (ii).
Finally we show that (iii) $\rightarrow$ (i). If we take $U = \mathbb{N}$, then by $\mathrm{DL}(1, U)$ there exists M unbounded subset of $\mathbb{N}$ such that $i < j \rightarrow \alpha(i) \leq \alpha(j)$, for every $i, j \in M$. With the use of DC a sequence $m_0 < m_1 < \ldots < m_n < m_{n+1} < \ldots$, is formed in M such that $\alpha(m_0) \leq \alpha(m_1) \leq \ldots \leq \alpha(m_n) \leq \alpha(m_{n+1}) \leq \ldots$. $\square$

In contrast to $\forall_{l\geq 2}(\mathrm{DL}(1, l))$, the infinite case $\mathrm{DL}(1, \infty)$ is not constructively acceptable. In [26] Veldman gave a Brouwerian counterexample to $\mathrm{DL}(1, \infty)$. Here we show its constructive equivalence to LPO, which is the following formula

$$\forall_{\alpha\in\mathbb{F}(\mathbb{N},\mathbf{2})}\bigg(\exists_{n\in\mathbb{N}}(\alpha(n) = 1) \vee \forall_{n\in\mathbb{N}}(\alpha(n) = 0)\bigg).$$

LPO is only classically true and a taboo for all varieties of constructive mathematics. Next we show that $\mathrm{DL}(1, \infty)$ implies LPO.

Proposition 4.2. $\mathrm{DL}(1, \infty) \rightarrow \mathrm{LPO}$.

Proof. We prove that if $\alpha \in \mathbb{F}(\mathbb{N}, \mathbf{2})$, then $\exists_{n\in\mathbb{N}}(\alpha(n) = 1) \vee \forall_{n\in\mathbb{N}}(\alpha(n) = 0)$, which is trivially equivalent to the original formulation of LPO. Applying $\mathrm{DL}(1, \infty)$ on α we get a sequence of indices $i_1 < i_2 < i_3 < \ldots$, such that $\alpha(i_1) \leq \alpha(i_2) \leq \alpha(i_3) \leq \ldots$. Note that if $\alpha(i_1) = 1$, then $\alpha(i_n) = 1$, for each $n \geq 1$. Through α we define a sequence $\beta \in \mathbb{F}(\mathbb{N}, \mathbf{2})$ by

$$\beta(n) = \begin{cases} 1 & \text{, if } \forall_{m\leq i_n}(\alpha(m) = 1) \\ 0 & \text{, if } \exists_{m\leq i_n}(\alpha(m) = 0). \end{cases}$$

By $\mathrm{DL}(1, \infty)$ on β, a sequence of indices $j_1 < j_2 < j_3 < \ldots$, is formed such that $\beta(j_1) \leq \beta(j_2) \leq \beta(j_3) \leq \ldots$. If $\beta(j_1) = 0$, then $\exists_{m\leq i_{j_1}}(\alpha(m) = 0)$, and the conclusion of LPO is reached. If $\beta(j_1) = 1$, then again $\beta(j_m) = 1$, for each $m \in \mathbb{N}$. In that case we show that $\forall_{n\in\mathbb{N}}(\alpha(n) = 1)$. Consider a fixed $n \in \mathbb{N}$. Then we can find $i_k > n$ and $j_l > k$. Since $\beta(j_l) = 1$, $\forall_{m\leq i_{j_l}}(\alpha(m) = 1)$. But $k < j_l$ implies that $n < i_k < i_{j_l}$, therefore $\alpha(n) = 1$. $\square$

In [21], p.148, Ratiu asked whether $\mathrm{DL}(k, U)$ implies LPO. By Propositions 4.1 and 4.2 we get an affirmative answer to this.

Proposition 4.3. *If $P(n)$ is a decidable predicate on $\mathbb{N}$, then* LPO $\rightarrow [\forall_{n\in\mathbb{N}}(P(n)) \vee \exists_{n\in\mathbb{N}}(\neg P(n))]$.

Proof. If we define

$$\alpha(n) = \begin{cases} 1 & \text{, if } \neg P(n) \\ 0 & \text{, if } P(n) \end{cases}$$

then LPO on α is exactly $\forall_{n\in\mathbb{N}}(P(n)) \vee \exists_{n\in\mathbb{N}}(\neg P(n))$. $\square$

Definition 4.4. *If $i \in \mathbb{N}$ and $\alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N})$, we call i a peak for α,* $\mathrm{Peak}_\alpha(i)$, *if and only if* $\forall_{n>i}(\alpha(i) > \alpha(n))$.

