Completely regular Bishop spaces

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Abstract. Bishop's notion of a function space, here called a Bishop space, is a constructive function-theoretic analogue to the classical set-theoretic notion of a topological space. Here we introduce the quotient, the pointwise exponential and the completely regular Bishop spaces. For the latter we present results which show their correspondence to the completely regular topological spaces, including a generalized version of the Tychonoff embedding theorem for Bishop spaces. All our proofs are within Bishop's informal system of constructive mathematics BISH.

1 Why Bishop spaces

The theory of Bishop spaces is so far the least developed approach to constructive topology with points. Bishop introduced them in [1], where he established their connection to his notion of neighborhood spaces, a set-theoretic constructive version of a topological space, and he defined the least Bishop space over a subbase, the product of Bishop spaces and a notion of a connected Bishop space. In [2], p.80, Bishop added some comments on them, while in [4] Bridges revived the subject, studying the morphisms between various metric spaces seen as Bishop spaces and relating Bishop spaces to apartness spaces. In [6] Ishihara related the subcategory **Fun** of the category **Bis** of Bishop spaces to the category of neighborhood spaces **Nbh**. In [8] we reported on our current development of the theory of Bishop spaces.
Our approach to topology is constructive, since we work within Bishop's informal system of constructive mathematics BISH, it is function-theoretic, since most of the notions involved are based on the concept of function, and we accept points from the beginning. Hence, the theory of Bishop spaces is an approach to constructive point-function topology, and its study is motivated by the following remarks:

(i) Function-based concepts are more suitable to constructive study than set-based ones. That's why Bishop, in [2] p.77, suggested to focus attention on Bishop spaces instead of on neighborhood spaces.
(ii) Bishop's topology of functions F corresponds to the ring of real-valued continuous functions $C(X)$ on a topological space X. This allows a direct "communication" between the two theories, which does not mean though, a direct translation, due to the classical set-theoretic character of $C(X)$.
(iii) The theory of Bishop spaces meets the standards of Bishop for a constructive mathematical theory: it has simple foundation and it follows the style of standard mathematics.

2 Basic definitions and facts

If X is an inhabited set, we denote by $\mathbb{F}(X)$ the set of all functions of type $X \to \mathbb{R}$, where $\mathbb{R}$ is the set of constructive reals. A constant function in $\mathbb{F}(X)$ with value $a \in \mathbb{R}$ is denoted by $\overline{a}$, and their set by $\mathrm{Const}(X)$. In BISH a compact metric space is a complete and totally bounded metric space, and a locally compact metric space is one in which every bounded subset is included in a compact one. If X is a locally compact metric space, $\mathrm{Bic}(X)$ denotes the subset of $\mathbb{F}(X)$ of all *Bishop-continuous* functions, where $\phi \in \mathrm{Bic}(X)$, if ϕ is uniformly continuous on every bounded subset of X. Since $\mathbb{R}$ with its standard metric is locally compact, $\mathrm{Bic}(\mathbb{R})$ denotes the Bishop-continuous functions on $\mathbb{R}$.
A *Bishop space* is a pair $\mathcal{F} = (X, F)$, where X is an inhabited set and $F \subseteq \mathbb{F}(X)$ satisfies the following conditions:
(BS_1) $\mathrm{Const}(X) \subseteq F$.
(BS_2) $f \in F \to g \in F \to f + g \in F$.
(BS_3) $f \in F \to \phi \in \mathrm{Bic}(\mathbb{R}) \to \phi \circ f \in F$.
(BS_4) $f \in \mathbb{F}(X) \to \forall_{\epsilon>0}\exists_{g\in F}\forall_{x\in X}(|g(x) - f(x)| \leq \epsilon) \to f \in F$.
Bishop used the term *function space* for $\mathcal{F}$ and *topology* for F. Since the former is used in many different contexts, we prefer the term Bishop space for $\mathcal{F}$, while we use the latter, since the *topology of functions* F on X corresponds nicely to the standard *topology of opens* $\mathcal{T}$ on X. A topology F is a ring and a lattice; by BS_2 and BS_3 if $f, g \in F$, then $f \cdot g$, $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$ and $|f| \in F$. The sets $\mathrm{Const}(X)$ and $\mathbb{F}(X)$ are topologies on X, called the *trivial* and the *discrete* topology, respectively. If F is a topology on X, then $\mathrm{Const}(X) \subseteq F \subseteq \mathbb{F}(X)$. It is straightforward that $\mathbb{F}_b(X) := \{f \in \mathbb{F}(X) \mid f \text{ is bounded}\}$ is a topology on X, and if $\mathcal{F} = (X, F)$ is a Bishop space, then $\mathcal{F}_b = (X, F_b)$ is a Bishop space, where $F_b = \mathbb{F}_b(X) \cap F$ corresponds to the ring $C^*(X)$ of the bounded elements of $C(X)$. If X is a locally compact metric space, it is direct to see that $\mathrm{Bic}(X)$ is a topology on X. The structure $\mathcal{R} = (\mathbb{R}, \mathrm{Bic}(\mathbb{R}))$ is the Bishop space of reals.
Most of the new Bishop spaces generated from old ones are defined through Bishop's inductive concept, found in [2] p.78, of the *least topology* $\mathcal{F}(F_0)$ generated by a given inhabited *subbase* F_0 of real-valued functions on X. Conditions BS_1-BS_4, seen as inductive rules, together with the rule $f_0 \in F_0 \to f_0 \in \mathcal{F}(F_0)$ induce the following induction principle $\mathrm{Ind}_{\mathcal{F}}$ on $\mathcal{F}(F_0)$:

$$
\begin{aligned}
&\forall_{f_0\in F_0}(P(f_0)) \to \\
&\forall_{a\in\mathbb{R}}(P(\overline{a})) \to \\
&\forall_{f,g\in\mathcal{F}(F_0)}(P(f) \to P(g) \to P(f+g)) \to \\
&\forall_{f\in\mathcal{F}(F_0)}\forall_{\phi\in\mathrm{Bic}(\mathbb{R})}(P(f) \to P(\phi\circ f)) \to \\
&\forall_{f\in\mathcal{F}(F_0)}(\forall_{\epsilon>0}\exists_{g\in\mathcal{F}(F_0)}(P(g) \ \wedge\ \forall_{x\in X}(|g(x)-f(x)| \leq \epsilon)) \to P(f)) \to \\
&\forall_{f\in\mathcal{F}(F_0)}(P(f)),
\end{aligned}
$$

where P is any property on $\mathbb{F}(X)$. Since the identity function $\mathrm{id}_{\mathbb{R}}$ on $\mathbb{R}$ belongs to $\mathrm{Bic}(\mathbb{R})$, we get that $\mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\mathrm{id}_{\mathbb{R}})$.

