Ludwig-Maximilians-Universität
München

Master Program Theoretical and Mathematical Physics

Master Thesis

On Relativistic Interactions in Quantum Theories

Advisor:
Prof. Dr. Detlef Dürr

Author:
Lukas Nickel

July 27, 2015

Abstract

This thesis is dedicated to the question: (How) Is a relativistic quantum theory with interaction possible? We discuss the problems of quantum field theories, especially the non-existence of dynamics, which lead us to consider alternative approaches. Multi-time wave functions are identified as the basic objects in relativistic quantum mechanics. Focusing on models with constant particle number, we present several possibilities how interaction can be achieved. We give an overview of the current state of research concerning relativistic quantum theory, followed by a number of own contributions.

For one method of implementing interaction, namely by boundary conditions, we present a model of N identical mass-less particles moving in one dimension. The multi-time wave function satisfies a system of N Dirac equations with boundary conditions on the space-time configurations where two particles meet, which leads to a relativistic contact interaction. Lorentz invariance as well as probability conservation are proven rigorously and the unique solution of this model is obtained by a generalized method of characteristics. The effective potential, which features δ-functions of the particle distances, is derived by self-adjoint extensions of the Dirac Hamiltonian for several particles.

Another promising way of generating interaction is the use of integral equations. As an example, we consider the Bethe-Salpeter equation, a multi-time integral equation supposed to describe bound states in quantum electrodynamics. Problems connected with this equation such as the lacking gauge invariance and the "abnormal solutions" are explained and some ideas to resolve them are mentioned. Lastly, we perform the non-relativistic limit of the Bethe-Salpeter equation in position space and thereby derive the Breit equation in a new way.

Contents

Part I
Possibilities for relativistically interacting quantum theories

1 Introduction and overview

> "So one of my missions in life is to get people to see that if they want to talk about the problems of quantum mechanics – the real problems of quantum mechanics – they must be talking about Lorentz invariance." - John Bell

There are two extremely deep insights about our world that were unraveled by physicists in the last century. The first one is relativity, the insight that the concept of space-time ought to replace the separate entities of space and time. The second fact is non-locality in the sense of Bell's theorem, instantaneous causal connections that carry over arbitrary distances without any attenuation.
Alas, there is a profound tension between these two concepts. There are well-established physical theories for both of them separately, Einstein's theory of relativity on the one hand and quantum mechanics on the other hand. But finding a combination of relativity and quantum mechanics is a tough business. If we follow the guideline that a physical theory ought to be conceptually clear and mathematically well-defined, all reputed solutions to the problem, especially quantum field theories, are not fully satisfactory. Alternative attempts without conceptual or mathematical flaws have been made, but they are so far only able to describe systems without interaction, so the real Bell correlations can not yet be explained[1]. Any new theory that is invented should be judged according to its ability to unite the relativistic space-time structure with the quantum-mechanical entanglement and non-locality.
The question asked here is therefore one of today's most important questions in fundamental physics: (How) Is a relativistic quantum theory with interaction possible?
Of course, it is impossible to conclusively answer this question in this master's thesis. Still, many thoughts are presented that may be helpful for the search. Many theoretical physicists all over the world try to find a theory of quantum gravity today, with little success so far. Every one of the various approaches has some value and we do not claim that our approach is the only one that can lead to a relativistically interacting quantum theory. What we do claim is that it is worthwhile to take a step back from the rather intricate approaches that are mostly discussed in the literature and start from the basic consideration what we wish for from a fundamental theory. If the aims of conceptual and mathematical clarity guide our search, the ideas in this thesis arise from natural

[1] To some extent, Bell correlations can be reproduced by a non-interacting theory if one assumes a suitable initial state (e.g. a singlet state) and then utilizes some measurement formalism. But this is of course an unfinished picture since the initial state often arises through interaction that is not described by that theory and also the measurement should be describable on a fundamental level by interactions of matter.

trains of thought. However, there are always other promising paths and we have to leave aside approaches such as non-linear models, rigorous quantum field theory and others in this thesis.

In part I, we investigate some general possibilities for relativistically interacting quantum theories. We start with clarifying the concepts in sec. 2 and are thus able to discuss in sec. 3 whether or not quantum field theory qualifies as the kind of theory we are aiming at. If we only call theories with well-defined dynamics *interacting*, it becomes clear that we cannot be satisfied with quantum electrodynamics or other theories of the Standard Model. The search for alternatives starts from non-interacting relativistic quantum theories in section 4, where we identify the concept of *multi-time wave functions* as constitutive for relativistic quantum theories. Therefore, the subsequent sec. 5 treats approaches to interaction in the multi-time picture of relativistic quantum mechanics. Since interaction potentials make the equations inconsistent, several other possibilities to obtain interaction are listed and discussed. The two most promising of them, boundary conditions and integral equations, are then discussed at length in the subsequent parts of this thesis.

In part II, based on the article [1] which is collaborate work with Matthias Lienert, we generalize a model by Lienert [2] which is to our knowledge the first rigorous, interacting multi-time model where probability conservation and Lorentz invariance are ensured. In our generalization, it describes N identical Dirac particles with mass $m = 0$ in $1 + 1$ dimensions. Interaction is generated by boundary conditions for the multi-time wave function on configurations where two particles meet. We prove probability conservation and Lorentz invariance of the model and show the existence of a unique solution, which can, in this simple case, be constructed explicitly by a generalized method of characteristics. Furthermore, we derive that the interaction generated by the boundary conditions can be effectively described by δ-function potentials of the particle distances. It is then discussed whether the model can be generalized and it becomes clear that in higher spatial dimensions, the boundary conditions do not realize interaction. Therefore, one cannot hope to describe realistic physics with such a model, but it can serve as a first step in order to understand the interplay of the involved mechanisms.

Part III of this thesis finally treats the Bethe-Salpeter equation as an example for a multi-time integral equation. After explaining how it comes about as an equation that should describe bound states in quantum electrodynamics, we discuss several open problems such as gauge invariance, current conservation and divergences in the Bethe-Salpeter equation. The context of multi-time wave functions discussed in this thesis gives rise to new ideas how some of the issues with the equation can be resolved. Besides, we derive the non-relativistic limit of the Bethe-Salpeter equation in position space in order to demonstrate how non-relativistic equations with potential terms may arise from relativistic multi-time integral equations.

2 Clarification of the concepts

The necessity of clarifying the notions and concepts under consideration is often only appreciated in philosophical works, albeit it is impossible to discuss certain aspects of fundamental physical theories without establishing a consensus about what the words one uses are supposed to mean. By the way, explicating what your theory is about, clarifying its connection to reality is indeed a task for physicists and should not be foisted on the philosophers, following the guideline: do not leave your mess for others.
This thesis is about relativistic interactions in quantum theories. Therefore, we illuminate in the next three subsections how the terms *relativistic*, *interaction* and *quantum theory* should or could be understood.

2.1 What is relativistic?

We will always use the word relativistic in the sense of *special relativity* (SR) in this thesis. In the original formulation by Einstein [3], special relativity was based on two postulates:

- The relativity principle: The physical laws are independent of the inertial frame that is used.
- $c = \text{const.}$: The velocity of light (in vacuum) is independent of the relative motion of the light sources.

In Einstein's argumentation, the velocity of light must then also be the same for all observers because it comes out of a physical law, so the relativity principle applies.
Two years later, Minkowski laid the foundation of a formulation of special relativity with the help of a 4-dimensional Lorentzian manifold called *space-time*. This formulation is purely geometric and introduces coordinate systems only as a secondary concept. It looks quite elegant and natural, and this was surely one of the reasons for the success of SR. It was also acknowledged by Einstein that the most natural way to think about relativity is in terms of space-time structure [4, p. 34]. Known relativistic theories such as electrodynamics can be formulated in a geometric (or *covariant*) way, only using the structures of Minkowski space-time. Therefore, it seems well motivated to define a *relativistic theory* like Maudlin did in [5]:

> "[A] theory is compatible with Relativity if it can be formulated without ascribing to space-time any more or different intrinsic structure than the [...] relativistic metric."

This rule implies Lorentz invariance *in all aspects* of the theory. Understood in this sense, Lorentz invariance is a secondary concept related to changes of coordinate systems on a manifold and a necessary condition for theories to be compatible with Minkowski space-time. This seems a more natural point of view than starting with Lorentz invariance because, as Maudlin formulates nonchalantly [5], "coordinate systems do not, in any deep sense, exist".
The above definition of a relativistic theory might nevertheless be too restrictive and

make it impossible to find a relativistic quantum theory. A weaker requirement that is surely necessary for a theory to be compatible with experience is the Lorentz invariance of predictions of experimental outcomes. Hence, the right way to obtain a new physical theory might still be to relax the relativity postulate and to go back to some notion of absolute space or a preferred frame of reference. There are no known methods how to discriminate one inertial frame from another, so a theory with additional space-time structure should at least be constructed in such a way that this structure cannot be detected. A good example to study this are the Hypersurface-Bohm-Dirac models which assume a preferred foliation of space-time, but for observers and detectors, this foliation is invisible [6, 7]. This is related to the Lorentz ether theory which is empirically equivalent to Einstein's SR but declares one frame of reference to be the "true" one (the rest frame of the aether) and the other frames measure, in some sense, wrong effects that have to be corrected by Lorentz transformations [8, p. 188 ff.][2]. It seems to be extremely difficult to formulate a realistic, deterministic relativistic quantum theory without additional space-time structure [5].

General relativity shows that the geometric formulation of SR is very powerful and allows for generalization, which is a good argument for taking the structure of Minkowski space-time seriously and rejecting additional structures. On the other hand, there are space-times in general relativity that are, by themselves, endued with a kind of preferred frame that might be used for a Hypersurface-Bohm-Dirac theory, for example the comoving coordinates in a Friedmann universe (compare [9, p. 715]).

Let us briefly summarize the points discussed above: Although weaker requirements could be necessary, our desire is a fully relativistic theory, which means that it can be formulated geometrically in Minkowski space-time. Ultimately, it is a difficult open question if this requirement can be fulfilled in a realistic quantum theory (with interaction).

2.2 What is interaction?

Even though this question sounds as if it should be easy to answer, it is not; and there is little literature about the general concept of interaction. In our description, it does not include cases where one particle is subject to an *external* force or field, effective treatments in which back-reactions are neglected.

In Newtonian physics, one would naturally say that two particles or bodies interact if and only if there is a force acting between them. Newton's third law "action equals reaction" ensures that every action, every force appearing in classical physics is an *inter*-action, there cannot be an action of A on B that does not *re*-act on A. According to Newton's first law, a non-interacting particle can be recognized by its movement on a straight line with constant velocity.

But in a new theory, even free particles might behave in another way. This becomes apparent, for example, in the quantum mechanical description of free identical fermions[3]. The mere fact that there is a fermion "there" occupying a state already implies that

[2]The important difference is, of course, that the particle world-lines for which the foliation is needed have an extremely high explanatory value while the Lorentzian aether does not have any.

[3]The term *free* is usually used, like here, as a synonym for non-interacting.

another fermion cannot be in that state. This mechanism of Pauli's exclusion principle is usually not called interaction.
The usual quantum mechanical way to talk about interaction is to divide the Hamiltonian into two parts, $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, where $\hat{H}_0$ is the free part. A system is interacting if the interaction part $\hat{H}_{int}$ is different from zero. The form of the free Hamiltonian can be restrained by symmetry arguments which implement what it means to be non-interacting (compare [10, §17]). This way of defining interaction can of course not be applied in theories which are not in Hamiltonian form, like the multi-time theories presented in parts II and III. But since there is still a wave function in such a theory, there is a method of generalization: In the paper by Matthias Lienert [2] that will be discussed and generalized in part II, a model is defined to be interacting iff there are wave functions that are initially product states and become entangled with time. We can motivate this criterion by applying it to a two-particle wave function in non-relativistic quantum mechanics. Assume we have a wave function obeying a Schrödinger equation

$$i\frac{\partial\psi(\mathbf{x}_1,\mathbf{x}_2,t)}{\partial t} = \hat{H}(\mathbf{x}_1,\mathbf{x}_2,t)\psi(\mathbf{x}_1,\mathbf{x}_2,t), \tag{2.1}$$

and let the Hamiltonian be such that there is *no interaction* in the sense that every initial product state stays a product state during time evolution. So there are solutions

$$\psi(\mathbf{x}_1,\mathbf{x}_2,t) = f_1(\mathbf{x}_1,t)f_2(\mathbf{x}_2,t), \tag{2.2}$$

which can be inserted above:

$$\frac{\partial f_1(\mathbf{x}_1,t)}{\partial t}f_2(\mathbf{x}_2,t) + f_1(\mathbf{x}_1,t)\frac{\partial f_2(\mathbf{x}_2,t)}{\partial t} = -i\hat{H}(\mathbf{x}_1,\mathbf{x}_2,t)f_1(\mathbf{x}_1,t)f_2(\mathbf{x}_2,t) \tag{2.3}$$

The left side is a product of functions of $\mathbf{x}_1$ and $\mathbf{x}_2$. This directly implies that the right side can only be such a product, too. Therefore, the Hamiltonian may not contain any functions (potentials) depending on both $\mathbf{x}_1$ and $\mathbf{x}_2$ that are not in product form. It may still contain a term depending explicitly on the time t, according to our comment above that external fields do not correspond to interaction. Therefore, we conclude that the criterion gives reasonable results in non-relativistic Schrödinger theory and is a sensible definition of interaction. So, as long as we consider theories where the term entanglement is meaningful, we may use the criterion from [2], bearing in mind that it might be provisional.