Proposition 4.5. *If $\alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N})$, then*

$$\mathrm{LPO} \rightarrow \forall_{i\in\mathbb{N}}\bigg(\mathrm{Peak}_\alpha(i) \vee \exists_{n>i}(\alpha(i) \leq \alpha(n))\bigg).$$

Proof. If $\mathbb{N}_{>i} = \{n \in \mathbb{N} : n > i\}$ and $e : \mathbb{N} \rightarrow \mathbb{N}_{>i}$ is the bijection defined by $e(n) = (n + 1) + i$, for every $n \in \mathbb{N}$, then for the decidable predicate

$$P_i(n) \leftrightarrow \alpha(e(n)) < \alpha(i) \leftrightarrow \alpha((n + 1) + i) < \alpha(i),$$

Proposition 4.3 gives

$$\forall_{n\in\mathbb{N}}(\alpha((n + 1) + i) < \alpha(i)) \vee \exists_{n\in\mathbb{N}}(\alpha((n + 1) + i) \geq \alpha(i)).$$

Therefore, either i is a peak for α, or there is an index after i of at least the same value as i under α, which is exactly what we need to prove. $\square$

Proposition 4.6. LPO $\to$ DL$(1,\infty)$.

Proof. Through the previous decidability of $\mathrm{Peak}_\alpha(i)$ we define a sequence $\beta \in \mathbb{F}(\mathbb{N}, \mathbf{2})$ by

$$\beta(n) = \begin{cases} 0 & , \text{ if } \exists_{m>n}(\alpha(n) \leq \alpha(m)) \\ 1 & , \text{ if } \mathrm{Peak}_\alpha(n) \end{cases}$$

By LPO, if $\forall_{n\in\mathbb{N}}(\beta(n) = 0) \leftrightarrow \forall_{n\in\mathbb{N}}\exists_{m>n}(\alpha(n) \leq \alpha(m))$, then, since 0 is positively not a peak for α, $\exists_{n_1>0}(\alpha(0) \leq \alpha(n_1))$. Similarly, $\exists_{n_2>n_1}(\alpha(n_1) \leq \alpha(n_2))$, and so on. By DC a sequence $0 = n_0 < n_1 < n_2 < \ldots$, is constructed such that $\alpha(n_0) \leq \alpha(n_1) \leq \alpha(n_2) \leq \ldots$. If $\exists_{n\in\mathbb{N}}(\beta(n) = 1) \leftrightarrow \exists_{n\in\mathbb{N}}(\mathrm{Peak}_\alpha(n))$, and if we consider the tail of α

$$\alpha(n+1), \alpha(n+2), \alpha(n+3), \ldots,$$

then

$$\alpha(j) \in \{0, 1, \ldots, \alpha(n) - 1\},$$

for every $j \geq n+1$. Since this tail of α is a new sequence, then either it has positively no picks, and the previous case is applied, or there is some index $n+m+1$ which is a peak for the sequence $\alpha(n+1), \alpha(n+2), \alpha(n+3), \ldots$. Since $\alpha(n+m+1) \in \{0, 1, \ldots, \alpha(n)-1\}$, then $\alpha(j) \in \{0, 1, \ldots, \alpha(n) - 2\}$, for every $j > n+m+1$. After at most $\alpha(n)$-1 number of steps we will have found a tail of α with no peaks. If we apply then the argument of the first case, we reach our conclusion. ☐

In analogy to Proposition 2.5 we show that DL$(1,\infty)$ implies Stolzenberg's principle PH$(2, \mathbb{N}, \mathbb{N})$.

Proposition 4.7. DL$(1,\infty) \to$ PH$(2, \mathbb{N}, \mathbb{N})$.

Proof. Suppose that $\alpha \in \mathbb{F}(\mathbb{N}, \mathbb{N})$ and that χ is a 2-coloring of $\{\alpha_n : n \in \mathbb{N}\}$. By DL$(1,\infty)$ on $\chi \circ \alpha : \mathbb{N} \to 2$ there are indices $i_1 < i_1 < i_3 < \ldots$, such that $\chi(\alpha_{i_1}) \leq \chi(\alpha_{i_2}) \leq \chi(\alpha_{i_3}) \leq \ldots$. Since DL$(1,\infty) \to$ LPO, either all terms of $[\chi(\alpha_{i_n})]_n$ are 0, or there is a term α_{i_n} such that $\chi(\alpha_{i_n}) = 1$. In the first case $(\alpha_{i_n})_n$ itself is monochromatic, while in the second the tail $\alpha_n, \alpha_{n+1}, \alpha_{n+2} \ldots$, of α is monochromatic. ☐

5 Concluding remarks

The extraction of a bound $M_{\alpha_1,\ldots,\alpha_k}(l)$ from our proof of DL(k,l) resembles the extraction of a term out of a proof in the field of program extraction. It is an example of term extracted in an informal system of mathematics, like BISH.

The following open questions, or tasks need to be addressed in future work.