If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, their *product* is defined as the structure $\mathcal{F} \times \mathcal{G} = (X \times Y, F \times G)$, where

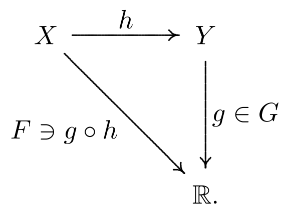
$$F \times G := \mathcal{F}(\{f \circ \pi_1 \mid f \in F\} \cup \{g \circ \pi_2 \mid g \in G\}).$$

It is direct to see that $\mathcal{F} \times \mathcal{G}$ satisfies the universal property for products and that $F \times G$ is the least topology which turns the projections π_1, π_2 into morphisms. If F_0 is a subbase of F and G_0 is a subbase of G, then using $\mathrm{Ind}_{\mathcal{F}}$ we get that

$$\mathcal{F}(F_0) \times \mathcal{F}(G_0) = \mathcal{F}(\{f_0 \circ \pi_1 \mid f_0 \in F_0\} \cup \{g_0 \circ \pi_2 \mid g_0 \in G_0\}).$$

Consequently, $\mathrm{Bic}(\mathbb{R}) \times \mathrm{Bic}(\mathbb{R}) = \mathcal{F}(\{\mathrm{id}_{\mathbb{R}} \circ \pi_1\} \cup \{\mathrm{id}_{\mathbb{R}} \circ \pi_2\}) = \mathcal{F}(\pi_1, \pi_2)$. If I is a given index set and $F_{0,i} \subseteq \mathbb{F}(X, \mathbb{R})$, for every $i \in I$, we define $\bigvee_{i \in I} F_{0,i} = \mathcal{F}(\bigcup_{i \in I} F_{0,i})$. If F_i is a topology on X_i, for every $i \in I$, the product topology on $\prod_{i \in I} X_i$ is defined by $\prod_{i \in I} F_i = \bigvee_{i \in I}(F_i \circ \pi_i)$, where $F_i \circ \pi_i = \{f \circ \pi_i \mid f \in F_i\}$. As expected, $\prod_{i \in I} \mathcal{F}(F_{0,i}) = \bigvee_{i \in I}(F_{0,i} \circ \pi_i)$. If $X_i = X$, for every $i \in I$, we use the notation $\mathcal{F}^I = (X^I, F^I)$. A *Euclidean* Bishop space is a product $\mathcal{R}^I$. Simplifying our notation, $\mathrm{Bic}(\mathbb{R})^I = \bigvee_{i \in I}(\mathrm{id}_{\mathbb{R}} \circ \pi_i) = \bigvee_{i \in I} \pi_i$.

If $\mathcal{F}, \mathcal{G}$ are Bishop spaces, a *Bishop morphism*, or simply a *morphism*, from $\mathcal{F}$ to $\mathcal{G}$ is a function $h : X \to Y$ such that $\forall_{g \in G}(g \circ h \in F)$



We denote the morphisms from $\mathcal{F}$ to $\mathcal{G}$ by $\mathrm{Mor}(\mathcal{F}, \mathcal{G})$. If $\mathrm{Const}(X, Y)$ denotes the constant functions from X to Y, then $\mathrm{Const}(X, Y) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Thus, the category **Bis** of Bishop spaces is formed with the Bishop morphisms as arrows. If $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ is onto Y, then h is called an *epimorphism*, and we denote their set by $\mathrm{Epi}(\mathcal{F}, \mathcal{G})$. If F is a topology on X, then clearly $F = \mathrm{Mor}(\mathcal{F}, \mathcal{R})$. If G_0 is a subbase of G, then using the induction principle we get that $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ iff $\forall_{g_0 \in G_0}(g_0 \circ h \in F)$, a fundamental property that we call the *lifting of morphisms*. We call a morphism h from $\mathcal{F}$ to $\mathcal{G}$ *open*, if $\forall_{f \in F} \exists_{g \in G}(f = g \circ h)$, and *strongly open*, if $\forall_{f \in F} \exists!_{g \in G}(f = g \circ h)$. Clearly, if $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$ such that h is 1-1 and onto Y, then $h^{-1} \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$ iff h is open. In this case h is called an *isomorphism* between $\mathcal{F}$ and $\mathcal{G}$.
Next we prove inductively the *lifting of openness*, a fundamental fact that we use here in the proof of the Theorem 4. First we need a lemma.

Lemma 1 (well-definability lemma). *Suppose that X, Y are inhabited sets and $h : X \to Y$ is onto Y. If $f : X \to \mathbb{R}$ such that for every $\epsilon > 0$ there exists some $g : Y \to \mathbb{R}$ such that $\forall_{x \in X}(|(g \circ h)(x) - f(x)| \leq \epsilon)$, then the function $\Phi : Y \to \mathbb{R}$ defined by $\Phi(y) = \Phi(h(x)) := f(x)$, for every $y \in Y$, is well-defined i.e., $\forall_{x_1, x_2 \in X}(h(x_1) = h(x_2) \to f(x_1) = f(x_2))$.*

Proof. We fix $x_1, x_2 \in X$ such that $h(x_1) = h(x_2) = y_0$, and some $\epsilon > 0$. By our hypothesis on f there exists some $g : Y \to \mathbb{R}$ such that $\forall_{x \in X}(|(g \circ h)(x) - f(x)| \leq \frac{\epsilon}{2})$. Hence, $|g(h(x_1)) - f(x_1)| = |g(y_0) - f(x_1)| \leq \frac{\epsilon}{2}$ and $|g(h(x_2)) - f(x_2)| = |g(y_0) - f(x_2)| \leq \frac{\epsilon}{2}$. Consequently, $|f(x_1) - f(x_2)| \leq |f(x_1) - g(y_0)| + |g(y_0) - f(x_2)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ is arbitrary, we get that $|f(x_1) - f(x_2)| \leq 0$, which implies that $f(x_1) = f(x_2)$.