2.3 What is quantum?

It seems difficult to state clearly which aspects are *essential* for a theory to be called a quantum theory. Still, there are some counterexamples, different formulations of quantum mechanics, that show which aspects are surely *not* essential or fundamental: all terms like *measurement, observables, operators*. A theory that uses words like this in its fundamental formulation is doomed to be ambiguous and cannot be made conceptually precise because there is no strict way in which *measurement* can be defined (compare

[11]). Luckily, there are formulations of quantum mechanics that do not need any of those suspicious words in their axioms. One of several examples is Bohmian mechanics, a theory in which it becomes clear in a very insightful way how operators as observables arise in special situations like the real measurements performed in laboratories [12].
Furthermore, we do not consider Hilbert spaces as too important, they are only mathematically helpful because they allow for a simple treatment of solutions to the Schrödinger equation and the application of the powerful methods of functional analysis. One can also do quantum mechanics without reference to Hilbert spaces and consider only strong (i.e. differentiable) solutions to the Schrödinger equation, compare [13].
So what is *quantum*? Historically, the starting point of quantum mechanics was the mixture of classical concepts with *quantization rules* as in the Bohr model of the atom. In the formulation used today, these rules are automatically encoded in the Schrödinger equation. The formulation of quantum theory with the help of a *wave function*, a function on configuration space, connects to the notion of *entanglement*. The description of interacting particles with one wave function leads to Bell-non-locality, a profound concept that can be used to characterize a quantum theory. The notion of Bell– or EPR–locality is defined by Maudlin as follows [14]:

> "A physical theory is EPR–local iff according to the theory procedures carried out in one region do not immediately disturb the physical state of systems in sufficiently distant regions in any significant way."

This is deliberately a weak statement in order to make its opposite even stronger. It was proven by the Bell-inequalities and their experimental violation that our world can not be described accurately by a Bell-local theory [15][4]. Therefore, a quantum theory in our sense is a theory that describes in some appropriate sense "quanta" which behave in such a way that the non-local Bell correlations occur.
And this is now the big tension mentioned in the introduction: Non-locality and relativity seem to fit together rather badly[5]. No reason to give up! One should expend some effort in the search for a theory of the world that is relativistic, but describes the violations of Bell-locality.

3 Remarks on Quantum Field Theory

3.1 Questions and questionable aspects

If one asks a standard textbook, the answer to our question whether there is a relativistically interacting quantum theory will be affirmative in the sense: Yes, of course, it is quantum field theory (QFT)! With the clarifications of section 2 at hand, let us

[4]We will not elucidate the different interpretations of the meaning of Bell's inequality here. In agreement with John Bell himself, it seems quite clear for the author that the only alternative to accepting non-locality as consequence of Bell's inequality is the renunciation of *describing our world at all*.

[5]They are, however, not in logical contradiction, as claimed some times, which will become apparent e.g.in section 4.4.

investigate in which sense QFT may or may not be called a relativistically interacting quantum theory that satisfies the requirements we impose on a fundamental theory. We will confine ourselves to quantum electrodynamics (QED), probably the most important QFT that describes basic phenomena such as radiation and electron scattering. Let us pose some questions about QED, which can in most cases be answered in the same way for a general quantum field theory[6].

- *Is QED relativistic?* With the works of Tomonaga and Schwinger [16, 17], a Lorentz invariant formulation of a quantum theory of fields was achieved on a formal level. But the statistical analysis of their theories has never been done successfully. Practical calculations in QED mainly yield (in perturbation theory) elements of the S-Matrix, which is the time evolution operator from $-\infty$ to ∞. For such an operator, Lorentz invariance is not a strict requirement because the notion of spatial and temporal infinity is the same in all inertial frames. It is good that the elements of the S-matrix are relativistically invariant, but a relativistic analysis on finite scales might require something different from the current framework.

- *Is QED interacting?* This is a hard question, since we would like to analyze the dynamics of the theory in order to see if our criterion for interaction from sec. 2.2 applies. But there is no dynamics, no well-defined "Schrödinger equation" of QED. The majority of the equations that are written down are ill-defined and there exist no solutions of the full theory with both fermion and photon field quantized.

- *Is QED quantum?* Quantum electrodynamics is of course quantum, not only because it is contained in its name, but also because one took the classical theory and then *quantized* it. This "quantization" may be a very formal procedure, but it yields a formalism with the help of which one can calculate many empirically well-established values. According to the above discussion, we require that a fully satisfactory quantum theory be able to predict the violation of Bell's inequalities. QED is not able to do so because there exists no dynamics on finite scales. Because of this shortcoming, it is also difficult to say in which sense other theories (non-relativistic QM or classical physics) are limiting cases of QED, compare [18].

Although QED provides a very powerful calculation scheme for making predictions of measurement outcomes, we deduce from the answers given above that it does not qualify as a fundamental quantum theory with relativistic interactions. The main point of criticism is the non-existence of dynamics. QED does indeed give a number of insights about the world and helps to understand experimental data especially in scattering situations, but it can only be seen as an effective description. There are two main reasons for this:

1. *Logical and mathematical consistency:* The elements of the S-matrix in QED can only be computed with the help of a perturbation series which is expected to

[6]There are some well-defined quantum field theories in 1+1 dimensions, but we will not discuss them here and focus on the standard model of particle physics and the quantum field theories that are indeed used for the description of our world.

diverge (compare [19]). Furthermore, the infinities that arise in calculations and have to be *renormalized* show that the theory is not even mathematically well-defined order by order. It is known that QED is renormalizable and one can get rid of the infinities in every case, but still, as Landau and Lifschitz put it [20, p. 4][7],

> "Nonetheless, these procedures have the character of half-empirical recipes, and our conviction that the results obtained this way are correct is eventually based on their outstanding agreement with experiment, but not on the inner consistency and the logical clarity of the theory's fundamental principles."

2. *Conceptual clarity:* It is unproblematic to regard the S-matrix formalism as an effective description. The distances of many measurements e.g. in scattering situations are really *very large* compared to scales of elementary particles, so the limit $x, t \to \infty$ is well motivated. Compare the thermodynamic limit $N \to \infty$, which is never truly fulfilled for a physical system, but the behaviour of systems with very large N is very close to it. A fundamental theory, in contrast, must not be restricted to giving an effective description; it ought to describe every phenomenon (or at least a wide range of phenomena) in the world in a clear way, without inconsistencies such as the measurement problem. QED is insufficient in this sense also because its application to bound systems is questionable, as will be discussed in more detail in sec. III, so every-day situations and the stability of tables or elephants cannot be explained by QED.
   Moreover – even if one is only interested in measurement outcomes – there are new experiments in reach, for example those at the ELI (extreme light infrastructure), where non-perturbative QED effects will become visible and the effective QED formalism will presumably not be able to predict the outcomes [21].

The discussion above should make clear that QED, despite its virtues, cannot serve as a fundamental theory or provide fundamental insight about nature. Because of the unsolved problems it contains, it is in the author's opinion not a good idea to try to build something new by extending its formalism and ideas in some way – as it is done in most approaches to quantum gravity. Instead, one should go back some steps and build a more stable foundation on which entirely new physical theories can be based. Quantum field theories should come out of these new theories as effective descriptions in several special situations, exactly like thermodynamics is derived from classical Newtonian mechanics.

3.2 No-go theorems about relativistic quantum mechanics of particles

Having expounded the problems of quantum field theories, one might now be tempted to look for alternatives. The first idea is to take particles as the basic concept of the

[7] My translation from the German version: "Trotzdem haben diese Verfahren weitgehend den Charakter halbempirischer Rezepte, und unsere Überzeugung von der Richtigkeit der auf diesem Wege erhaltenen Ergebnisse beruht letzten Endes auf ihrer hervorragenden Übereinstimmung mit dem Experiment, aber nicht auf der inneren Konsistenz und der logischen Klarheit der Grundprinzipien der Theorie."

theory and not only view it in some vague way as an excitation of a field. Edwin Jaynes explains in [22] why the quantization of both matter and electromagnetic field is the source of many problems in QFT:

> "One can hardly imagine a better way to generate infinities in physical predictions than by having a mathematical formalism with $(\infty)^2$ more degrees of freedom than are actually used by Nature."

Therefore, a theory dealing only with particles could be worthwhile. This may entail both models with a fixed number of particles and a variable number of particles. However, there is a number of arguments that are supposed to show that a particle theory of relativistic quantum mechanics is impossible. Therefore, before going on with our task, we consider those no-go theorems and explain their implications. In the following, we discuss

- a superficial argument found in a QFT textbook,
- the theorem against relativistic quantum theories of particles by Malament,
- the "no interaction"-theorem by Currie, Jordan and Sudarshan.

The popular textbook about QFT by Peskin and Schroeder starts out with a chapter on "The necessity of the field viewpoint" [23, p. 13f.]. Their argument for this "necessity" is a calculation of a propagator for one particle, which does not vanish outside the light-cone. This is considered to be in disagreement with causality (as Maudlin points out [24], this is also not necessarily true). However, their calculation uses the Hamiltonian $H = \sqrt{\mathbf{p}^2 + m^2}$ which is a pseudo-differential operator and *not* the right Hamiltonian for relativistic quantum mechanics. If one uses the Dirac Hamiltonian, these problems do not arise, because the propagator is indeed equal to zero outside the light-cone (as proven in [25, p. 15] and [26]). Therefore, the whole argument of Peskin and Schroeder only shows that one should use the Dirac equation for relativistic quantum mechanics (of one particle, at least). A field viewpoint has nothing to do with this.
A well-known article featuring a no-go theorem is David Malament's "In defense of dogma: Why there cannot be a relativistic quantum mechanics of (localizable) particles" [27]. Malament claims that his theorem proves the impossibility of the concept of *localization* in a relativistic quantum theory. It is, however, only a theorem about *operators* that do not exist under certain conditions. As operators need not be considered the basic objects in a quantum theory, but can be derived as bookkeeping objects in situations that resemble measurements, this theorem says nothing definite about the possibilities to construct a relativistic quantum mechanical theory of particles. In Christian Beck's PhD thesis [28], a generalized version of Malament's theorem is proven and it is explained that the real implications of the no-go theorem are not about ontological or fundamental questions, but about the possibilities of certain measurements and detectors in relativistic quantum mechanics. This is clear because all talk about operators is talk about measurements in the end. Still, the theorem has striking consequences: In a world with one particle, it is always possible with probability greater than zero

that two spatially separated detectors click (i.e. detect a particle) at the same time [28]. Ideal measurements like in non-relativistic quantum mechanics are therefore impossible in principle. Beck argues that this is due to unavoidable effects of pair creation (vacuum fluctuations) induced by the presence of the detector. We are thus forced to acknowledge that detectors are not (and have never been) able to measure something without changing it, as they themselves are physical entities that interact with the rest of the world (compare the philosophical treatment in [29]).
There is something else to discuss about Malament's theorem that is a generic feature of many no-go theorems: It needs quite restrictive assumptions that are sold as normal and natural, but usually exactly hide the real problem. Malament needs the two important assumptions that

- physical states are defined on some Hilbert space $\mathscr{H}$,
- there is a Hamiltonian operator $H : \mathscr{H} \to \mathscr{H}$ generating time evolution which is bounded from below.

We have argued above that Hilbert spaces are rather unimportant and there is no reason why one should take for granted that a new theory is formulated with the help of those objects. The timelessness of $\mathscr{H}$ and the implementation of time evolution with a unitary time operator fit perfectly to the non-relativistic case, but are highly asymmetric in their treatment of space and time and should be replaced in a relativistic setting. Moreover, the boundedness from below is not as innocent an assumption as it may seem: The usual Dirac Hamiltonian is not bounded from below. If we move to the usual Fock space where the negative energy states are reinterpreted as positive energy states of antiparticles, the Hamiltonian becomes bounded from below, so the theorem holds. Its implication is then that a localized measurement cannot ideally determine the particle number, but not more.
Another example of an often-cited no-go result is the so-called "no interaction"-theorem [30] which is supposed to show that relativistic theories of particles, both classical and quantum, cannot be interacting (and therefore, another time, we have to consider fields). Its assumptions contain again the generation of time evolution by a Hamiltonian operator on Hilbert space. Additionally, it is made use of a kind of relativistic position operator whose existence and status is unclear in the quantum case. The authors come to the misleading result that particles are the problem that make relativistic theories necessarily non-interacting. But in fact, the real implication of their theorem is the other way around: if one looks for interacting relativistic quantum theories of particles (why not?), then one should probably overcome the Hamiltonian formalism, or consider models with particle creation or annihilation, which are not covered by the theorem.
In general, the problem of no-go theorems is that they have to make some severe assumptions about the structure of the theory that are usually too restrictive. Consequently, Federico Laudisa argues in his article "Against the 'no-go' philosophy of quantum mechanics" [31] that instead of proving quite generic results with little importance, one should think about the *actual* alternatives to the present formulations of theories (or invent new alternatives) and discuss to what extent they have the desired properties.

