TOPOLOGICAL CHARACTERISATION OF THE COMPACT ELEMENTS OF THE LATTICE $<T^c, \subset>$, FOR ANY TOPOLOGICAL SPACE $<X, T>$

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Abstract *Let $<X, T>$ be any topological space. We prove that x is a compact element of the lattice $<T^c, \subset>$ if and only if $x = \overline{x_0}$, where x_0 is a finite subset of X with the property $(\forall t \in x_0)\ (\forall a \subset T^c)\ (t \in \overline{\cup a} \Rightarrow t \in \cup a)$.*

We also prove that in T_1-spaces, x is a compact element of the lattice $<T^c, \subset>$ if and only if x is a finite set of isolated points of $<X, T>$.

§1. Introduction - Preliminaries. Our sets are these of *ZF Set Theory* and are in general denoted by small letters. Still we preserve the topological notations *X, T, T^c*; $<X, T>$ is any topological space and T^c is the set of its closed subsets. We consider the elementary notions from Topology and their properties known to the reader. We denote by $\overline{x}$ and $\underline{x}$ the closure and the interior of the subset x of $<X, T>$, respectively. Compact sets of $<X, T>$ are refered to as *topologically compact sets,* in order not to be confused with the compact elements of the lattices $<T, \subset>$ and $<T^c, \subset>$.

A partially ordered (*p.o.*) set is denoted by $<a, \leq>$, and a *lattice* is understood to be a *p.o.* set $<a, \leq>$, such that every binary subset of it has an infimum and a supremum. If a subset b of $<a, \leq>$ has an infimum and a supremum, then $<a, \leq>$ is called a *complete lattice.* For every topological space $<X, T>$, one can easily show that *$<T, \subset>$ is a complete lattice and*

A.M.S. 93T-06-05, January 1993

that $(\forall b \subset T\text{-}\{\emptyset\})$ $(\vee b = \cup b$ *and* $\wedge b = \underline{\cap b})$. One can also easily show that $<T^c, \subset>$ *is a complete lattice and that* $(\forall b \subset T^c\text{-}\{\emptyset\})$ $(\vee b = \overline{\cup b}$ *and* $\wedge b = \cap b)$.

A non-empty set b of subsets of a complete lattice $<a, \leq>$ is called a *cover* of an element x of $<a, \leq>$ if $x \leq \vee b$. An element x of $<a, \leq>$ is called *compact* if every cover of x has a finite subset which is a cover of x. A complete lattice $<a, \leq>$ is called *algebraic*, if every element of it is the supremum of a set of compact elements of it. The property of *being algebraic* is shared by most of the significant for algebraists lattices (see [1]). For instance the lattice of subuniverses of any algebra, the lattice of congruences on any algebra, the lattice of normal subgroups of a group, the lattice of ideals of a ring or a lattice are all algebraic.

Algebraic lattices are rare in Topology. Specifying the compact elements of the lattices $<T, \subset>$ and $<T^c, \subset>$ is still of an interest and helps us trace the particular topological spaces whose either of the above mentioned lattices is algebraic.

The problem of topologically characterising the compact elements of the lattices $<T, \subset>$ *and* $<T^c, \subset>$, *for a given topological space* $<X, T>$, *was posed to the 1991-92 Undergraduate Seminar of Set Theory and Lattices theory of the Aristotle University of Thessaloniki-Greece, by Dr. C. Kalfa, who had the responsibility of the seminar. The three of us arrived at a solution, helped by Prof. D. Jancovic. We thank him for his valuable help.*

§2. Characterisation of the compact elements of the lattices $<T, \subset>$ and $<T^c, \subset>$.

2.0. Theorem Let $<X, T>$ be a topological space. x is a compact element of the lattice $<T, \subset>$ if and only if x is a topologically compact open set of $<X,T>$.

Proof Immediate through the fact that $(\forall b \in P(T)-\{\emptyset\})$ $(\vee b = \overline{\cup b})$ □

2.1. Lemma If y_0 is a finite subset of a topological space $<X, T>$, then there exists a subset x_0 of y_0 with the properties

a. $\overline{x_0} = \overline{y_0}$

b. $(\forall t \in x_0)$ $(\forall s \in x_0)$ $(t \neq s \Rightarrow t \notin \overline{\{s\}} \wedge s \notin \overline{\{t\}})$

Proof Consider the equivalence relation r on y_0, given by the rule $<t,s> \in r \Leftrightarrow \overline{\{s\}} = \overline{\{t\}}$. Let z_0 be any set of representatives of the corresponding equivalence classes. It obviously holds that **(i)** $\overline{y_0} = \overline{z_0}$ and **(ii)** $(\forall t \in z_0)$ $(\forall s \in z_0)$ $(t \neq s \Rightarrow \overline{\{t\}} \neq \overline{\{s\}})$.