Proposition 1 (lifting of openness). *If $\mathcal{F} = (X, \mathcal{F}(F_0))$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h \in \mathrm{Epi}(\mathcal{F}, \mathcal{G})$, then*

$$\forall_{f_0 \in F_0} \exists_{g \in G}(f_0 = g \circ h) \to \forall_{f \in \mathcal{F}(F_0)} \exists_{g \in G}(f = g \circ h).$$

Proof. If $f = f_0 \in F_0$, then we just use our premiss. Of course, a constant function $\overline{a} : X \to \mathbb{R}$ is written as the composition $\overline{a} \circ h$, where we use the same notation for the constant function of type $Y \to \mathbb{R}$ with value a. If $f = f_1 + f_2$ such that $f_1 = g_1 \circ h$ and $f_2 = g_2 \circ h$, for some $g_1, g_2 \in G$, then $f = (g_1 + g_2) \circ h$, where $g_1 + g_2 \in G$ by BS_2. If $f = \phi \circ f'$, where $\phi \in \mathrm{Bic}(\mathbb{R})$, and there is some $g \in G$ such that $f' = g \circ h$, then $f = (\phi \circ g) \circ h$, where $\phi \circ g \in G$ by BS_3. Suppose next that $\epsilon > 0$ and $f' \in \mathcal{F}(F_0)$ such that $f' = g \circ h$, for some $g \in G$, and $\forall_{x \in X}(|f'(x) - f(x)| = |g(h(x)) - f(x)| \leq \epsilon)$. If $\Phi : Y \to \mathbb{R}$ is the function determined by the well-definability lemma such that $f = \Phi \circ h$, we get, since h is onto Y, that $\forall_{y \in Y}(|g(y) - \Phi(y)| \leq \epsilon)$. Since $\epsilon > 0$ is arbitrary, we conclude by condition BS_4 that $\Phi \in G$.

A morphism h from $\mathcal{F}$ to $\mathcal{G}$ induces the mapping $h^* : G \to F$, $g \mapsto h^*(g)$, where $h^*(g) := g \circ h$, which is a ring and a lattice homomorphism. If h is an epimorphism, then h^* is a *partial isometry* i.e., $||h^*(g)||$ exists whenever $||g||$ exists and moreover $||h^*(g)|| = ||g||$. Recall that constructively $||g|| = \sup\{|g(y)| \mid y \in Y\}$ does not always exist for some bounded element of a topology G.
The *pointwise exponential* Bishop space $\mathcal{F} \to \mathcal{G} = (\mathrm{Mor}(\mathcal{F}, \mathcal{G}), F \to G)$ corresponds to the point-open topology within the category of topological spaces **Top** and it is defined by

$$F \to G := \mathcal{F}(\{e_{x,g} \mid x \in X, g \in G\}), \quad e_{x,g} : \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \to \mathbb{R}, \quad e_{x,g}(h) = g(h(x)),$$

for every $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. A simple induction shows that if G_0 is a subbase of G, then $F \to \mathcal{F}(G_0) = \mathcal{F}(\{e_{x,g_0} \mid x \in X, g_0 \in G_0\})$. The *dual* Bishop space of $\mathcal{F}$ is the space $\mathcal{F}^* = (F, F^*)$, where

$$F^* := \mathcal{F}(\{\hat{x} \mid x \in X\}), \quad \hat{x} : F \to \mathbb{R}, \quad \hat{x}(f) = f(x),$$

for every $f \in F$. Clearly, $\mathcal{F}^* = \mathcal{F} \to \mathcal{R} = (\mathrm{Mor}(\mathcal{F}, \mathcal{R}), F \to \mathrm{Bic}(\mathbb{R}))$.
Although Ishihara and Palmgren constructed in [7] the quotient topological space using predicative methods, our definition of the quotient Bishop space is straightforward and permits a smooth translation of the standard classical theory of quotient topological spaces into the theory of Bishop spaces. If $\mathcal{F} = (X, F)$ is a Bishop space, Y is an inhabited set and $e : X \to Y$ is onto Y, it is direct to see that the set of functions G_e, defined as

$$G_e := \{g \in \mathbb{F}(Y) \mid g \circ e \in F\},$$

is a topology on Y. We call $\mathcal{G}_e = (Y, G_e)$ the *quotient Bishop space*, and G_e the *quotient* topology on Y, with respect to e. As in standard topology, the quotient topology G_e is the largest topology on Y which makes e a morphism, while if $\mathcal{H} = (Z, H)$ is a Bishop space, a function $h : Y \to Z \in \mathrm{Mor}(\mathcal{G}_e, \mathcal{H})$ iff $h \circ e \in \mathrm{Mor}(\mathcal{F}, \mathcal{H})$. The next proposition is direct to prove.

Proposition 2. *Let $\mathcal{F} = (X, F)$ be a Bishop space and $\sim$ be the equivalence relation on X defined by $x_1 \sim x_2 \leftrightarrow \forall_{f\in F}(f(x_1) = f(x_2))$. If $\pi : X \to X/{\sim}$ is the function $x \mapsto [x]_\sim$ and $\mathcal{F}/{\sim} = (X/{\sim}, G_\pi)$ is the quotient Bishop space, then π is a strongly open morphism from $\mathcal{F}$ to $\mathcal{F}/{\sim}$, and the function $\rho : F \to G_\pi$, $f \mapsto \rho(f)$, where $\rho(f)([x]_\sim) := f(x)$, is a ring and a lattice homomorphism and a partial isometry onto G_π.*