4 Non-interacting relativistic quantum theories

"A great deal of my work is just playing with equations and seeing what they give. [...] I think it's a peculiarity of myself that I like to play about with equations, just looking for beautiful mathematical relations which maybe don't have any physical meaning at all. Sometimes they do." - Paul Dirac

4.1 The Dirac equation for one particle

When Dirac looked for a relativistic generalization of the Schrödinger equation, he realized that this could not be done without the introduction of a wave function with several components, a *spinor*. In three spatial dimensions, the Dirac equation is a relativistic wave equation for a 4-component spinor ψ and reads ($x = (t, \mathbf{x})$)

$$(i\partial\!\!\!/ - m)\psi(x) = 0, \tag{4.1}$$

where $\partial\!\!\!/ := \gamma^\mu \partial_\mu$ and γ^μ are the famous Dirac matrices. In Hamiltonian form, to see the connection to the Schrödinger equation more clearly, one can write

$$i\partial_t \psi(x) = (-i\gamma^0\gamma^j\partial_j + \gamma^0 m)\psi(x) =: H_D(x)\psi(x) \tag{4.2}$$

So far, this is a one-particle equation for a spin-$\frac{1}{2}$ particle. In contrast to the non-relativistic equation, which can be formulated for N particles without any problems as

$$i\partial_t \varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N) = \left(\sum_{j=1}^{N} H_j^{\text{free}}(\mathbf{x}_j) + \sum_{j \neq k} V(\mathbf{x}_j, \mathbf{x}_k) \right) \varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N), \tag{4.3}$$

the Dirac equation has no obvious generalization to multi-particle situations. (In the next subsection, we will see that it has to be generalized to a *system* of equations.) Indeed, the equation has a number of problems even in the one-particle case so that the even harder question of generalization to interaction between several particles is normally not discussed in textbooks[8]. The spectrum of the free Dirac operator is given by (see e.g. [25])

$$\sigma(H_D) = (-\infty, -m] \cup [m, \infty)\,, \tag{4.4}$$

which has been puzzling physicists for nearly a century by now. The negative energy solutions of the Dirac equation are a severe problem because one could obtain an infinite amount of energy from a particle if it were coupled to a radiation field or something similar, which goes under the name of *radiation catastrophe*. The ad-hoc solution for this problem was the introduction of the *Dirac sea* [32]. Dirac postulated that all negative energy states are occupied and thus the Pauli exclusion principle forbids a particle to fall down to the negative energy states. Holes in the Dirac sea can then be interpreted as anti-particles: Dirac's approach led to the prediction of the existence of the positron,

[8] It is often heard that one is forced to change to a field description in order to go beyond the one-particle Dirac equation, but as QFT has, as stated above, numerous problems, this cannot be the be-all and end-all in relativistic quantum theory.

which was a great success for his formulation. But there are many open questions and conceptual problems with this approach that remain unsolved. Beyond others there is the problem of the infinite number of particles necessary to fill the Dirac sea[9] and the question of how their dynamics looks like and how it can become effectively negligible. To date, one can say that the status of the Dirac equation is still not fully clear.
One should remark, however, that the free Dirac equation without external fields is still physically reasonable without any reference to negative energies because an initial state with positive energy stays forever in the positive energy sector.
For the one-particle Dirac equation, one can easily prove that the current

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x), \tag{4.5}$$

where $\bar{\psi} := \psi^\dagger\gamma^0$, is conserved in the sense of

$$\partial_\mu j^\mu = 0. \tag{4.6}$$

This conserved current is closely related to the probabilistic interpretation of the wave function, which will be discussed in detail in section 7.1. We only note here that the zero-component is exactly given by $j^0 = \psi^\dagger\psi$, the well-known probability density from non-relativistic quantum mechanics.

4.2 Multi-time wave functions

When we consider a relativistic quantum theory with more than one particle, the first question is how the idea of a wave function can be adopted. Dirac was the first one to discuss this and proposed the idea of a multi-time wave function [33, 34]. A wave function in non-relativistic QM for N particles in d spatial dimensions is a map

$$\varphi : \mathbb{R} \times (\mathbb{R}^d)^N \to \mathcal{S}, \quad (t, \mathbf{x}_1, ..., \mathbf{x}_N) \mapsto \varphi(t, \mathbf{x}_1, ..., \mathbf{x}_N), \tag{4.7}$$

with the appropriate spin space, usually

$$\mathcal{S} = \underbrace{\mathbb{C}^{2s+1} \otimes \cdots \otimes \mathbb{C}^{2s+1}}_{N \text{ times}} \equiv (\mathbb{C}^{2s+1})^{\otimes N}, \tag{4.8}$$

for N particles with spin s. So φ has arguments in configuration space for each fixed t which is seen as an external time parameter. This is as non-relativistic as can be, because time and space are treated very differently. A first step is to replace the point $(t,\ \mathbf{x}_1, ..., \mathbf{x}_N)$ by the synonymous collection of space-time events $((t, \mathbf{x}_1), (t, \mathbf{x}_2), \ldots, (t, \mathbf{x}_N))$. This makes it possible to think of the wave function as a function on Galilean configuration space-time, which maps each configuration $(t, \mathbf{x}_1, ..., t, \mathbf{x}_N)$ with equal time coordinates to a vector in $\mathcal{S}$. From this viewpoint, it is rather easy to see how to generalize the notion of a wave function to relativity. There, we have no absolute notion of simultaneity at hand, there are only light-cones and the concepts of space-like

[9] This problem does not vanish just by changing to the mathematically equivalent picture of the vacuum as is customary in QFT, because this vacuum has infinite energy.

and time-like directions. Therefore, we should expect that a relativistic wave-function is a map on configuration space-time,

$$\psi : \Omega \subset (\mathbb{R}^{d+1})^N \to \mathcal{S}, \quad (t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N) \mapsto \psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N), \tag{4.9}$$

that depends on N space-time points $x_j = (t_j, \mathbf{x}_j)$, and thereby on N time coordinates – thus the name *multi-time wave function*. The subset Ω of configuration space-time that should be considered is the set of space-like configurations

$$\mathscr{S} := \left\{ (x_1, ..., x_N) \in (\mathbb{R}^{d+1})^N \middle| \, (t_j - t_k)^2 - |\mathbf{x}_j - \mathbf{x}_k|^2 < 0 \; \forall j \neq k \right\}, \tag{4.10}$$

because space-likeness is the geometric relativistic generalization of simultaneity. Even more basically speaking, the notion of an N-particle configuration does only make sense on a space-like hypersurface, because the world-line of a particle will cross every space-like hypersurface exactly once, but any other arbitrarily often. We can also expect that the wave-function should be integrable over a space-like hypersurface, but not over a time-like one, because this is also not possible in the non-relativistic case ($\int_{\mathbb{R}} dt \; \psi(t, \mathbf{x})$ typically diverges). Further evidence for this restriction to space-like configurations being reasonable is provided by several multi-time models which are only consistent on these configurations and not on the whole of configuration space-time, see [35], and by the comparison with the Tomonaga-Schwinger formulation of QED, where the wave-function is defined on space-like hypersurfaces.

As discussed by Bloch [36], one might conjecture that a natural statistical interpretation of the multi-time wave function $\psi(t_1, \mathbf{x}_1, ..., t_N, \mathbf{x}_N)$ on a space-like configuration is that its square describes the probability of particle 1 being at place $\mathbf{x}_1$ at time t_1 and particle 2 being at $\mathbf{x}_2$ at time t_2 and so on. However, there is neither experimental evidence for this nor a deeper theoretical understanding; which would require a definite theory of relativistic quantum mechanics.

In a nutshell, the multi-time picture which uses a wave function $\psi(x_1, ..., x_N)$ constitutes the Schrödinger picture of relativistic QM. In [37], where many basic ideas of multi-time dynamics were explicated for the first time, it is also explained in which way one can see its equivalence to the relativistic Heisenberg picture, given by field operators $\hat{\Phi}(x)$ acting on some Fock space. This equivalence only holds under certain conditions for the dynamics of the multi-time wave function. The Schrödinger picture does not only provide a more direct approach on what happens, it also has the decisive advantages of not using the questionable operators as basic objects and of allowing for a more general class of dynamics such as the one discussed in part III.

4.3 Free multi-time evolution

Because a multi-time wave function $\psi(x_1, ..., x_N)$ has to be evolved in N time coordinates, one may now guess how to generalize the one-particle Dirac equation (4.2). Adding up the N single-time Hamiltonians will not help at all because one has to evolve N time arguments, so one should better impose a *system of N equations*:

$$i\frac{\partial}{\partial t_j}\psi(x_1, ..., x_N) = \left(-i\gamma_j^0 \gamma_j^a \frac{\partial}{\partial x_j^a} + \gamma_j^0 m_j \right) \psi(x_1, ..., x_N), \quad j = 1, ..., N. \tag{4.11}$$

The right hand side is defined as the *partial Hamiltonian* H_j. For comparison with the familiar single-time models, one may set all times equal and use the chain rule to get

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(t,\mathbf{x}_1,...,t,\mathbf{x}_N) &= \sum_{j=1}^{N} i\frac{\partial}{\partial t_j}\psi(t_1,\mathbf{x}_1,...,t_N,\mathbf{x}_N)\Big|_{t_1=...=t_N=t} \\ &= \sum_{j=1}^{N} H_j(x_j)\psi(t,\mathbf{x}_1,...,t,\mathbf{x}_N). \end{aligned} \tag{4.12}$$

In case of the free Dirac Hamiltonians, the full system (4.11) admits unique solutions for given initial values on a Cauchy surface (compare [38]), in the simplest case $\psi(0,\mathbf{x}_1,...,0,\mathbf{x}_N)$. This is because the partial Hamiltonians H_j are all self-adjoint on a common domain and commute with each other, which implies the existence of a unitary N-parameter group $U(t_1,...t_N)$ such that

$$\psi(t_1,\mathbf{x_1},...,t_N,\mathbf{x}_N) = U(t_1,...t_N)\psi(0,\mathbf{x}_1,...,0,\mathbf{x}_N) \tag{4.13}$$

is a solution of (4.11). Details are found in [37, sec. 1.3].

4.4 Relativistic Collapse Theory

Before proceeding to the question of interaction, we present a way how the multi-time wave functions can be used to build a quantum theory which may be called "fully relativistic" according to our criterion in sec. 2.1. This is the relativistic version of the collapse model of Ghirardi, Rimini and Weber due to Roderich Tumulka [39]. The general idea of these models is to modify of the Schrödinger equation by adding a stochastic term that describes the *collapse* of the wave function as a real, objective process, not only an unclear prescription connected with *measurements*. The relativistic GRW model (rGRW) uses a multi-time wave function that satisfies the system of N Dirac equations (4.11), only with the modification that at random times, there is a collapse of the wavefunction, modeled by multiplication with a Gaussian centered around a random place (which is picked according to the $|\psi|^2$-distribution). With this evolution of the wavefunction at hand, one can define a way matter behaves in order to obtain a connection with our real world. There are two possibilities:

- The flash ontology (rGRWf), where the space-time points around which the collapses are centered, so-called *flashes*, are taken to be the only thing there is [39]. An object as we see it would then be explained dynamically by our perception of a *galaxy of flashes*.

- The matter density ontology (rGRWm), where *primitive stuff* (matter) is distributed according to the $|\psi|^2$-distribution. In the relativistic case, one must of course not take the distribution at equal times, but instead evaluate $|\psi|^2$ at the past light-cone of each event [40].

Both rGRWf and rGRWm are relativistic quantum theories, but so far without interaction. The models introduce two new parameters, the average time between two collapses, τ, and the width of the Gaussian, a. Because the collapse is not complete (the tails of the Gaussian extend up to infinity), there is a slight deviation from orthodox quantum mechanics. But if one chooses the parameters of order $a \sim 10^{-7}$ m and $\tau \sim 10^{15}$ s, no present experiment can distinguish the GRW models from orthodox quantum mechanics or Bohmian mechanics. Still, an improvement of experimental techniques can confine the parameters further and could lead to an empirical decision between collapse models and models that retain the Schrödinger equation [41].
Moreover, one can attack the relativistic GRW model from several directions by basic arguments. The first obvious point is the fundamental randomness that is hard to grasp. Second, one may criticize the two ontologies. The flash ontology is rather unintuitive and leads to seemingly paradoxical statements like: there are some times, even small time intervals, when there is *nothing* there of my chair. (Because the flashes are randomly distributed discrete space-time events, this will happen some time.) Still, the model reminds us that our perception of the world comes only from averaging over quite large scales in space and time and all effects have to be explained dynamically, and the dynamics is such that I do not have to fear falling to the ground when sitting on my chair. The matter density ontology has the unpleasant feature that the stuff moving around in our world does not do that in a continuous way, but "jumps around" when a collapse of the wave function occurs [40]. Models with a continuous way of collapsing the wave function like the CSL (continuous spontaneous localization) model exist, too, but suffer from divergences if one tries to make them relativistic [42].
Although these are all valid points that deserve some attention, the relativistic GRW-theory should not be discarded: it is the first quantum theory that is fully relativistic, i.e. adds no external structure to Minkowski space-time.

5 Approaches to interaction in multi-time quantum theories

In this section, several approaches to generate interaction in relativistic multi-time quantum theories are discussed. In sec. 5.1, we show that the well-known mechanism of interaction via potentials is not possible in this case. Consequently, we discuss a variety of other conceivable mechanisms of interaction in sec. 5.2. Let us remark that the question of interaction will only be treated on the level of the wave function. If one is able to construct a suitable interacting (multi-time) wave function equation with a suitable conserved current, one will also be able to construct a realistic relativistic quantum theory out of it, e.g. a Hypersurface-Bohm-Dirac theory or a theory with objective collapse like the one described in section 4.4. The problems with the construction of such theories are independent of the question discussed here.

5.1 Consistency conditions and no-go theorem for potentials

In non-relativistic quantum mechanics, interactions are usually modeled by addition of an interaction potential

$$\hat{H}_{int} = \sum_{i \neq j} V(\mathbf{x}_i, \mathbf{x}_j) \tag{5.1}$$

to the Hamiltonian, for example a Coulomb potential $V(\mathbf{x}_i, \mathbf{x}_j) \propto |\mathbf{x}_i - \mathbf{x}_j|^{-1}$. Doing this for the Dirac equation leads to an explanation of the fine structure of hydrogen. Evidently, one would first try doing something similar in the case of relativistic multi-time equations, writing down a system of N equations similar to (4.11)

$$i\frac{\partial}{\partial t_j}\psi(x_1, ..., x_N) = H_j\psi(x_1, ..., x_N), \quad j = 1, ..., N, \tag{5.2}$$

where H_j could be the Dirac Hamiltonian plus a suitable interaction potential. But the evolution in *several* time components does not work that easily. It should be assured by the equations that the time evolution is not path-dependent, i.e. one arrives at the same result if one evolves *first* in t_j-direction and *then* in t_k-direction or the other way around. This leads to so-called *consistency conditions* which are in the simplest case given by the following lemma.