1. To study further these terms $M_{\alpha_1,\ldots,\alpha_k}(l)$, since by Berger's constructive proof in [2] of Higman's lemma for words of an alphabet with two letters by the finite cases of Dickson's lemma, a bound for this case of Higman's lemma can be formulated.

2. Results like Proposition 2.3 have already been implemented in MINLOG. The implementation forced the inductive formulation of appropriate lemmas that cover the repetitive arguments used in the informal proofs. It will be interesting to codify formally the more complex repetitive arguments found in the rest constructive proofs presented here.

3. To extend the tools found in the proofs of Theorems 3.2 and 3.4 in order to prove these results in complete generality.

4. To extend our study of the finite and infinite cases of Dickson's lemma to a similar study of the finite and infinite cases of combinatorial theorems like Higman's lemma, or Kruskal's theorem.

References

[1] P. Aczel, and M. Rathjen: *Notes on Constructive Set Theory*, Book draft, 2010.

[2] J. Berger: Dickson's lemma and Higman's lemma are equivalent, South American Journal of Logic, to appear, 2016.

[3] J. Berger, and H. Schwichtenberg: A bound for Dickson's Lemma, Logical Methods in Computer Science, to appear, 2016.

[4] U. Berger, H. Schwichtenberg, and M. Seisenberger: The Warshall algorithm and Dickson's lemma: Two examples of realistic program extraction. Journal of Automated Reasoning, 26, 2001, 205-221.

[5] U. Berger, W. Buchholz, and H.Schwichtenberg: Refined program extraction from classical proofs, Annals of Pure and Applied Logic, 114, 2002, 3-25.

[6] M. Bezem, and V. Veldman: Ramsey's theorem and the pigeonhole principle in intuitionistic mathematics. J. London Math. Soc, 47, 1993, 193-211.

[7] E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.

[8] E. Bishop, and D. Bridges: *Constructive Analysis*, Grundlehren der Math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.

[9] D. S. Bridges, and F. Richman: *Varieties of Constructive Mathematics*, Cambridge University Press, 1987.

[10] H. Burkill, and L. Mirsky: Monotonicity, Journal of Mathematical Analysis and Applications, 41, 1973, 391-410.

[11] T. Coquand, and H. Persson: Gröbner Bases in Type Theory, in *TYPES'98*, T. Altenkirch et. al. (eds.), LNCS 1657, 1999, 33-46.

[12] D. Cox, J. Little, and D. O'Shea: *Ideal, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Undergraduate Texts in Mathematics. Springer Verlag, 1992.

[13] L. Dickson: Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. Am. J. Math, 35, 1913, 413-422.

[14] H. M. Friedman: Finite functions and the necessary use of large cardinals, Ann. of math. (2) 148, no.3, 1998. 803-893.

[15] R. L. Graham, B. L. Rotschild, and J. H. Spencer: *Ramsey Theory*, Wiley, 1980

[16] J. M. Harris, J. L. Hirst, and M. J. Mossinghoff: *Combinatorics and Graph Theory*, Springer, 2008.

[17] A. Hertz: *A constructive version of the Hilbert basis theorem*, Master's Thesis, Carnegie Mellon University, 2004.

[18] H. Ishihara: Reverse Mathematics in Bishop's Constructive Mathematics, Philosophia Scientiæ, 2006, 43-59.

[19] F. Martin-Mateos, J. Alonso, M. Hidalgo, and J. Ruiz-Reina: A Formal Proof of Dickson's Lemma in ACL2, in M. Y. Vardi, and A. Voronkov (Eds.) *10th International Conference on Logic for Programming Artificial Intelligence and Reasoning*, LNAI 2850, Springer-Verlag 2003, 49-58.

[20] J. Myhill: Constructive Set Theory, J. Symbolic Logic 40, 1975, 347-382.

[21] D. Ratiu: *Refinement of Classical Proofs for Program Extraction*, PhD Thesis, LMU, Munich 2011.

[22] F. Richman, and G. Stolzenberg: Well Quasi-Ordered Sets, Advances in Mathematics 97, 1993, 145-153.

[23] C. Schwarzweller: Groebner bases - theory refinement in the Mizar system, in *Lecture Notes in Artificial Intelligence*, volume 3863, Springer Verlag, 2005, 299-314.

[24] H. Schwichtenberg, and S. Wainer: *Proofs and Computations*, Cambridge University Press, 2011.

[25] D. Vytiniotis, T. Coquand, and D. Wahlstedt: Stop When You Are Almost-Full, Adventures in Constructive Termination, in L. Beringer and A. Felty (Eds.) ITP 2012, LNCS 7406, Springer-Verlag Berlin Heidelberg, 2012, 250-265.

[26] Wim Veldman: An intuitionistic proof of Kruskal's theorem, Archives of Mathematical Logic 43, 2004, 215-264.