If $\mathcal{F} = (X, F)$ is a Bishop space and $A \subseteq X$, the *relative* Bishop space of $\mathcal{F}$ on A is the structure $\mathcal{F}_{|A} = (A, F_{|A})$, also called a *subspace* of $\mathcal{F}$, where

$$F_{|A} := \mathcal{F}(\{f_{|A} \mid f \in F\}).$$

If F_0 is a subbase of F, we get inductively that $F_{|A} = \mathcal{F}(\{f_{0|A} \mid f_0 \in F_0\})$. The topology $F_{|A}$ is the smallest topology G on A such that $\mathrm{id}_A \in \mathrm{Mor}(\mathcal{G}, \mathcal{F})$. If $\mathcal{G} = (Y, G)$ is a Bishop space and $e : X \to B \subseteq Y$, then $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \leftrightarrow e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_{|B})$, and if e is open as a morphism from $\mathcal{F}$ to $\mathcal{G}$, it is trivially open as a morphism from $\mathcal{F}$ to $\mathcal{G}_{|B}$. An isomorphism between $\mathcal{F}$ and a subspace of $\mathcal{G}$ is called a *topological embedding* of $\mathcal{F}$ into $\mathcal{G}$.
A topology F on X induces the *canonical* apartness relation on X, which is introduced in [4] and it is defined, for every $x_1, x_2 \in X$, by

$$x_1 \bowtie_F x_2 :\leftrightarrow \exists_{f\in F}(f(x_1) \bowtie_{\mathbb{R}} f(x_2)),$$

where $a \bowtie_{\mathbb{R}} b :\leftrightarrow a > b \vee a < b \leftrightarrow |a - b| > 0$, for every $a, b \in \mathbb{R}$. Moreover, $a \bowtie_{\mathrm{Bic}(\mathbb{R})} b \leftrightarrow a \bowtie_{\mathbb{R}} b$; if $a \bowtie_{\mathrm{Bic}(\mathbb{R})} b$, then $\phi(a) \bowtie_{\mathbb{R}} \phi(b)$, for some $\phi \in \mathrm{Bic}(\mathbb{R})$. By the obvious pointwise continuity of ϕ at a we have that if $0 < \epsilon = |\phi(b) - \phi(a)|$, $\exists_{\delta(\frac{\epsilon}{2})>0}\forall_{x\in\mathbb{R}}(|x - a| < \delta(\frac{\epsilon}{2}) \to |\phi(x) - \phi(a)| \leq \frac{\epsilon}{2})$. Hence, $\neg(|a - b| < \delta(\frac{\epsilon}{2}))$ i.e., $|a - b| \geq \delta(\frac{\epsilon}{2}) > 0$. For the converse we just use the equivalence $a \bowtie_{\mathbb{R}} b \leftrightarrow \mathrm{id}_{\mathbb{R}}(a) \bowtie_{\mathbb{R}} \mathrm{id}_{\mathbb{R}}(b)$. An apartness relation $\bowtie$ on X is called *tight*, if $\neg(x_1 \bowtie x_2) \to x_1 = x_2$, for every $x_1, x_2 \in X$. It is direct to see that if F is a topology on X, then $\bowtie_F$ is tight iff

$$\forall_{x_1,x_2\in X}(\forall_{f\in F}(f(x_1) = f(x_2)) \to x_1 = x_2).$$

The sufficiency is mentioned in [4] and for its proof one uses the obvious fact that $\bowtie_{\mathbb{R}}$ is tight. If F_0 is a subbase of F and the restriction of every $f_0 \in F_0$ to some $A \subseteq X$ is constant, then an induction shows that the restriction of every $f \in \mathcal{F}(F_0)$ to A is constant. If $F = \mathcal{F}(F_0)$, we get that $\bowtie_F$ is tight iff

$$\forall_{x_1,x_2\in X}(\forall_{f_0\in F_0}(f_0(x_1) = f_0(x_2)) \to x_1 = x_2),$$

applying the previous lifting to the set $A = \{x_1, x_2\}$, where $x_1, x_2 \in X$.

3 Completely regular topologies of functions

A completely regular topological space $(X, \mathcal{T})$ is one in which any pair (x, B), where B is closed and $x \notin B$, is separated by some $f \in C(X, [0,1])$. A completely regular and T_1-space satisfies classically the property

$$\forall_{x_1, x_2 \in X}(\forall_{f \in C(X)}(f(x_1) = f(x_2)) \rightarrow x_1 = x_2).$$

The importance and the "sufficiency" of the completely regular topological spaces in the theory of $C(X)$ is provided by the Stone-Čech theorem according to which, for every topological space X there exists a completely regular space ρX and a continuous mapping $\tau : X \rightarrow \rho X$ such that the induced function $g \mapsto \tau^*(g)$, where $\tau^*(g) = g \circ \tau$, is a ring isomorphism between $C(\rho X)$ and $C(X)$ (see [5] p.41).

We call a Bishop space $\mathcal{F} = (X, F)$ *completely regular*, if its canonical apartness relation $\bowtie_F$ is tight, hence the equality of X is determined by F, which we call a *completely regular* topology. Since $\bowtie_{\mathrm{Bic}(\mathbb{R})} \leftrightarrow \bowtie_{\mathbb{R}}$ and $\bowtie_{\mathbb{R}}$ is tight, $\mathcal{R}$ is completely regular. It is direct that $\mathcal{F}$ is completely regular iff $\mathcal{F}_b$ is completely regular, while if X has more than two points, then $\mathrm{Const}(X)$ is not completely regular.

In this section we prove some first fundamental results on completely regular Bishop spaces. The following version of the Stone-Čech theorem expresses the corresponding "sufficiency" of the completely regular Bishop spaces within **Bis**. Its proof is a translation of the classical one, since the quotient Bishop spaces behave as the quotient topological spaces.