Lemma 5.1 *(Corollary 1.2.3 in [37]) Let H_j be self-adjoint operators with common domain on a Hilbert space $\mathscr{H}$ for all $j = 1, ..., N$. For initial data $\psi(\mathbf{x}_1, ..., \mathbf{x}_N) \in \mathscr{H}$ given at $t_1 = \cdots = t_N = 0$, there exists a unique $\mathscr{H}$-valued solution of the system* (5.2) *if and only if the Hamiltonians commute, i.e.*

$$[H_j, H_k] = 0 \quad \forall j, k. \tag{5.3}$$

Idea of proof: The lemma is a corollary of theorem VIII.12 in [43]. The main idea is that invoking

$$e^{-it_jH_j}e^{-it_kH_k}\psi(0, \mathbf{x}_1, ...0, \mathbf{x}_N) \overset{!}{=} e^{-it_kH_k}e^{-it_jH_j}\psi(0, \mathbf{x}_1, ...0, \mathbf{x}_N) \tag{5.4}$$

for all ψ in the domain of the operators necessarily leads to $[H_j, H_k] = 0$. For more details, we refer to [37].

The case treated in lemma 5.1 is simple because the Hamiltonians can be considered operators on the time-less Hilbert space and do therefore themselves not depend on time. An actual relativistic potential must of course also depend on time coordinates. In such a setting, the consistency condition (5.3) generalizes to [26]:

$$\left[\frac{\partial}{\partial t_j} - H_j, \frac{\partial}{\partial t_k} - H_k\right] = 0 \quad \forall j, k. \tag{5.5}$$

It turns out that these consistency conditions are quite restrictive and do not allow interaction potentials, as is proven by Petrat and Tumulka in [26]. Their theorem assumes

that the Hamiltonians H_j are the sum of a free Dirac (or Schrödinger) Hamiltonian and a smooth interaction potential V_j which is acting only on the spin component of the j-th particle. Then it follows that the consistency conditions (5.5) are satisfied if and only if the potential is equivalent (via a phase transformation) to a potential that only depends on the coordinates of one particle, $\tilde{V}_j(x_j)$. Therefore, interactions between the particles are excluded.
One should remark that the assumptions of Petrat's and Tumulka's theorem are too restrictive, because the potentials one usually considers in relativistic QM (e.g. the Breit potential) are neither smooth nor only dependent on the spin components of one particle. Nevertheless, the no-go theorem shows that interaction potentials are not well-fit to relativistic quantum mechanics. There may be some residual possibilities to explore, but it is much more promising to focus on new mechanisms of relativistic interaction.

5.2 Possible mechanisms of interaction

We are now ready to tackle the main question: Which mechanisms of relativistic interaction may exist in a quantum theory? Of course, we cannot give an exhaustive list (and often, the solution to such a difficult problem is in the end something that no one has ever thought of before). The following list, however, features a number of possibilities for relativistic interactions in a quantum theory. We will stay in the context of multi-time wave functions, of course noticing that there may also be other possibilities for RQM, but none as natural and promising as the use of multi-time wave functions.

- **Generalized potentials:** The potentials excluded by the no-go theorem are assumed to be multiplication operators, so a generalization to "potentials" which also include differential operators might come into one's mind. And indeed, this path has been taken for the two-body Dirac equations of constraint theory (see [44] for a treatment from the multi-time perspective). These are multi-time Dirac equations for two particles with potential terms dependent not only on spatial coordinates, but also on the total momentum operator. By the correspondence $p_\mu \to i\partial_\mu$, we see that this means that differentials with respect to both time and space coordinates are contained in the potential, usually up to infinite order. Indeed, it is possible to create interaction in this way, but the equations suffer from other problems: the free Dirac current is not conserved, hence one has to find modified conserved currents. But these can be chosen with some arbitrariness and do often not allow a probabilistic interpretation because they do not provide a positive density [44]. We will not further pursue this option here also because the approach seems to be limited to a particle number of $N = 2$.

- **Pair creation:** In their paper [26], Petrat and Tumulka state that the exclusion of interaction potentials in multi-time evolution equations "naturally leads us to considering particle creation and annihilation". The empirical success of QFT with its picture of particles and anti-particles as non-conserved entities as well as the possibility of obtaining effective potentials (e.g. Yukawa potentials) via particle

exchange may well promote this thinking. But we note that, to our knowledge, no one has succeeded in writing down a relativistic theory of particle creation and annihilation in three dimensions that does not have ultraviolet divergences. We agree with Tumulka's view that the problem of divergences is best attacked in the Schrödinger picture, which means by use of multi-time wave functions, but the divergences lead us to focus on other approaches in this thesis.
For skeptics, we should mention that in order to explain empirical effects like electron-positron-pair creation, a theory does not necessarily have to feature a changing particle number. Pair creation can also be well understood with the help of the Dirac sea picture, where a positron is only a hole in the "sea" of negative-energy electrons, and therefore, the particle number is indeed conserved, there are only N electrons[10] which may change from negative to positive energies and back.

- **Boundary conditions:** In sec. 4.2, we saw that the multi-time wave function has as natural domain the space-like configurations $\mathscr{S}$. Since this is a set with non-empty boundary, the question arises whether one should or could impose boundary conditions on (parts of) its boundary and whether these boundary conditions have any impact on the dynamics. A rigorous analysis of these questions will be done in part II of this thesis, starting with a 1+1-dimensional model where interaction by boundary conditions can indeed be achieved. Generalizations to higher dimensions and the question whether the interaction can be retained are then discussed in sec. 10.
  A combination of the two previous points is the approach of so-called *interior boundary conditions* (IBC) by Teufel and Tumulka [45]. The idea is to write down divergence-free Hamiltonians for pair creation by using boundary conditions on the set of coincidence points (where the space-time coordinates of two particles agree). Insofar, their method is in some sense a generalization of our model presented in part II to pair creation. Up to now, their method was only applied to non- or semi-relativistic situations and the question whether relativistic interactions can be created by IBCs is open. We will further comment on this in 10.3.

- **Integral equations:** The issue of consistency only appears when we consider a system of N differential equations for the multi-time wave function. If the dynamics is contained in only one equation, there is possibly more freedom to implement interactions. This naturally leads to the concept of integral or integro-differential equations. Let us consider a simple example, a system of two free Dirac equations,

$$\begin{aligned} i\frac{\partial}{\partial t_1}\psi(x_1,x_2) &= H_D(x_1)\psi(x_1,x_2) \\ i\frac{\partial}{\partial t_2}\psi(x_1,x_2) &= H_D(x_2)\psi(x_1,x_2), \end{aligned} \tag{5.6}$$

[10] To date, one really uses infinitely many particles to fill the Dirac sea, but one may hope to reduce this number to a finite N.

which can be rewritten as one integral equation quite easily because the particles 1 and 2 evolve independently. Using the free Dirac propagator defined by

$$\left(i\frac{\partial}{\partial t_j} - H_D(x_j)\right) S_D(x_j, x_j') = \delta^{(4)}(x_j - x_j'), \quad j = 1, 2, \tag{5.7}$$

we can write (5.6) as [46, p. 333]

$$\psi(x_1, x_2) = \int_{S_1 \times S_2} d\sigma(y_1) d\sigma(y_2)\ iS_D(y_1, x_1)\ iS_D(y_2, x_2)\ \not{n}_1(y_1)\not{n}_2(y_2)\ \psi(y_1, y_2), \tag{5.8}$$

where the integration is over arbitrary surfaces $S_j,\ j = 1, 2$, enclosing x_j, and n_j is the normal vector to the surface S_j.

It is now viable to write down equations similar to (5.8) that are not reducible to a differential equation and thus behave completely differently from the systems of multi-time differential equations discussed so far. Indeed, there exists one such equation that is known since 1951, the Bethe-Salpeter equation. It is supposed to describe bound states in QED and has a number of interesting physical and mathematical properties that will be discussed in part III of this thesis. We will see that it has problems we are not yet able to overcome, but it serves as a promising example of how integral equations may generate relativistic interactions in a quantum theory.

Part II
A simple interacting multi-time model of N fermions in one dimension

In his paper "A relativistically interacting exactly solvable multi-time model for two mass-less Dirac particles in 1 + 1 dimensions" [2], Matthias Lienert gave an example for a multi-time model that has the desirable features: interaction, Lorentz-invariance, probability conservation and compatibility with antisymmetry (i.e. identical particles). The follow-up paper [1] by Lienert and the present author generalized these results to N particles and also showed that a generalized domain is not helpful because it does not allow for non-trivial solutions. For this, we refer also to Lienert's PhD thesis [37]. This part is a slightly modified and more detailed version of the second paper, where the relevant results from the first paper are stated and explained again. Only section 9.3 about self-adjoint extensions in the single-time model is exclusively found in this thesis and the introduction is replaced by a shortened motivation. The main points that motivate the model were already worked out in part I of this thesis.

6 General properties of the model

6.1 Motivation

In section 5.2, we discussed several mechanisms of interaction that are possible in the multi-time formalism. One of those was the choice of certain boundary conditions for the multi-time wave function ψ that will turn out to generate entanglement in the time evolution. Even if we did not hope that interaction was possible this way, the question of boundary conditions would naturally arise because we consider the wave function on a domain with non-empty boundary, namely $\mathscr{S}$ (or possibly a subset of $\mathscr{S}$).
The model aims at understanding the interplay between boundary conditions, interaction, Lorentz invariance and probability conservation. We focus on a strongly simplified setting of mass-less particles in one spatial dimension, which allows us to solve the model explicitly even in the case of N particles and thereby grasp the relation of the mentioned concepts.
We need to briefly comment on the usual method of introducing boundary conditions in quantum systems, common especially in the field of zero-range physics (see [47]). There, the boundary conditions are prescribed via the domains of the respective Hamiltonians. It is addressed in depth in [2] that this method is not suitable here, since the Hilbert spaces are time-less and the time coordinates come as external parameters in that formalism. Relativistic boundary conditions may depend on multiple time and space coordinates, so we have to impose them not in domains of operators, but in a more direct way as supplementary conditions for the solutions of partial differential equations. Therefore, we will encounter the methods of functional analysis only in section 9.3, where an effective single-time model is constructed out of our fully relativistic model.

This part is organized as follows: After defining the model, we specialize on identical particles (sec. 6.3) and give the general solution in sec. 6.4. A general discussion about probability conservation in relativistic QM follows (sec. 7.1) and its implication, the uniqueness of solutions, is proven (sec. 7.2). Then we investigate which boundary conditions are compatible with both probability conservation (sec. 7.3) and Lorentz invariance (sec. 7.4). The main theorem providing the explicit unique solution of the model is stated and proven in section 8. Afterwards, we demonstrate in sec. 9 that the model is interacting and construct the corresponding single-time model, which shows that our model is the fully relativistic version of an N-particle Dirac equation with δ-function potential. This is done rigorously via self-adjoint extensions of the Dirac Hamiltonian. Lastly, section 10 is dedicated to the question of whether generalizations of the model are feasible.

6.2 Definition of the model

Our model is based on a multi-time wave function for N mass-less Dirac particles on the set of space-like configurations

$$\mathscr{S} \;:=\; \{(t_1, z_1, ..., t_N, z_N) \in \mathbb{R}^{2N} : (t_j - t_k)^2 - (z_j - z_k)^2 < 0 \;\forall j \neq k\} \tag{6.1}$$

in $1+1$-dimensional space-time, with metric $g = \mathrm{diag}(1, -1)$. The appropriate spin space is $(\mathbb{C}^2)^{\otimes N}$. Thus, ψ has 2^N spin components $\psi_i,\ i = 1, ..., 2^N$.
As multi-time evolution equations we use a system of N mass-less Dirac equations

$$i\gamma_k^\mu \partial_{k,\mu}\, \psi(x_1, ..., x_N) \;=\; 0, \quad k = 1, ..., N. \tag{6.2}$$

Here, $x_k = (t_k, z_k)$, $\partial_{k,\mu} = \frac{\partial}{\partial x_k^\mu}$ and γ_k^μ is an abbreviation for

$$\mathbb{1} \otimes \cdots \underbrace{\otimes\, \gamma^\mu \,\otimes}_{\text{k-th place}} \cdots \otimes \mathbb{1}. \tag{6.3}$$

The familiar Gamma matrices from $3+1$ dimensions are 4×4-matrices, but their size depends on the dimension of space-time, as the relevant condition on the matrices is the Clifford algebra relation (see e.g. [48, p. 85])

$$[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}\mathbb{1}. \tag{6.4}$$

In one spatial dimension, this can already be fulfilled by (2×2)-matrices. We choose the representation

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \sigma_1\sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6.5}$$

($\sigma_i,\ i = 1, 2, 3$, denote the Pauli matrices), which is extremely practical because it diagonalizes the multi-time Dirac equations without mass. If one multiplies eq. (6.2) with γ_k^0 from the left, this results in

$$\left(\frac{\partial}{\partial t_k} + \sigma_{3,k}\frac{\partial}{\partial z_k}\right) \psi(t_1, z_1, ..., t_N, z_N) \;=\; 0, \quad k = 1, ..., N, \tag{6.6}$$

with the usual shorthand notation $\sigma_{3,k}$ standing for a tensor product of identity matrices with a σ_3 matrix at the k-th place. We will soon see that the general solution of this system of equations is easily found because of its diagonal form.
Initial data are prescribed on the set

$$\mathcal{I} \;:=\; \{(t_1, z_1, ..., t_N, z_N) \in \overline{\mathscr{S}} : t_1 = \cdots = t_N = 0\}. \tag{6.7}$$