Consider, now, the set

$$x_0 = \{t \in z_0 : \overline{\{t\}} \text{ is a maximal of } <\{\overline{\{t\}} : t \in z_0\}, \subset>\}.$$

x_0 has the required properties because:

a. If $t \in z_0$, then $\{t\}$ predeceeds or coinsides with a maximal of the finite *p.o.* set $<\{\overline{\{t\}}: t \in z_0\}, \subset>$ - easy proof by induction. Thus $t \in \overline{\{s\}}$ for some $s \in x_0$. It follows that $t \in \overline{x_0}$ and that a. holds.

b. Let t and s be two distinct elements of x_0. Because of the maximality of the also distinct $\overline{\{t\}}$ and $\overline{\{s\}}$, we have that $\overline{\{t\}} \not\subset \overline{\{s\}}$ and $\overline{\{s\}} \not\subset \overline{\{t\}}$; so b. holds □

2.2. Theorem Let $<X, T>$ be a topological space. x is a compact element of the lattice $<T^c, \subset>$ if and only if there exists a finite subset x_0 of X having the property $(\forall t \in x_0)(\forall a \subset T^c)(t \in \overline{\cup a} \Rightarrow t \in \cup a)$, such that $x = \overline{x_0}$.

Proof

Let x be a compact element of $<T^c, \subset>$. Obviously $x \subset \cup\{\overline{\{t\}}: t \in x\} \subset \vee\{\overline{\{t\}}: t \in x\}$. Since $\{\overline{\{t\}}: t \in x\}$ is a cover of x, there exists a finite subset y_0 of x, such that $\{\overline{\{t\}}: t\in y_0\}$ is a cover of x. Consequently $x\subset \vee\{\overline{\{t\}}: t\in y_0\} = \cup\{\overline{\{t\}}: t\in y_0\} = y_0$. From lemma 2.1. it follows that *there exists a finite set x_0, such that $x_0 = x$ and $(\forall t\in x_0)$ $(\forall s\in x_0)$ $(t\neq s \Rightarrow t\notin \overline{\{s\}}$ and $s\notin \overline{\{t\}})$.*

We prove that, for any $t\in x_0$, $\overline{\{t\}}$ is a compact element of the lattice $<T^c, \subset>$: Let a be a cover of $\overline{\{t\}}$. The set

$$a' = a \cup \{\overline{x_0-\{t\}}\}$$

is a cover of x, because $t\in\overline{\{t\}}\subset\overline{\cup a}\subset\overline{\cup a'}$ and $(\forall s\in x_0-\{t\})(s\in\overline{x_0-\{t\}})\subset\overline{\cup a'}$. There exists, thus, a finite subset a_0' of a', such that $x\subset \vee a_0' = \cup a_0'$. Because of the property of x_0, $t\notin \cup\{\overline{\{s\}}: s\in x_0-\{t\}\} = \overline{x_0-\{t\}}$. Since $t\in\cup a_0'$, t belongs to some $z\in a$, $\overline{\{t\}}\subset z=\vee\{z\}$ and $\{z\}$ is a cover of $\overline{\{t\}}$. This proves that $\overline{\{t\}}$ is compact.

We prove that $(\forall t\in x_0)(\forall a\subset T^c)(t\in\overline{\cup a} \Rightarrow t\in\cup a)$: If $t\in x_0$, $a\subset T^c$ and $t\in\overline{\cup a}$, then $\overline{\{t\}}\subset\vee a$. Since $\overline{\{t\}}$ is compact, there is a finite $a_0\subset a$, such that $\overline{\{t\}}\subset\vee a_0 = \cup a_0\subset\cup a$. It follows that $t\in\cup a$.

We have proved direction "$\Rightarrow$" of Theorem 2.2. The converse direction is left to the reader.

□

In any topological space <X, T> an isolated point t has the property

$$(\forall a\subset T^c)(t\in\overline{\cup a}\Rightarrow t\in\cup a) \qquad (1)$$

because $\{t\}$ is an open neighbourhood of t, so $t\in\overline{\cup a} \Rightarrow \{t\}\cap(\cup a) \neq \emptyset \Rightarrow t\in\cup a$. In general, property (1) doesn't characterise the isolated points of any topological space - for instance, both x and y are not isolated points of $<\{x, y\}, \{\emptyset, \{x, y\}\}>$, but have property (1). We observe that *when <X, T> is a T_1-space, property (1) is a characteristic property of the isolated* *points of it.* This is so, because if t has property (1) and t is not isolated, then $\{t\} \notin T$ and $X-\{t\} \notin T^c$. The set $\{\{s\}: s \in X-\{t\}\}$ contains only closed sets and $t \in X = \overline{X-\{t\}} = \overline{\cup\{\{s\}: s \in X-\{t\}\}} \Rightarrow t \in \cup\{\{s\}: s \in X-\{t\}\} = X-\{t\}$, which is absurd. So, if t has property (1) , then it is isolated.

This remark, together with the fact that finite sets of T_1-spaces are closed, helps us simplify *in the case of T_1-spaces* the characterisation of the compact elements of $<T^c, \subset>$ provided by theorem 2.2., as follows:

2.3. Corollary Let $<X, T>$ be a T_1-space. x is a compact element of the lattice $<T^c, \subset>$ if and only if x is a finite set of isolated points of $<X, T>$.

REFERENCES

[1] **Burris, S. and Sankappanavar, H.P.** *A course in Universal Algebra,* Springel-Verlag, 1981.

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