Theorem 1 (Stone-Čech theorem for Bishop spaces). *For every Bishop space $\mathcal{F} = (X, F)$ there exists a completely regular Bishop space $\rho\mathcal{F} = (\rho X, \rho F)$ and a mapping $\tau : X \rightarrow \rho X \in \mathrm{Mor}(\mathcal{F}, \rho\mathcal{F})$ such that the induced mapping τ^* is a ring isomorphism between ρF and F.*

Proof. We use the equivalence relation $x_1 \sim x_2 \leftrightarrow \forall_{f \in F}(f(x_1) = f(x_2))$, for every $x_1, x_2 \in X$, and if $\tau = \pi : X \rightarrow X/{\sim}$, where $x \mapsto [x]_{\sim}$, we consider the quotient Bishop space $\rho\mathcal{F} = \mathcal{F}/{\sim} = (X/{\sim}, G_\pi) = (\rho X, \rho F)$. By the Proposition 2 we have that π is a morphism from $\mathcal{F}$ to $\mathcal{F}/{\sim}$ and $\rho : F \rightarrow G_\pi$ is a ring homomorphism onto G_π. We also know that $\pi^* : G_\pi \rightarrow F$ is a ring homomorphism. Since $\rho(g \circ \pi)([x]_{\sim}) = (g \circ \pi)(x) = g([x]_{\sim})$, for every $[x]_{\sim} \in X/{\sim}$, we get that $\rho \circ \pi^* = \mathrm{id}_{G_\pi}$. Since $\pi^*(\rho(f)) = \rho(f) \circ \pi = f$, for every $f \in F$, we get that $\pi^* \circ \rho = \mathrm{id}_F$. Hence, π^* is a bijection (see [2] p.17). Finally, $\rho\mathcal{F}$ is completely regular; if $\forall_{g \in \rho F}(g([x_1]_{\sim}) = g([x_2]_{\sim}))$, then $\forall_{f \in F}(f(x_1) = f(x_2))$, since $\rho(f) \circ \pi = f$ and $\rho(f) \in \rho F$, therefore $x_1 \sim x_2$ i.e., $[x_1]_{\sim} = [x_2]_{\sim}$.

Proposition 3. *Suppose that $\mathcal{F} = (X, F), \mathcal{G} = (Y, G)$ are Bishop spaces.*

(i) If $\mathcal{G}$ is isomorphic to the completely regular $\mathcal{F}$, then $\mathcal{G}$ is completely regular.
(ii) If $A \subseteq X$ and $\mathcal{F}$ is a completely regular, then $\mathcal{F}_{|A}$ is completely regular.
(iii) $\mathcal{F}$ and $\mathcal{G}$ are completely regular iff $\mathcal{F} \times \mathcal{G}$ is completely regular.
(iv) $\mathcal{F} \rightarrow \mathcal{G}$ is completely regular iff $\mathcal{G}$ is completely regular.
(v) The dual space $\mathcal{F}^*$ of $\mathcal{F}$ is completely regular.

Proof. (i) Suppose that e is an isomorphism between $\mathcal{F}$ and $\mathcal{G}$, and $y_1, y_2 \in Y$. Since e is onto Y and e^* is onto G, we get that $\forall_{g \in G}(g(y_1) = g(y_2)) \leftrightarrow \forall_{g \in G}(g(e(x_1)) = g(e(x_2))) \leftrightarrow \forall_{f \in F}(f(x_1) = f(x_2)) \to x_1 = x_2$. Hence, $y_1 = y_2$.
(ii) If $a_1, a_2 \in A$, it suffices to show that $\forall_{f \in F}(f_{|A}(a_1) = f_{|A}(a_2)) \to a_1 = a_2$. The premiss is rewritten as $\forall_{f \in F}(f(a_1) = f(a_2))$, and since F is completely regular, we conclude that $a_1 = a_2$.
(iii) The hypotheses $\forall_{f \in F}((f \circ \pi_1)(x_1, y_1) = (f \circ \pi_1)(x_2, y_2))$ and $\forall_{g \in G}((g \circ \pi_2)(x_1, y_1) = (g \circ \pi_2)(x_2, y_2))$ imply $\forall_{f \in F}(f(x_1) = f(x_2))$ and $\forall_{g \in G}(g(y_1) = g(y_2))$. For the converse we topologically embed in the obvious way $\mathcal{F}, \mathcal{G}$ into $\mathcal{F} \times \mathcal{G}$ and we use (i) and (ii).
(iv) If $\mathcal{G}$ is completely regular and $\forall_{h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})} \forall_{x \in X} \forall_{g \in G}(e_{x,g}(h_1) = e_{x,g}(h_2))$ i.e., $\forall_{h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})} \forall_{x \in X} \forall_{g \in G}(g(h_1(x)) = g(h_2(x)))$, then the tightness of $\bowtie_{\mathcal{G}}$ implies that $\forall_{x \in X}(h_1(x) = h_2(x))$ i.e., $h_1 = h_2$. For the converse we suppose that $\forall_{h_1, h_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})} \forall_{x \in X} \forall_{g \in G}(e_{x,g}(h_1) = e_{x,g}(h_2)) \to h_1 = h_2$. We fix $y_1, y_2 \in G$ and we suppose that $\forall_{g \in G}(g(y_1) = g(y_2))$. Since $\overline{y_1}, \overline{y_2} \in \mathrm{Const}(X, Y) \subseteq \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, we have that $\forall_{x \in X} \forall_{g \in G}(e_{x,g}(\overline{y_1}) = g(\overline{y_1}(x)) = g(y_1) = g(y_2) = g(\overline{y_2}(x)) = e_{x,g}(\overline{y_2}))$. Hence, $\overline{y_1} = \overline{y_2}$ i.e., $y_1 = y_2$.
(v) Since $\mathcal{F}^* = \mathcal{F} \to \mathcal{R}$ and $\mathcal{R}$ is completely regular, we use (iv).

The proof of the Proposition 3(iii) works for an arbitrary product of Bishop spaces too. As in classical topology one can show that the quotient of a completely regular Bishop space need not be completely regular. If $\mathcal{F} = (X, F)$ and $\mathcal{G}_i = (Y_i, G_i)$ are Bishop spaces, for every $i \in I$, the family $(h_i)_{i \in I}$, where $h_i : X \to Y_i$, for every $i \in I$, *separates the points* of X, if $\forall_{x,y \in X}(\forall_{i \in I}(h_i(x) = h_i(y)) \to x = y)$.