Since $\mathscr{S}$ has a non-empty boundary $\partial\mathscr{S}$, one should expect that boundary conditions are needed to ensure the uniqueness of a solution. At this point, we leave open the exact nature of the boundary conditions. It will be clarified by further considerations about Lorentz invariance and probability conservation.
All in all, the model consists of the following three ingredients:

$$\begin{cases} \text{the system of equations (6.6) on } \mathscr{S}, \\ \text{initial conditions on } \mathcal{I}, \\ \text{boundary conditions on } \partial\mathscr{S}. \end{cases} \tag{6.8}$$

6.3 Antisymmetry and reduction of the domain

We will see that antisymmetry of the wave function for identical particles opens up the possibility to reduce the domain from $N!$ disconnected parts to a single connected one. It is natural to consider identical particles as they are not dynamically distinguished by eqs. (6.2) alone.
We will introduce a notation which is tailor-made for the multi-time equations (6.6). Denote the spin components of ψ by $\psi_{s_1 \dots s_N}$ where each s_i can take the values ± 1. We write

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \\ \psi_{2^N} \end{pmatrix} \equiv \begin{pmatrix} \psi_{--\cdots--} \\ \psi_{--\cdots-+} \\ \psi_{--\cdots+-} \\ \vdots \\ \psi_{++\cdots++} \end{pmatrix}. \tag{6.9}$$

Indistinguishability implies the following antisymmetry condition for the wave function. Let $\pi \in S^N$ be a permutation. Then

$$\psi_{s_{\pi(1)} \dots s_{\pi(N)}}(x_{\pi(1)}, ..., x_{\pi(N)}) \;\overset{!}{=}\; (-1)^{\mathrm{sgn}(\pi)} \psi_{s_1 \dots s_N}(x_1, ..., x_N). \tag{6.10}$$

This condition can be used to relate a solution of (6.8) on the different parts of the domain $\mathscr{S}$. Note that in one spatial dimension, $\mathscr{S}$ separates into $N!$ disjoint parts which can be classified according to the relation of the spatial coordinates z_k, for example $z_2 < z_1 < z_5 < z_3 < \cdots$. Using the permutation group S^N, we write $\mathscr{S}$ as the disjoint union of open sets,

$$\begin{aligned} \mathscr{S} \;&=\; \bigsqcup_{\pi \in \mathrm{S^N}} \mathscr{S}_\pi, \\ \text{where} \quad \mathscr{S}_\pi \;&:=\; \left\{(t_1, z_1, ..., t_N, z_N) \in \mathscr{S} : z_{\pi(1)} < \cdots < z_{\pi(N)}\right\}. \end{aligned} \tag{6.11}$$

The crucial point is the following: given a solution of the model, as defined by (6.8) on $\mathscr{S}_1$ (corresponding to $z_1 < \cdots < z_N$), antisymmetric continuation via eq. (6.10) yields a solution on $\mathscr{S}_\pi$ *provided* the boundary and initial conditions are chosen to be compatible with antisymmetry. Note that this restricts the possible classes of initial boundary value problems (IBVPs) (6.8) to an autonomous IBVP on $\mathscr{S}_1$. We shall employ this strategy in the following. Our new model may be summarized according to (6.8) with $\mathscr{S}$ replaced by $\mathscr{S}_1$.

6.4 Multi-time characteristics and general solution

Let us look at the diagonalized multi-time Dirac equations (6.6) for a fixed component $\psi_{s_1 \ldots s_N}$. It contains nearly only $\mathbb{1}$s, so only the spin index of the k-th particle decides whether there is a $+$ or $-$ sign in the equation. Using the notation (6.9), this can be expressed as

$$\left(\frac{\partial}{\partial t_k} - s_k \frac{\partial}{\partial z_k} \right) \psi_{s_1 \ldots s_k \ldots s_N} \;=\; 0, \quad k = 1, ..., N. \tag{6.12}$$

Note that (6.12) imposes N equations for each of the 2^N spin components $\psi_{s_1 \ldots s_N}$. The general solution is obtainable because the system of Dirac equations looks like a system of coupled linear transport equations with constant prefactors. The transport equation has the form

$$\frac{\partial}{\partial x} u(x, y) + b \cdot \frac{\partial}{\partial y} u(x, y) = 0 \tag{6.13}$$

and one recognizes, using the coordinate transformation $v = x + bt, w = x - bt$, that it becomes

$$\frac{\partial}{\partial v} u\left(x(v, w), y(v, w)\right) = 0, \tag{6.14}$$

so solutions may only depend on w, so we have $u(x, y) = u(w) = u(x - bt)$. This is the simplest case for the so-called method of characteristics [49, p. 96–99].

Lemma 6.1 *The general solution of eqs.* (6.12) *is given by*

$$\psi_{s_1 \ldots s_N}(t_1, z_1, ..., t_N, z_N) \;=\; f_{s_1 \ldots s_N}\left(z_1 + s_1 t_1, ..., z_N + s_N t_N\right) \tag{6.15}$$

where $f_{s_1 \ldots s_N} \colon \mathbb{R}^N \to \mathbb{C}$ *are* C^1*-functions,* $s_1 = \pm 1, ..., s_N = \pm 1$.

Proof: The result is a consequence of the simple method of characteristics, which becomes obvious if one is familiar with the notation. Simply write out eq. (6.12) for $\psi_{s_1 \ldots s_N}$ separately: $\left(\frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial z_1} \right) \psi_{s_1 \ldots s_N} = 0, \cdots, \left(\frac{\partial}{\partial t_N} - s_N \frac{\partial}{\partial z_N} \right) \psi_{s_1 \ldots s_N} = 0$. So the solution may only depend on the combinations $z_1 + s_1 t_1$ and so on. This directly implies the form (6.15). □

The form of the general solution motivates the following definition.

Definition: Let $p = (t_1, z_1, \dots t_N, z_N) \in \mathbb{R}^{2N}$. Then we call

$$c_k \;:=\; z_k + s_k t_k \tag{6.16}$$

the *characteristic values* at p associated with the component $\psi_{s_1 \dots s_N}$.
Furthermore, we define the *multi-time characteristic* of component $\psi_{s_1 \dots s_N}$ by

$$S_{s_1 \dots s_N}(c_1, ..., c_N) \;:=\; \{(t_1, z_1, ..., t_N, z_N) \in \mathbb{R}^{2N} : z_k + s_k t_k = c_k\}. \tag{6.17}$$

With these definitions, one can reformulate lemma 6.1 as follows: the components $\psi_{s_1 \dots s_N}$ of solutions of (6.12) are constant on the respective multi-time characteristics (6.17). Note that this implies existence and uniqueness on the domain $\mathbb{R}^{2N}$ for an initial value problem at $t_1 = ... = t_N = 0$, the functions $f_{s_1 \dots s_N}$ being given by the initial values. However, as known from [2], this is in general not true for a domain with boundary such as $\mathscr{S}_1$.

7 Probability conservation and Lorentz invariance

This section is all about the hunt for the right boundary conditions. As a start, we state an adequate notion of probability conservation on space-like hypersurfaces (sec. 7.1). A general theorem which shows that probability conservation implies the uniqueness of solutions is proven (sec. 7.2) and thereafter, a condition on the components of the wave-function that leads to probability conservation is derived (sec. 7.3). Then we deal with the question of Lorentz invariance of our model, especially with the constraints this puts on the boundary conditions (sec. 7.4). The fruits of our labor will be reaped in the following section 8 where it is shown that the Lorentz invariant and probability conserving boundary conditions we construct also lead to the existence of a unique solution.

7.1 A geometric formulation of probability conservation

It is clear that the non-relativistic notion of probability conservation,

$$\int d^d x_1 \cdots d^d x_N |\psi|^2(t, \mathbf{x}_1, ..., t, \mathbf{x}_N) = 1 \;\forall t, \tag{7.1}$$

which heavily draws on a notion of simultaneity, has to be generalized in a relativistic theory. Following the usual path of accomplishing relativistic generalizations, we need to identify a geometric quantity that reduces to it in some cases. The Dirac equation provides a natural tensor current which generalizes the one-particle current of eq. (4.5),

$$j^{\mu_1 \cdots \mu_N} \;:=\; \overline{\psi} \gamma_1^{\mu_1} \dots \gamma_N^{\mu_N} \psi, \tag{7.2}$$

which satisfies the conservation equation

$$\partial_{k,\mu_k} j^{\mu_1 \cdots \mu_k \cdots \mu_N} \;=\; 0 \;\forall k. \tag{7.3}$$

With the help of this current tensor, one can rewrite the above equation (7.1) as

$$\int d^d x_1 \cdots d^d x_N \; j^{0\ldots 0}(t, \mathbf{x}_1, ..., t, \mathbf{x}_N) = 1 \; \forall t,$$

and one recognizes that this is the equal-time version of the more general formula

$$\int_\Sigma d\sigma_1(x_1) n_{\mu_1}(x_1) \ldots \int_\Sigma d\sigma_N(x_N) n_{\mu_N}(x_N) j^{\mu_1 \ldots \mu_N}(x_1, \ldots, x_N) = 1 \; \forall \Sigma, \tag{7.4}$$

where Σ is understood to be a (smooth) space-like hypersurface. This is now a purely geometrical formulation of probability conservation that treats all space-like hypersurfaces equally and reduces to (7.1) for equal-time hypersurfaces where the normal vector is $n = (1, 0, ..., 0)$. If the domain Ω is only a subset of configuration space-time, like $\mathscr{S}$, we should confine the integration to $\Sigma^N \cap \Omega$.
We will later apply Stokes' theorem to obtain conditions for probability conservation, thus it is useful to rewrite equation (7.4) with the help of differential forms.
One defines a *current form* as follows:

$$\begin{aligned} \omega_j \; &:= \sum_{\mu_1, \ldots, \mu_N = 0}^{d} (-1)^{\mu_1 + \cdots + \mu_N} j^{\mu_1 \ldots \mu_N} (dx_1^0 \wedge \ldots \widehat{dx_1^{\mu_1}} \cdots \wedge dx_1^d) \\ &\qquad \wedge \cdots \wedge (dx_N^0 \wedge \ldots \widehat{dx_N^{\mu_N}} \cdots \wedge dx_N^d) \\ &= \sum_{\mu_1 \ldots \mu_N = 0}^{d} (-1)^{\mu_1 + \cdots + \mu_N} \cdot j^{\mu_1 \mu_2 \ldots \mu_N} \bigwedge_{k=1}^{N} \left(\bigwedge_{\substack{\nu_k = 0 \\ \nu_k \neq \mu_k}}^{d} dx_k^{\nu_k} \right) \end{aligned} \tag{7.5}$$

where $\widehat{(\cdot)}$ denotes omission from the wedge product. The rank of this form is Nd. The continuity equations (7.3) imply that its exterior derivative vanishes, i.e. $d\omega_j = 0$. The relativistic notion of probability conservation on a domain Ω then reads [2]

$$\int_{\Sigma^N \cap \Omega} \omega_j \; = \; \int_{(\Sigma')^N \cap \Omega} \omega_j \tag{7.6}$$

for all pairs of space-like hypersurfaces Σ, Σ'. Equation (7.6) reduces to the form in (7.4) by use of the formula from analysis that expresses the surface element $d\sigma$ as a differential form [50, p. 435]:

$$d\sigma = \sum_{\mu=0}^{d} n_\mu (-1)^\mu dx^0 \wedge \ldots \widehat{dx^\mu} \cdots \wedge dx^d \tag{7.7}$$

It is noteworthy that the form of the tensor current (7.2) strongly depends on the evolution equations that are used. It is known, for example, that for Klein-Gordon equations, one has, using the abbreviation $\overset{\leftrightarrow}{\partial} := \overset{\rightarrow}{\partial} - \overset{\leftarrow}{\partial}$ where the derivatives act to the direction depicted by the arrow, the conserved current form

$$j_{\mu_1 \ldots \mu_k \ldots \mu_N} = \phi^* \left(i \overset{\leftrightarrow}{\partial}_{1,\mu_1} \right) \ldots \left(i \overset{\leftrightarrow}{\partial}_{N,\mu_N} \right) \phi. \tag{7.8}$$

This fact should be appreciated to not confuse the logical structure of building a new quantum theory: First comes the evolution equation, then it dictates the conserved current and the notion of probability conservation, which may then be used to define Hilbert spaces.

7.2 Probability conservation implies uniqueness of solutions

The notion (7.6) of probability conservation is very powerful. In this section, we consider a general domain Ω and work out in detail that $\int_{(\Sigma^N \cap \Omega)} \omega_j$ is a so-called *energy integral*. That is a functional that is conserved for solutions of the partial differential equation and allows us to prove uniqueness of solutions. Similar methods are ubiquitous in the study of PDEs (see [49]). In this case we do not need to go deeply into the theory because it is rather obvious that uniqueness of generalized (weak) solutions directly implies that there can also be at most one solution in the classical sense.

Definition: Let Σ be a space-like hypersurface. We define function spaces

$$\mathcal{H}_\Sigma^{(N)} \;:=\; L^2(\Sigma^N \cap \Omega) \otimes (\mathbb{C}^2)^{\otimes N}. \tag{7.9}$$

Furthermore, we call the solution of the IBVP (6.8) *weakly unique* iff for any two solutions ψ, φ and any space-like hypersurface Σ the restrictions $\psi_{|\Sigma}$, $\varphi_{|\Sigma}$ of ψ, φ to arguments in $\Sigma^N \cap \Omega$ are equal as elements of $\mathcal{H}_\Sigma^{(N)}$.