Theorem 2 (embedding lemma for Bishop spaces). *Suppose that $\mathcal{F} = (X, F)$ and $\mathcal{G}_i = (Y_i, G_i)$ are Bishop spaces and $h_i : X \to Y_i$, for every i in some index set I. If the family of functions $(h_i)_{i \in I}$ separates the points of X, $h_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_i)$, for every $i \in I$, and $\forall_{f \in F} \exists_{i \in I} \exists_{g \in G_i}(f = g \circ h_i)$, then the evaluation map $e : X \to Y = \prod_{i \in I} Y_i$, defined by $x \mapsto (h_i(x))_{i \in I}$, is a topological embedding of $\mathcal{F}$ into $\mathcal{G} = \prod_{i \in I} \mathcal{G}_i$.*

Proof. First we show that e is $1-1$; $e(x_1) = e(x_2) \leftrightarrow (h_i(x_1))_{i \in I} = (h_i(x_2))_{i \in I} \leftrightarrow \forall_{i \in I}(h_i(x_1) = h_i(x_2)) \to x_1 = x_2$. By the lifting of morphisms we have that $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}) \leftrightarrow \forall_{g \in G}(g \circ e \in F) \leftrightarrow \forall_{i \in I} \forall_{g \in G_i}((g \circ \pi_i) \circ e = g \circ (\pi_i \circ e) \in F) \leftrightarrow \forall_{i \in I} \forall_{g \in G_i}(g \circ h_i \in F) \leftrightarrow \forall_{i \in I}(h_i \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_i))$. Next we show that e is open i.e., $\forall_{f \in F} \exists_{g \in G}(f = g \circ e)$. If $f \in F$, by hypothesis (iii) there is some $i \in I$ and some $g \in G_i$ such that $f = g \circ h_i$. Since $g \circ \pi_i \in G = \prod_{i \in I} G_i$ we have that $(g \circ \pi_i)(e(x)) = (g \circ \pi_i)((h_i(x))_{i \in I}) = g(h_i(x)) = f(x)$, for every $x \in X$, hence $f = (g \circ \pi_i) \circ e$. Thus, e is open as a morphism from $\mathcal{F}$ to $\mathcal{G}$, therefore it is open as a morphism from $\mathcal{F}$ to $\mathcal{G}_{|e(X)}$.

According to the classical Tychonoff embedding theorem, the completely regular topological spaces are precisely those which can be embedded in a product of the closed unit interval $\mathcal{I}$. In the following characterization of the tightness of the canonical apartness relation it is $\mathcal{R}$ which has the role of $\mathcal{I}$.

Theorem 3 (Tychonoff embedding theorem for Bishop spaces). *Suppose that $\mathcal{F} = (X, F)$ is a Bishop space. Then, $\mathcal{F}$ is completely regular iff $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^F$.*

Proof. If $\mathcal{F}$ is completely regular, using the embedding lemma we show that the mapping $e : X \to \mathbb{R}^F$, defined by $x \mapsto (f(x))_{f \in F}$, is a topological embedding of $\mathcal{F}$ into $\mathcal{R}^F$. The topology F is a family of functions of type $X \to \mathbb{R}$ that separates the points of X, since the separation condition is exactly the tightness of $\bowtie_F$. That every $f \in F$ is in $\mathrm{Mor}(\mathcal{F}, \mathcal{R})$ is already mentioned. If we fix some $f \in F$, then $f = \mathrm{id}_{\mathbb{R}} \circ f$, and since $\mathrm{id}_{\mathbb{R}} \in \mathrm{Bic}(\mathbb{R})$, the condition (iii) of the embedding lemma is satisfied. If $\mathcal{F}$ is topologically embedded into $\mathcal{R}^F$, then $\mathcal{F}$ is completely regular, since by Proposition 3 a Euclidean Bishop space is completely regular and $\mathcal{F}$ is isomorphic to a subspace of a completely regular space.

If $\mathcal{F} = (X, F)$, $\mathcal{G} = (Y, G)$ are Bishop spaces and $h \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then using Theorem 3 one shows, as in the classical case, the existence of a mapping $\rho h : \rho X \to \rho Y \in \mathrm{Mor}(\rho\mathcal{F}, \rho\mathcal{G})$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ h\ } & Y \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau'} \\ \rho X & \xrightarrow[\ \rho h\ ]{} & \rho Y, \end{array}$$

where τ, τ' are the morphisms determined by Theorem 1. Next we prove directly a generalized form of the Tychonoff embedding theorem.

Theorem 4 (general Tychonoff embedding theorem). *Suppose that $\mathcal{F} = (X, \mathcal{F}(F_0))$ is a Bishop space. Then, $\mathcal{F}$ is completely regular iff $\mathcal{F}$ is topologically embedded into the Euclidean Bishop space $\mathcal{R}^{F_0}$.*

Proof. If $\mathcal{F}$ is completely regular, we show directly that the mapping $e : X \to \mathbb{R}^{F_0}$, defined by $x \mapsto (f_0(x))_{f_0 \in F_0}$, is a topological embedding of $\mathcal{F}$ into $\mathcal{R}^{F_0}$. Since the tightness of $\bowtie_{\mathcal{F}(F_0)}$ is equivalent to $\forall_{x_1, x_2 \in X}(\forall_{f_0 \in F_0}(f_0(x_1) = f_0(x_2)) \to x_1 = x_2))$, we get that e is $1{-}1$. Using our remark on the relative topology given with a subbase, we get that, since $\mathrm{Bic}(\mathbb{R})^{F_0} = \bigvee_{f_0 \in F_0} \pi_{f_0}$, its restriction to $e(X)$ is $(\mathrm{Bic}(\mathbb{R})^{F_0})_{|e(X)} = (\bigvee_{f_0 \in F_0} \pi_{f_0})_{|e(X)} = \bigvee_{f_0 \in F_0} (\pi_{f_0})_{|e(X)}$. By the lifting of morphisms we have that $e \in \mathrm{Mor}(\mathcal{F}, (\mathcal{R}^{F_0})_{|e(X)})$ iff $\forall_{f_0 \in F_0}((\pi_{f_0})_{|e(X)} \circ e = f_0 \in \mathcal{F}(F_0))$, which holds trivially. In order to prove that e is open it suffices by the lifting of openness on the epimorphism $e : X \to e(X)$ to show that $\forall_{f_0 \in F_0} \exists_{g \in (\mathrm{Bic}(\mathbb{R})^{F_0})_{|e(X)}} (f_0 = g \circ e)$. Since $f_0 = (\pi_{f_0})_{|e(X)} \circ e$ and $(\pi_{f_0})_{|e(X)} \in (\mathrm{Bic}(\mathbb{R})^{F_0})_{|e(X)}$, for every $f_0 \in F_0$, we are done. The converse is proved as in the proof of the Theorem 3.

If we consider $F_0 = F$, then $\mathcal{F}(F_0) = F$ and the general Tychonoff embedding theorem implies Theorem 3. If $X = \mathbb{R}^n$, then $\mathrm{Bic}(\mathbb{R})^n = \mathcal{F}(\pi_1, \ldots, \pi_n)$, and the above embedding e is $\mathrm{id}_{\mathbb{R}^n}$, since $\boldsymbol{x} \mapsto (\pi_1(\boldsymbol{x}), \ldots, \pi_n(\boldsymbol{x})) = \boldsymbol{x}$.

Proposition 4. *Suppose that $\mathcal{F} = (X, F)$ is a completely regular Bishop space, $\mathcal{G} = (Y, G)$ is Bishop space, and $\tau : X \to Y \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$. Then, τ^* is onto F iff τ is a topological embedding of $\mathcal{F}$ into $\mathcal{G}$ such that $G_{|\tau(X)} = \{g_{|\tau(X)} \mid g \in G\}$.*

Proof. We suppose that τ^* is onto F and we show first that τ is 1−1; suppose that $\tau(x_1) = \tau(x_2)$, for some $x_1, x_2 \in X$. We have that $\forall_{f \in F}(f(x_1) = f(x_2))$, since by the onto hypothesis of τ^*, if $f \in F$, there is some $g \in G$ such that $f(x_1) = (g \circ \tau)(x_1) = g(\tau(x_1)) = g(\tau(x_2)) = (g \circ \tau)(x_2) = f(x_2)$. By the complete regularity of $\mathcal{F}$ we conclude that $x_1 = x_2$. By the lifting of morphisms we get directly that if $\tau \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $\tau \in \mathrm{Mor}(\mathcal{F}, \mathcal{G}_{|\tau(X)})$. The onto hypothesis of τ^* i.e., $\forall_f \exists_{g \in G}(f = g \circ \tau)$, implies that $\forall_f \exists_{g' \in G_{|\tau(X)}}(f = g' \circ \tau)$, where $g' = g_{|\tau(X)}$. Hence, $\tau : X \to \tau(X)$ is an isomorphism between $\mathcal{F}$ and $\mathcal{G}_{|\tau(X)}$. Next we show that $\{g_{|\tau(X)} \mid g \in G\}$ is a topology on $\tau(X)$, therefore by the definition of the relative topology we get that $G_{|\tau(X)} = \{g_{|\tau(X)} \mid g \in G\}$. Clearly, $\overline{a}_{|\tau(X)} = \overline{a}$, $(g_1 + g_2)_{|\tau(X)} = g_{1|\tau(X)} + g_{2|\tau(X)}$ and $(\phi \circ g)_{|\tau(X)} = \phi \circ g_{|\tau(X)}$, where $\phi \in \mathrm{Bic}(\mathbb{R})$. Suppose that $h : \tau(X) \to \mathbb{R}$, $\epsilon > 0$ and $g \in G$ such that $\forall_{y \in \tau(X)}(|g(y) - h(y)| \leq \epsilon) \leftrightarrow \forall_{x \in X}(|g(\tau(x)) - h(\tau(x))| \leq \epsilon)$. Since $g \circ \tau \in F$ and ϵ is arbitrary, we conclude by the condition BS_4 that $h \circ \tau \in F$, hence, by our onto hypothesis of τ^*, there is some $g \in G$ such that $g \circ \tau = h \circ \tau$ i.e., $g_{|\tau(X)} = h$. For the converse we fix some $f \in F$ and we find $g \in G$ such that $f = \tau^*(g)$. Since $\tau : X \to \tau(X)$ is open, there exists some $g' \in G_{|\tau(X)}$ such that $f = g' \circ \tau$, and since $G_{|\tau(X)} = \{g_{|\tau(X)} \mid g \in G\}$, there is some $g \in G$ such that $g' = g_{|\tau(X)}$, hence $f = g_{|\tau(X)} \circ \tau = g \circ \tau = \tau^*(g)$.

As in [5] p.155 for $C(X)$, the Proposition 4 implies the Tychonoff embedding theorem; if F is completely regular and $e : X \to \mathbb{R}^F$ is defined by $x \mapsto (f(x))_{f \in F}$, then $e^* : \bigvee_{f \in F} \pi_f \to F$ is onto F, since $e^*(\pi_f) = \pi_f \circ e = f$, therefore e is an embedding of $\mathcal{F}$ into $\mathcal{R}^F$ such that $(\bigvee_{f \in F} \pi_f)_{|e(X)} = \{g_{|e(X)} \mid g \in \bigvee_{f \in F} \pi_f\}$.

Proposition 5. *If $\mathcal{F} = (X, F)$ and $\mathcal{G} = (Y, G)$ are Bishop spaces, then:*
(i) If $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, then $\hat{x} \circ e^* = \widehat{e(x)}$, for every $x \in X$, and $e^* \in \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*)$.
(ii) The mapping $\hat{}: X \to F^*$, defined by $x \mapsto \hat{x}$, is in $\mathrm{Mor}(\mathcal{F}, \mathcal{F}^{**})$ and it is 1−1 iff $\mathcal{F}$ is completely regular.
(ii) If $\mathcal{G}$ is completely regular, then $\forall_{e_1, e_2 \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})}(e_1^* = e_2^* \to e_1 = e_2)$.