Theorem 7.1 *The solution of the IBVP* (6.8) *with boundary conditions ensuring probability conservation* (7.6) *and initial values on* $\mathcal{I} = (\Sigma_0)^N \cap \Omega$, *i.e.* $\psi_{|\mathcal{I}} \equiv g \in \mathcal{H}_{\Sigma_0}^{(N)}$ *is weakly unique.*

Proof: Consider the expression

$$\|\psi\|_\Sigma^2 \;:=\; \int_{\Sigma^N \cap \Omega} \omega_j(\psi), \tag{7.10}$$

where $\omega_j(\psi)$ is the current form constructed from ψ according to eqs. (7.2) and (7.5). Because the Dirac tensor current j is positive-definite and sesquilinear in the wave function, $\|\cdot\|_\Sigma$ defines a norm on $\mathcal{H}_\Sigma^{(N)}$[11] .
Let ψ, φ be solutions of the IBVP. Then: $\psi_{|\Sigma_0} \equiv \varphi_{|\Sigma_0} \equiv g \in \mathcal{H}_{\Sigma_0}^{(N)}$ and therefore $\|\psi_{|\Sigma_0} - \varphi_{|\Sigma_0}\|_{\Sigma_0} = 0$. Now let Σ be an arbitrary space-like hypersurface. Probability conservation (7.6) yields:

$$\|\psi_{|\Sigma} - \varphi_{|\Sigma}\|_\Sigma \;=\; \|\psi_{|\Sigma_0} - \varphi_{|\Sigma_0}\|_{\Sigma_0} \;=\; 0 \tag{7.11}$$

and it follows that $\psi_{|\Sigma} \equiv \varphi_{|\Sigma}$ as elements of $\mathcal{H}_\Sigma^{(N)}$. □

[11] One should recognize that this works exactly because we only consider space-like hypersurfaces. On a time-like hypersurface, the expression (7.10) could even become negative.

Remark: The proof of thm. 7.1 suggests that the map

$$U_{\Sigma\to\Sigma'} : \mathcal{H}_\Sigma^{(N)} \to \mathcal{H}_{\Sigma'}^{(N)}, \quad \psi_{|\Sigma} \mapsto \psi_{|\Sigma'}, \tag{7.12}$$

sending the restriction of a solution ψ of the IBVP to $\Sigma^N \cap \Omega$ to its restriction to $(\Sigma')^N \cap \Omega$, defines a unitary evolution from one space-like hypersurface to another (see also [51, sec. 3]). The analogous view in quantum field theory constitutes the Tomonaga-Schwinger picture [16, 17].
Note that having in mind a functional-analytic view on time evolution, it might seem natural to take the reverse way to define a multi-time evolution, i.e. first defining the spaces $\mathcal{H}_\Sigma^{(N)}$ and a unitary map $U_{\Sigma\to\Sigma'}$. However, this is not convincing because then there may exist $\Sigma \neq \Sigma'$ with $\Sigma \cap \Sigma' \neq \emptyset$ such that $\psi_{|\Sigma}(q) \neq \psi_{|\Sigma'}(q)$ even for $q \in \Sigma \cap \Sigma'$ (see [35]). Without additional conditions to enforce $\psi_{|\Sigma}(q) = \psi_{|\Sigma'}(q)$ for $q \in \Sigma \cap \Sigma'$, this would mean that one could not regard the multi-time wave functions and the tensor current j as geometrical objects. This, however, may be necessary for a physical interpretation (see e.g. [6]).

7.3 Boundary conditions from probability conservation

The formulation via the Nd-form ω_j together with the property $d\omega_j = 0$ makes it possible to use Stokes' theorem to extract conditions on ω_j and thereby on j which ensure probability conservation.

Lemma 7.2 *Let the wave function ψ be compactly supported on all sets of the form $\Sigma^N \cap \mathcal{S}_1$. Then probability conservation on $\mathcal{S}_1$ in the sense of*

$$\int_{\Sigma^N \cap \mathcal{S}_1} \omega_j \;=\; \int_{(\Sigma')^N \cap \mathcal{S}_1} \omega_j \tag{7.13}$$

for all space-like hypersurfaces Σ, Σ' holds if

$$\omega_j|_{\mathscr{C}_1} \;=\; 0, \tag{7.14}$$

where

$$\mathscr{C}_1 := \{(t_1, z_1, \ldots t_N, z_N) \in \partial\mathcal{S}_1 |\; \exists k : t_k = t_{k+1} \wedge z_k = z_{k+1}\}\,. \tag{7.15}$$

Remark:

1. The assumption of compact support of the wave function with respect to spatial directions is needed as a technical assumption in the proof. It is reasonable because the multi-time Dirac equations have finite propagation speed (see eq. (6.15)). So if the initial data are compactly supported, the wave function will be compactly supported for all times. It would in fact be enough to demand that the wave function vanishes fast enough for any $z_k \to \infty$, but this would technically be more complicated.

2. Note that the wave function is, strictly speaking, not defined on $\partial\mathscr{S}_1$. When using values of the wave function at the boundary (such as in eq. (7.14)), we mean the corresponding limit in $\mathscr{S}_1$. In this way, jumps of the wave function across the boundaries of different $\mathscr{S}_\pi$ are admitted. In fact, singularities of this kind are typical for zero-range interactions [47].

Proof: Let Σ, Σ' be space-like hypersurfaces. We construct a suitable submanifold with boundary in order to be able to use Stokes' theorem.
Let $t_\Sigma(z)$ denote the time coordinate of the unique point $p = (t_\Sigma(z), z) \in \Sigma$. Let $R > 0$ and consider the following set:

$$V_R := \left\{ (t_1, z_1, \ldots, t_N, z_N) \in \overline{\mathscr{S}}_1 \;\middle|\; \begin{array}{c} \exists \tau \in [0,1] : \forall k : t_k = t_\Sigma(z_k) + \tau\,(t_{\Sigma'}(z_k) - t_\Sigma(z_k)) \\ \text{and } |z_k| \leq R \end{array} \right\} \tag{7.16}$$

V_R is a bounded and closed, thus compact, $(N+1)$-dimensional submanifold of $\mathbb{R}^{2N}$ with boundary

$$\partial V_R = (\Sigma^N \cap \mathscr{S}_1) \cup ((\Sigma')^N \cap \mathscr{S}_1) \cup M_1 \cup M_2, \tag{7.17}$$

where M_2 is the subset of V_R with $|z_k| = R$ for some k and

$$M_1 \;=\; V_R \cap \partial\mathscr{S}_1. \tag{7.18}$$

Because of the first condition in the definition of V_R, a configuration in V_R is always an element of $\mathcal{S}^N$ where $\mathcal{S}$ is a space-like hypersurface. Therefore, it can only be an element of $M_1 \subset \partial\mathscr{S}_1$ (i.e. light-like) if $\exists k : t_k = t_{k+1}$ and $z_k = z_{k+1}$. This implies $M_1 \subset \mathscr{C}_1$.
In the limit $R \to \infty$, the integral $\int_{M_2} \omega_j$ vanishes because of the compact support of the wave function. Thus, it follows from the theorem of Stokes, together with $d\omega_j = 0$, that

$$0 \;=\; \lim_{R\to\infty} \int_{V_R} d\omega_j = \lim_{R\to\infty} \int_{\partial V_R} \omega_j = -\int_{\Sigma^N \cap \mathscr{S}_1} \omega_j + \int_{(\Sigma')^N \cap \mathscr{S}_1} \omega_j + \int_{M_1} \omega_j. \tag{7.19}$$

The minus sign in front of the first integral on the r.h.s. is due to orientation conventions. Thus, probability conservation in the sense of eq. (7.13) holds iff $\int_{M_1} \omega_j = 0$. In order to make this integral vanish for all possible choices of Σ, Σ', the condition

$$\omega_j|_{\mathscr{C}_1} \;\stackrel{!}{=}\; 0. \tag{7.20}$$

has to be satisfied. □

We should study now what the condition (7.14) implies for the components of the wave function. Since this condition is not so easy to analyze, we will simplify our lives by first concentrating on the special case of equal-time hypersurfaces in a fixed (but otherwise arbitrary) Lorentz frame.

Lemma 7.3 *Let $\mathscr{C}_{1,t} := \{(t_1, z_1, ..., t_N, z_N) \in \mathscr{C}_1 : t_1 = \cdots = t_N\}$. Then the condition for probability conservation on equal-time hypersurfaces Σ_t in a particular Lorentz frame, i.e.* (7.14) *with $\mathscr{C}_1$ replaced by $\mathscr{C}_{1,t}$, holds if and only if*

$$\psi^\dagger(p)\,(\sigma_{3,k} - \sigma_{3,k+1})\,\psi(p) \;=\; 0 \quad \forall p \in \mathscr{C}_{1,t}^{(k)} \quad \forall k = 1, ..., N-1, \tag{7.21}$$

where $\mathscr{C}_{1,t}^{(k)} := \{(t_1, z_1, ..., t_N, z_N) \in \mathscr{C}_{1,t} : z_k = z_{k+1}\}$.
Furthermore, equation (7.21) *can be rewritten as*

$$\sum_{\substack{(s_1,...,s_N)\in\{-,+\}^N \\ s_k \neq s_{k+1}}} s_{k+1} |\psi_{s_1 \dots s_N}|^2(p) \;=\; 0 \quad \forall p \in \mathscr{C}_{1,t}^{(k)}. \tag{7.22}$$

Proof: We have to evaluate the condition $\omega_j|_{\mathscr{C}_{1,t}} = 0$. Note that for $p \in \mathscr{C}_{1,t}$ there exists a $k \in \{1, ..., N-1\}$ such that $p = (t, z_1, ..., t, z_k = z, t, z_{k+1} = z, ..., t, z_N)$.
Next, we calculate $\omega_j|_{\mathscr{C}_{1,t}}$ according to eq. (7.5), recalling that in this case $x_k^0 = t$ and $x_j^1 = z_j$ as well as $z_k = z_{k+1} = z$. All terms with more than one index $\mu_l = 1$ in $j^{\mu_1 \dots \mu_l \dots \mu_N}$ vanish because they contain $dt \wedge dt = 0$. Moreover, the terms with $\mu_k = \mu_{k+1} = 0$ do not contribute, either, as they contain $dz \wedge dz = 0$. We are left with terms where all indices μ_j are equal to zero apart from the k-th or the $(k+1)$-th:

$$\begin{aligned}
\omega_j(p) \;&=\; -j^{0\dots(\mu_k=0)(\mu_{k+1}=1)\dots 0}(p)\, dz_1 \wedge \dots \wedge dz_{k-1} \wedge dz \wedge dt \wedge dz_{k+2} \wedge \dots \wedge dz_N \\
&\quad -j^{0\dots(\mu_k=1)(\mu_{k+1}=0)\dots 0}(p)\, dz_1 \wedge \dots \wedge dz_{k-1} \wedge dt \wedge dz \wedge dz_{k+2} \wedge \dots \wedge dz_N \\
&=\; \left(j^{0\dots 10\dots 0} - j^{0\dots 01\dots 0}\right)(p)\, dz_1 \wedge \dots \wedge dz_{k-1} \wedge dz \wedge dt \wedge dz_{k+2} \wedge \dots \wedge dz_N
\end{aligned} \tag{7.23}$$

This expression vanishes if and only if the bracket is zero. This gives condition (7.21):

$$0 \;=\; (j^{0\dots 10\dots 0} - j^{0\dots 01\dots 0})(p) \;=\; \psi^\dagger(p)\,(\sigma_{3,k} - \sigma_{3,k+1})\,\psi(p). \tag{7.24}$$

Written out in components, eq. (7.24) reads:

$$0 \;=\; \sum_{(s_1,...,s_N)\in\{-,+\}^N} \left(-s_k |\psi_{s_1 \dots s_N}|^2(p) + s_{k+1} |\psi_{s_1 \dots s_N}|^2(p)\right). \tag{7.25}$$

Summands with $s_k = s_{k+1}$ cancel out. We are left with

$$0 \;=\; \sum_{\substack{(s_1,...,s_N)\in\{-,+\}^N \\ s_k \neq s_{k+1}}} 2 s_{k+1} |\psi_{s_1 \dots s_N}|^2(p), \tag{7.26}$$

which yields (7.22). □

Of course, we would like to find adequate boundary conditions that lead to probability conservation on general space-like hypersurfaces. At the moment, we only have the necessary conditions (7.22). The question of whether there is a subclass of them that is Lorentz invariant and ensures the existence of a solution is attacked in the next subsection.

7.4 Lorentz invariance

Apart from the boundary conditions, the Lorentz invariance of our model (6.8) is already manifest since the Dirac equations are known to transform covariantly and the domain is a Lorentz-invariant set. In order to ascertain which of the boundary conditions have the same functional form in all frames, we should evaluate the transformation behavior of the spinor ψ. Under a Lorentz transformation $\Lambda : x \mapsto x'$ in the proper Lorentz group $\mathcal{L}_+^\uparrow$, it is

$$\psi'(x_1, ..., x_N) \;=\; S(\Lambda) \otimes \cdots \otimes S(\Lambda) \psi(\Lambda^{-1} x_1, ..., \Lambda^{-1} x_N), \tag{7.27}$$

where

$$S(\Lambda) \;=\; \exp\left(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}\right), \quad \sigma^{\mu\nu} = \frac{i}{2} \left[\gamma^\mu, \gamma^\nu\right]. \tag{7.28}$$

Here, ω is an antisymmetric $(1+d) \times (1+d)$ matrix which is the infinitesimal version of Λ.
For $d = 1$, there is only one free parameter $\beta \in \mathbb{R}$ corresponding to a boost in z-direction. One can calculate that

$$S(\Lambda) \;=\; \begin{pmatrix} \cosh\beta + \sinh\beta & 0 \\ 0 & \cosh\beta - \sinh\beta \end{pmatrix}. \tag{7.29}$$

As the matrix is diagonal due to our choice of γ-matrices, it is easy to calculate the N-fold tensor product in eq. (7.27) – just take the respective powers of the diagonal elements. We conclude that any component of ψ transforms as

$$\psi'_{s_1 ... s_N}(x_1, ..., x_N) \;=\; \prod_{k=1}^{N} \left(\cosh\beta - s_k \sinh\beta\right) \psi_{s_1 ... s_N}(\Lambda^{-1} x_1, ..., \Lambda^{-1} x_N). \tag{7.30}$$

This means that one gets a factor of $(\cosh\beta - \sinh\beta)$ for every plus and a factor of $(\cosh\beta + \sinh\beta)$ for every minus in the index $(s_1 ... s_N)$. Hence, components with an equal number of $+$ and $-$ signs transform alike.