Proof. (i) By the lifting of morphisms we have that $e^* \in \mathrm{Mor}(\mathcal{G}^*, \mathcal{F}^*) \leftrightarrow \forall_{x \in X}(\hat{x} \circ e^* \in G^*)$. But $\hat{x} \circ e^* = \widehat{e(x)} \in G^*$, since $(\hat{x} \circ e^*)(g) = \hat{x}(e^*(g)) = \hat{x}(g \circ e) = (g \circ e)(x) = g(e(x)) = \widehat{e(x)}(g)$, for every $g \in G$.
(ii) Since $\mathcal{F}^{**} = (F^*, F^{**})$, where $F^{**} = \mathcal{F}(\{\hat{f} \mid f \in F\})$, $\hat{f} : F^* \to \mathbb{R}$, and $\hat{f}(\theta) = \theta(f)$, for every $\theta \in F^*$, by the lifting of morphisms we have that $\hat{} \in \mathrm{Mor}(\mathcal{F}, \mathcal{F}^{**}) \leftrightarrow \forall_{f \in F}(\hat{f} \circ \hat{} \in F)$. But $\hat{f} \circ \hat{} = f$, since $(\hat{f} \circ \hat{})(x) = \hat{f}(\hat{x}) = \hat{x}(f) = f(x)$, for every $x \in X$. Since $\hat{x} = \hat{y} \leftrightarrow \forall_{f \in F}(\hat{x}(f) = \hat{y}(f)) \leftrightarrow \forall_{f \in F}(f(x) = f(y))$, the injectivity of $\hat{}$ implies the tightness of $\bowtie_F$ and vice versa.
(iii) By (i) we have that $\hat{x} \circ e_1^* = \widehat{e_1(x)}$ and $\hat{x} \circ e_2^* = \widehat{e_2(x)}$, for every $x \in X$. Since $e_1^* = e_2^*$, we get that $\widehat{e_1(x)} = \widehat{e_2(x)}$, for every $x \in X$. By (ii) and the complete regularity of $\mathcal{G}$ we get that $e_1(x) = e_2(x)$, for every $x \in X$.

Proposition 6. *Suppose that* $\mathcal{F} = (X, F)$ *and* $\mathcal{G} = (Y, G)$ *are completely regular Bishop spaces and* $\mathcal{E} : G \to F$ *is an isomorphism between* $\mathcal{G}^*$ *and* $\mathcal{F}^*$*. Then, there exists an (unique) isomorphism* $e : X \to Y$ *between* $\mathcal{F}$ *and* $\mathcal{G}$ *such that* $\mathcal{E} = e^*$ *iff* $\forall_{x \in X} \exists_{y \in Y} (\hat{x} \circ \mathcal{E} = \hat{y})$ *and* $\forall_{y \in Y} \exists_{x \in X} (\hat{y} = \hat{x} \circ \mathcal{E})$*.*

Proof. The necessity follows by applying the Proposition 5(i) on e and e^{-1}. For the converse we suppose that $\forall_{x \in X} \exists_{y \in Y} (\hat{x} \circ \mathcal{E} = \hat{y})$ and we show that $\forall_{x \in X} \exists!_{y \in Y} (\hat{x} \circ \mathcal{E} = \hat{y})$; if $\hat{x} \circ \mathcal{E} = \hat{y_1} = \hat{y_2}$, then by the complete regularity of $\mathcal{G}$ we get that $y_1 = y_2$. We define $e : X \to Y$ by $x \mapsto y$, where y is the unique element of Y such that $\hat{x} \circ \mathcal{E} = \hat{y}$. Similarly, we suppose that $\forall_{y \in Y} \exists_{x \in X} (\hat{y} = \hat{x} \circ \mathcal{E})$ and we show that $\forall_{y \in Y} \exists!_{x \in X} (\hat{y} = \hat{x} \circ \mathcal{E})$; if $\hat{y} = \hat{x_1} \circ \mathcal{E} = \hat{x_2} \circ \mathcal{E}$, then $\hat{y} \circ \mathcal{E}^{-1} = \hat{x_1} = \hat{x_2}$ and by the complete regularity of $\mathcal{F}$ we conclude that $x_1 = x_2$. We define $j : Y \to X$ by $y \mapsto x$, where x is the unique element of X such that $\hat{x} \circ \mathcal{E} = \hat{y}$. Next we show that $j = e^{-1}$, or equivalently that $e \circ j = \mathrm{id}_Y$ and $j \circ e = \mathrm{id}_X$; for the first equality we have that $\hat{y} = \widehat{j(y)} \circ \mathcal{E}$ and also that $\widehat{j(y)} \circ \mathcal{E} = \widehat{e(j(y))}$, which implies that $\hat{y} = \widehat{e(j(y))}$. By the complete regularity of $\mathcal{G}$ we get that $y = e(j(y))$. For the second equality we have that $\hat{x} \circ \mathcal{E} = \widehat{e(x)}$ and $\widehat{e(x)} = \widehat{j(e(x))} \circ \mathcal{E}$, which implies that $\hat{x} \circ \mathcal{E} = \widehat{j(e(x))} \circ \mathcal{E}$, and consequently $\hat{x} = \widehat{j(e(x))}$. By the complete regularity of $\mathcal{F}$ we get that $x = j(e(x))$. Hence, e is a bijection. Next we show that $\mathcal{E}(g) = g \circ e$, for every $g \in G$; since the first part of our hypothesis can be written as $\forall_{x \in X} (\hat{x} \circ \mathcal{E} = \widehat{e(x)})$, we get that $(\hat{x} \circ \mathcal{E})(g) = \widehat{e(x)}(g) \leftrightarrow \mathcal{E}(g)(x) = g(e(x)) \leftrightarrow \mathcal{E}(g)(x) = (g \circ e)(x)$, for every $g \in G$ and $x \in X$. Since $g \circ e = \mathcal{E}(g) \in F$, for every $g \in G$, we conclude that $e \in \mathrm{Mor}(\mathcal{F}, \mathcal{G})$, while if $f \in F$, since $\mathcal{E}$ is onto F, there exists some $g \in G$ such that $\mathcal{E}(g) = g \circ e = f$ i.e., e is open.

A formalization of the previous proof requires Myhill's axiom of nonchoice, which is considered compatible with BISH (see [3], p.75).

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