Example: We have a look at the easy case $N = 3$ in order to motivate the following lemma. Let us consider for example a boundary point $p = (t, z_1, t, z, t, z) \in \mathscr{C}_{1,t}$. We use eq. (7.22) to compute explicitly what the condition of probability conservation amounts to:

$$\omega_j(p) \;=\; 0 \quad \Leftrightarrow \quad |\psi_{--+}|^2(p) - |\psi_{-+-}|^2(p) + |\psi_{+-+}|^2(p) - |\psi_{++-}|^2(p) \;=\; 0. \tag{7.31}$$

Now what is the form of this equation in another frame? We Lorentz transform according to eq. (7.30) using the identity $(\cosh\beta - \sinh\beta)(\cosh\beta + \sinh\beta) = 1$:

$$\begin{aligned} 0 \;=\;& \left(\cosh\beta - \sinh\beta\right) \left(|\psi_{--+}|^2 - |\psi_{-+-}|^2\right)(p') \\ &+ \left(\cosh\beta + \sinh\beta\right) \left(|\psi_{+-+}|^2 - |\psi_{++-}|^2\right)(p'), \end{aligned} \tag{7.32}$$

where $p' = (\Lambda^{-1}(t,z_1), \Lambda^{-1}(t,z), \Lambda^{-1}(t,z))$. We see that (7.32) is not invariant as a whole, but if we split it up into two separate conditions that relate only components with the same number of plus and minus signs in their indices, i.e.

$$|\psi_{-+-}|^2(p) - |\psi_{--+}|^2(p) \;=\; 0 \text{ and } |\psi_{+-+}|^2(p) - |\psi_{++-}|^2(p) \;=\; 0, \tag{7.33}$$

these equations are the same in all Lorentz frames. They can be rewritten as

$$\psi_{--+}(p) \;=\; e^{i\varphi_-}\psi_{-+-}(p), \qquad \psi_{++-}(p) \;=\; e^{i\varphi_+}\psi_{+-+}(p). \tag{7.34}$$

A priori, the phases $\varphi_\pm$ could be functions of all particle coordinates. But we demand invariance under Poincaré transformations, not only Lorentz transformations, as physics should not depend on the choice of the origin of coordinates. Then the phases $\varphi_\pm$ may only depend on the Minkowski distances of pairs of particles, $(x_i - x_j)^\mu (x_i - x_j)_\mu$. In the 3-particle case, there exists only one such variable, $s^2 := (t_1 - t)^2 - (z_1 - z)^2$. The Minkowski distance s^2 changes along a multi-time characteristic although the solution has to be constant along the characteristic (see lemma 6.1), so $\varphi_\pm$ must be constant in order for solutions to exist. Further investigation of the existence and uniqueness problem shows that even $\varphi_+ = \varphi_-$ is necessary.
This yields the following general picture: let us define the sets

$$\mathscr{C}_{k,k+1} \;:=\; \left\{(t_1, z_1, ..., t_N, z_N) \in \overline{\mathscr{S}}_1 |\, t_k = t_{k+1} \wedge z_k = z_{k+1}\right\}. \tag{7.35}$$

so we can write $\mathscr{C}_1 = \bigcup_{k=1}^{N-1} \mathscr{C}_{k,k+1}$. Then it seems that Lorentz invariant boundary conditions are such that exchanging $s_k \leftrightarrow s_{k+1}$ in $\psi_{s_1 ... s_k s_{k+1} ... s_N}$ on $\mathscr{C}_{k,k+1}$ only yields a phase factor which must not depend on the other spin indices (but may depend on k). These insights motivate the choice of boundary conditions (7.36). It will be retained for the rest of this part. The following lemma shows that this choice is indeed Lorentz invariant.

Lemma 7.4 *Let $\varphi^{(k)} \in (-\pi, \pi]$ for $k = 1, ..., N-1$. Then the boundary conditions*

$$\psi_{s_1 ... s_{k-1} + - s_{k+2} ... s_N} \;\stackrel{!}{=}\; e^{i\varphi^{(k)}} \psi_{s_1 ... s_{k-1} - + s_{k+2} ... s_N} \text{ on } \mathscr{C}_{k,k+1}, \quad k = 1, ..., N-1 \tag{7.36}$$

are Lorentz invariant.

Proof: According to eq. (7.30), eq. (7.36) has the same form in every Lorentz frame. Besides, the sets $\mathscr{C}_{k,k+1}$ on which the condition is prescribed are Lorentz invariant. □

Remark: One might ask if there are other possible choices of boundary conditions which lead to $\omega_j = 0$ at the boundary and are Lorentz invariant. The example shows that for $N = 3$ we have already found the only one. For $N \geq 4$ there may exist more complicated boundary conditions with the desired properties, but our aim is a model that can be used for any $N \geq 2$, so we do not further pursue this question here.

The boundary conditions (7.36) were constructed such that they should fulfill equation (7.22), so probability conservation on equal-time hypersurfaces in one frame is ensured. By Lorentz-invariance as proven in lemma 7.4, this follows for any Lorentz frame. In a next step, probability conservation is proven in even greater generality, on every pair of space-like hypersurfaces.

Lemma 7.5 *The boundary conditions from lemma 7.4, i.e.*

$$\psi_{s_1 \dots s_{k-1} + - s_{k+2} \dots s_N} \stackrel{!}{=} e^{i\varphi^{(k)}} \psi_{s_1 \dots s_{k-1} - + s_{k+2} \dots s_N} \text{ on } \mathscr{C}_{k,k+1} \tag{7.37}$$

imply probability conservation on all space-like hypersurfaces in the sense of eq. (7.13)*.*

Proof: It was shown in lemma 7.2 that equation (7.13) follows if

$$\omega_j|_{\mathscr{C}_1} = 0. \tag{7.38}$$

We show that this equation indeed holds. Pick a point $p \in \mathscr{C}_1$. Then $\exists!\ k : p \in \mathscr{C}_{k,k+1}$. Condition (7.37) at this point yields

$$|\psi_{s_1 \dots s_{k-1} + - s_{k+2} \dots s_N}|^2(p) = |\psi_{s_1 \dots s_{k-1} - + s_{k+2} \dots s_N}|^2(p). \tag{7.39}$$

It follows that

$$j^{\mu_1 \mu_2 \dots 01 \dots \mu_N}(p) = j^{\mu_1 \mu_2 \dots 10 \dots \mu_N}(p) \tag{7.40}$$

because the expression for the current is diagonal in the components. In the formula for $\omega_j(p)$ (eq. (7.5)), we can first sum over the indices (μ_k, μ_{k+1}) and afterwards over the rest. Then there are four possibilities in the summands:

- $(\mu_k, \mu_{k+1}) = (0,0)$ or $(1,1)$: These do not contribute because the coordinates of the k-th and $k+1$-th particles are equal, say to (t, z), so either $dz \wedge dz = 0$ or $dt \wedge dt = 0$ appears as a factor in the wedge product.

- $(\mu_k, \mu_{k+1}) = (0,1)$ or $(1,0)$. One can see that these two factors cancel each other because (abbreviating all other, irrelevant factors in the wedge product by A and B)

$$\begin{aligned} & j^{\mu_1 \dots (\mu_k = 0)(\mu_{k+1} = 1) \dots \mu_N} A \wedge dt \wedge dz \wedge B + j^{\mu_1 \dots (\mu_k = 1)(\mu_{k+1} = 0) \dots \mu_N} A \wedge dz \wedge dt \wedge B \\ = & \left(j^{\mu_1 \dots 01 \dots \mu_N}(p) - j^{\mu_1 \dots 10 \dots \mu_N}(p) \right) A \wedge dt \wedge dz \wedge B \stackrel{(7.40)}{=} 0. \end{aligned} \tag{7.41}$$

Therefore, the probability conserving property $\omega_j(p) = 0$ holds. □

8 Existence of solutions on the space-like configurations

We come to the main result of this part: a theorem about existence and uniqueness of solutions for the boundary conditions that make sense according to our reasoning above (thm. 8.1). Uniqueness will be a direct consequence of theorem 7.1, but existence requires more work and will be shown by providing an explicit formula for the solutions. A priori, one might even think that the boundary conditions on the $(d+1)(N-1)$-dimensional set $\mathscr{C}_1$ do not fit to the Nd dimensions of the initial data surface (the dimensions are unequal for $N \neq d+1$) and thus over-determine the solution, but purely dimensional arguments are misleading in this case – the existence really depends on the way the characteristics intersect with the boundary.
Let us recollect and state again clearly what we are considering.

- The multi-time system of equations (6.2) on the domain $\mathscr{S}_1 \subset \mathscr{S}$.
- Initial conditions prescribed on the set

$$\mathscr{I}_1 := \left\{(t_1, z_1, \dots t_N, z_N) \in \overline{\mathscr{S}}_1 | t_1 = t_2 = \cdots = t_N = 0\right\}. \tag{8.1}$$

  We require

$$\psi_{s_1 s_2 \dots s_N}(0, z_1, 0, z_2, \dots, 0, z_N) = g_{s_1 s_2 \dots s_N}(z_1, z_2, ..., z_N) \text{ for } z_1 \leq \cdots \leq z_N. \tag{8.2}$$

  for all components of ψ.

- Boundary conditions on the sets defined by

$$\mathscr{C}_{k,k+1} := \left\{(t_1, z_1, \dots t_N, z_N) \in \overline{\mathscr{S}}_1 | t_k = t_{k+1}, z_k = z_{k+1}\right\}. \tag{8.3}$$

  We impose that on a set where the k-th and $(k+1)$-th particle are at the same space-time point, the respective $+-$ and $-+$ spin components are equal up to a phase factor,[12]

$$\psi_{s_1 s_2 \dots s_{k-1} + - \dots s_N} = \psi_{s_1 s_2 \dots s_{k-1} - + \dots s_N} \cdot e^{i\varphi^{(k)}} \text{ on } \mathscr{C}_{k,k+1}. \tag{8.4}$$

  We have remarked above that the phase factors $\varphi^{(k)}$ must be constant in space-time in order for solutions to exist. They may be different for different k, though.

These three conditions together form our *initial boundary value problem (IBVP)*. Before stating the result and proving it, we should gain some intuition for the processes that happen in our model. To achieve that, we provide a pictorial approach in a simple case.

[12] We remind that this equation does in fact mean that a limit approaching the sets $\mathscr{C}_{k,k+1} \not\subset \mathscr{S}_1$ from the interior of $\mathscr{S}_1$ should satisfy the required property.

Example: For $N = 3$ the wave function has $2^3 = 8$ components. According to lemma 6.1, these are constant along their respective multi-time characteristics. We visualize the multi-time characteristics as follows (see fig. 1). One can see from eq. (6.17) that the multi-time characteristics are the Cartesian product of $N = 3$ lines. These lines are plotted in the same space-time diagram. Any combination of three points on the different lines constitutes an element of the respective multi-time characteristic. The slopes of the various lines characterize the associated component $\psi_{s_1 \dots s_N}$. More precisely, a line to the left (right) for particle k is associated with the appearance of $s_k = +1$ ($s_k = -1$) in the index of ψ.

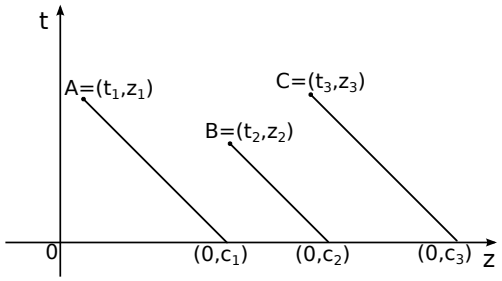

Figure 1: A multi-time characteristic $S_{+++}(c_1, c_2, c_3)$ for the component ψ_{+++}. $S_{+++}(c_1, c_2, c_3)$ is the Cartesian product of three lines which are plotted in the same space-time diagram. Every triple of points on different lines, e.g. (A, B, C), is contained in $\mathscr{S}_1$.

Fig. 1 shows a multi-time characteristic $S_{+++}(c_1, c_2, c_3)$ for the component ψ_{+++} where the c_k are defined by a certain point $p = (A, B, C) \in \mathscr{S}_1$ (see eq. (6.16)). ψ_{+++} is determined by initial data at p because the whole characteristic $S_{+++}(c_1, c_2, c_3)$ is contained in $\mathscr{S}_1$. This can be seen from the fact that any three points on different lines are space-like related. Besides, the value $\psi(p)$ is uniquely determined by initial data as there exists a unique intersection point with the surface $t_1 = t_2 = t_3 = 0$, given by $(0, c_1, 0, c_2, 0, c_3)$. Thus, we obtain

$$\psi_{+++}(t_1, z_1, t_2, z_2, t_3, z_3) \;=\; g_{+++}(0, c_1, 0, c_2, 0, c_3). \tag{8.5}$$

For a component $\psi_{s_1 s_2 s_3}$ containing plus as well as minus signs in the index, for example ψ_{+-+}, the situation is different (see fig. 2). One can see that intersections of the lines defining a characteristic S_{+-+} appear in the diagram. When this happens, the multi-time characteristic leaves $\mathscr{S}_1$. Therefore, a situation like in fig. 2 can occur: tracing back the multi-time characteristic to the initial data surface, one leaves the domain. Thus, ψ_{+-+} is not defined solely by initial values. It is the role of the boundary conditions to provide a value for ψ_{+-+} at the boundary point (P, P, C) (and thus along the whole characteristic). This works as follows: the component ψ_{-++} is defined at the boundary point by initial values. Thus, relating ψ_{-++} with ψ_{+-+} via eq. (7.37), we obtain:

$$\psi_{+-+}(t_1, z_1, t_2, z_2, t_3, z_3) \;=\; e^{i\varphi^{(1)}} g_{-++}(0, c_2, 0, c_1, 0, c_3). \tag{8.6}$$

Pictorially speaking, this amounts to exchanging particle labels and picking up a phase while leaving the domain on the way back to the initial data surface.

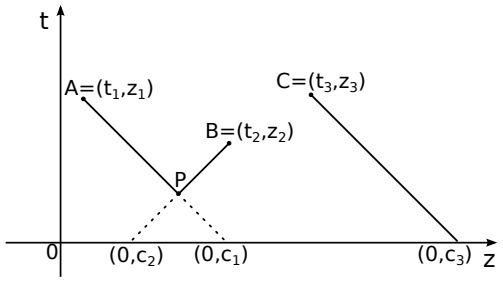

Figure 2: A multi-time characteristic S_{+-+} for ψ_{+-+}, depicted for the same configuration as in fig. 1. One cannot trace back the lines to the initial data surface $\mathcal{I}$ because one leaves $\mathscr{S}_1$ at the point P. One only obtains a definite value for $\psi_{+-+}(A,B,C)$ by first realizing $\psi_{+-+}(A,B,C) = \psi_{+-+}(P,P,C)$ and then employing the boundary conditions to obtain $\psi_{+-+}(A,B,C) = \psi_{+-+}(P,P,C) = e^{i\varphi^{(1)}}\psi_{-++}(P,P,C)$. The component ψ_{-++} is now determined at (P,P,C) by initial data. It corresponds to the same lines but where the particle labels 1 and 2 are exchanged. Thus, we can trace the dotted lines below (P,P,C) back to $t=0$, finding $\psi_{-++}(P,P,C) = g_{-++}(0,c_2,0,c_1,0,c_3)$.

The considerations above hint at a general idea: it is possible to obtain an explicit formula for the solutions of the IBVP (see eq. (8.8)) by a process of successively tracing back components to collisions, using the boundary conditions to switch the component, tracing back to the next collision and finally arriving at the initial data. In this way, one can even determine values of components with multiple intersections of the lines constituting the multi-time characteristic (see fig. 3). This motivates the following theorem.

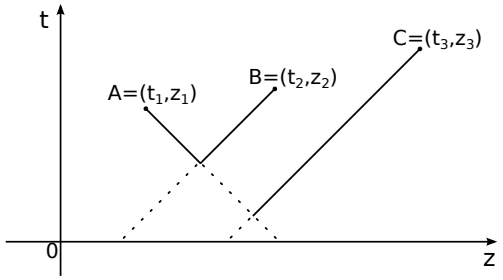

Figure 3: A multi-time characteristic with several intersections for the component ψ_{+--}.

Theorem 8.1 *Let $m \in \mathbb{N}$ and initial value functions $g_j \in C^m(\mathcal{I}_1, \mathbb{C})$ $\forall j \in \left\langle 1, ..., 2^N \right\rangle$ such that they also satisfy the boundary conditions, i.e.*

$$g_{s_1 ... s_{k-1} + - s_{k+2} ... s_N} = e^{i\varphi^{(k)}} g_{s_1 ... s_{k-1} - + s_{k+2} ... s_N} \quad \text{on } \mathcal{I} \cap \mathscr{C}_{k,k+1}, \quad k+1, ..., N-1, \quad (8.7)$$

and let this transition be C^m.
Then there exists a unique solution $\psi \in C^m\left(\overline{\mathscr{S}_1}, (\mathbb{C}^2)^{\otimes N}\right)$ of the IBVP (6.8) with domain

$\mathscr{S}_1$, boundary conditions (7.37) *and initial values* (8.2).
If all characteristic values $c_k = z_k + s_k t_k$ are different, the components of the solution are explicitly given by

$$\psi_{s_1 \dots s_N}(t_1, z_1, ..., t_N, z_N) \;=\; e^{i\phi^{\pi}_{s_1 \dots s_N}} g_{s_{\pi(1)} \dots s_{\pi(N)}} \left(c_{\pi(1)}, ..., c_{\pi(N)} \right), \tag{8.8}$$

where π is the permutation with

$$c_{\pi(1)} < \cdots < c_{\pi(N)}. \tag{8.9}$$

$\phi^{\pi}_{s_1 \dots s_N}$ is the phase which is uniquely determined by the $\varphi^{(k)}$ via the definition below.
If the some of the c_k are equal, $\psi_{s_1 \dots s_N}$ is given by the continuation of eq. (8.8).
In addition, the model ensures probability conservation on general space-like hypersurfaces and is Lorentz invariant.

Definition: A pair $(k, l) \in \{1, ..., N\}^2$ with $k < l$ is said to be a *collision* of a transposition $\pi \in S_N$ iff $\pi(k) > \pi(l)$.

Definition: The phases $\phi^{\pi}_{s_1 \dots s_N}$ appearing in theorem 8.1 are defined by the conditions:

1. $\phi^{\mathrm{id}}_{s_1 \dots s_N} = 0$.
2. Let τ_k be the transposition of k and $k+1$. If π can be decomposed as $\pi = \tau_k \circ \sigma$ where σ is a permutation with fewer collisions than π, then

$$\phi^{\pi}_{s_1 \dots s_N} \;=\; \phi^{\sigma}_{s_1 \dots s_{k+1} s_k \dots s_N} + s_k \varphi^{(k)}. \tag{8.10}$$

Lemma 8.2 *The phases $\phi^{\pi}_{s_1 \dots s_N}$ exist and are uniquely determined.*

Proof: We proceed via induction over the number of collisions.
Base case: If π has no collision, it follows that $\pi = \mathrm{id}$. The phase $\phi^{\mathrm{id}}_{s_1 \dots s_N}$ is determined by the first condition in the definition. If π has exactly one collision, then it is just a transposition of neighboring elements, so there exists some k with $\pi = \tau_k = \tau_k \circ \mathrm{id}$ and the phase is uniquely determined by (8.10) as $\phi^{\pi}_{s_1 \dots s_N} = s_k \varphi^{(k)}$.
Induction step: Assume that all phases $\phi^{\pi'}_{s_1 \dots s_N}$ for permutations with $n \geq 1$ collisions are uniquely determined and let π have $n+1$ collisions. It is known from the general theory of permutations that there exists at least one permutation σ with n collisions and a neighboring transposition τ_s such that $\pi = \tau_s \circ \sigma$. However, it may be possible to decompose π in two different ways:

$$\pi \;=\; \tau_s \circ \sigma \;=\; \tau_k \circ \kappa, \tag{8.11}$$

where $s, k \in \{1, \dots, N\}$, $s \neq k$ and σ, κ are permutations with at least n collisions. In order for these two permutations to have one collision less than π, we see that $(k, k+1)$ and $(s, s+1)$ must be collisions of π.
In order to show that despite the different ways of decomposition, the corresponding

phases are uniquely defined, we make use of the fact that the phases need only be defined for a certain type of permutation. To characterize them, we prove an auxiliary claim: in the above situation, τ_s commutes with τ_k because $|s-k| \neq 1$.

Claim: Let $\psi_{s_1 \dots s_N}$ and $(t_1, z_1, \dots, t_N, z_N) \in \mathscr{S}_1$ such that there is a collision, i.e. a pair (a, b) with $a < b$ and $c_a > c_b$. Then one of the following two possibilities holds:

$$\begin{cases} \text{either } s_a = +1 \ \wedge \ s_b = -1 \ \wedge \ t_a > 0 \ \wedge \ t_b > 0 \\ \quad \text{or} \quad s_a = -1 \ \wedge \ s_b = +1 \ \wedge \ t_a < 0 \ \wedge \ t_b < 0 \end{cases} \tag{8.12}$$

Proof of the Claim: We know that $a < b$, $c_a > c_b$ and $z_a < z_b$. We show that $s_a = +1$ implies $s_b = -1 \ \wedge \ t_a > 0 \ \wedge \ t_b > 0$; the second case follows analogously.
If $s_a = +1$, then

$$\begin{aligned} & c_a \ > \ c_b \\ \Leftrightarrow \quad & z_a + t_a \ > \ z_b + s_b t_b \\ \Leftrightarrow \quad & t_a - (s_b t_b) \ > \ z_b - z_a \ = \ |z_b - z_a|. \end{aligned} \tag{8.13}$$

If now $s_b = +1$, this would be a contradiction to the points (t_a, z_a) and (t_b, z_b) being space-like separated. Hence $s_b = -1$, so we have

$$|z_b - z_a| \ < \ t_a + t_b \ = \ |t_a + t_b|. \tag{8.14}$$

This implies that t_a and t_b cannot both be negative. So assume one of them is negative, w.l.o.g. $t_a > 0, t_b < 0$. But then $|t_a - t_b| > |t_a + t_b|$ and

$$|z_k - z_a| \ < \ |t_a + t_b| \ < \ |t_a - t_b| \tag{8.15}$$

is again a contradiction to the points being space-like. Thus, one must have $t_a > 0, t_b > 0$, which proves the claim.

Because of the specific sign combinations that allow for collisions, the claim shows that if $(s, s+1)$ is a collision, neither $(s-1, s)$ nor $(s+1, s+2)$ can be one. Therefore, $|k - s| \geq 2$.
We use the commutability of τ_k and τ_s to define a third permutation

$$\rho \ := \ \tau_k \circ \tau_s \circ \pi \ = \ \tau_k \circ \sigma \ = \ \tau_s \circ \kappa, \tag{8.16}$$

which, by construction, has $n-1$ collisions, i.e. one less than σ and κ. This means the seemingly different representations of $\phi^{\pi}_{s_1 \dots s_N}$,

$$\begin{aligned} \phi^{\pi}_{s_1 \dots s_N} \ &= \ \phi^{\sigma}_{s_1 \dots s_{s+1} s_s \dots s_N} + s_s \varphi^{(s)} \quad \text{and} \\ \phi^{\pi}_{s_1 \dots s_N} \ &= \ \phi^{\kappa}_{s_1 \dots s_{k+1} s_k \dots s_N} + s_k \varphi^{(k)} \end{aligned} \tag{8.17}$$

are in fact equal. This can be seen from the fact that the different ways of decomposing ρ via eq. (8.10) yield

$$\begin{aligned} \phi^{\sigma}_{s_1 \ldots s_{s+1} s_s \ldots s_N} + s_s \varphi^{(s)} &= \phi^{\rho}_{s_1 \ldots s_{k+1} s_k \ldots s_{s+1} s_s \ldots s_N} + s_k \varphi^{(k)} + s_s \varphi^{(s)} \\ &= \phi^{\kappa}_{s_1 \ldots s_{k+1} s_k \ldots s_N} + s_k \varphi^{(k)}. \end{aligned} \tag{8.18}$$

This finishes the proof of uniqueness of the phases because by the induction hypothesis, the phases associated with ρ, σ and κ exist and are uniquely determined. □

Proof of the Theorem: Lorentz invariance and probability conservation are clear from lemma 7.4 and lemma 7.5. We already know that uniqueness of solutions in a weak sense follows from probability conservation by virtue of thm. 7.1. If the function defined by eq. (8.8) is indeed m times continuously differentiable, it follows from continuity that it is also unique as a C^m-function.
Thus, it remains to show that the function given by (8.8) is indeed a classical (or 'strong') solution of the IBVP. In order to prove this, the following four points have to be verified:

1. *Differentiability:* We need to prove that $\psi \in \mathrm{C}^m\left(\mathscr{S}_1, (\mathbb{C}^2)^{\otimes N}\right)$. As the initial values satisfy $g_j \in \mathrm{C}^m(\mathcal{I}, \mathbb{C})\ \forall j = 1, ..., 2^N$, this property is inherited by ψ_j via the characteristics. To see this, note that eq. (8.8) just makes use of a translation of the initial values along straight lines in the (t_k, z_k) spaces.
   However, we need to consider the points separately where the permutation π changes. This exactly happens when at least two of the characteristic values c_j are equal. But then the C^m-property of ψ is just assured by the requirement that the initial values must satisfy the boundary conditions (eq. (8.7)) and that the transition shall be C^m.

2. The function defined by eq. (8.8) *solves the system of Dirac equations in* $\mathscr{S}_1$*:* This follows from lemma 6.1 because the components of the solution are indeed constant along the respective multi-time characteristics and do only depend on the characteristic values c_k.

3. *The initial conditions* (8.2) *are satisfied:* At a point $(0, z_1, 0, z_2, ..., 0, z_N) \in \mathcal{I} \cap \overline{\mathscr{S}_1}$, we have $c_k = z_k\ \forall k$ and thus $c_{\pi(1)} \leq c_{\pi(2)} \leq \cdots \leq c_{\pi(N)}$ is fulfilled for $\pi = \mathrm{id}$. Therefore, formula (8.8) reduces to

$$\psi_{s_1 \ldots s_N}(0, z_1, ..., 0, z_N) = g_{s_1 \ldots s_N}(c_1, ..., c_N) \tag{8.19}$$

   which is equivalent to (8.2).

4. *The boundary conditions* (7.37) *are satisfied:* Let $k \in \{1, ..., N\}$ and $(t_1, z_1, ..., t_k = t, z_k = z, t_{k+1} = t, z_{k+1} = z, ..., t_N, z_N) \in \mathscr{C}_{k,k+1}$. We consider two components of ψ where only the k-th and $(k+1)$-th sign is exchanged, or more formally: Let $(s_1, \ldots s_N)$, $(\tilde{s}_1, \ldots \tilde{s}_N) \in \{+1, -1\}^N$ with $s_l = \tilde{s}_l\ \forall l \notin \{k, k+1\}$ and $(s_k, s_{k+1}) = (+, -) = (\tilde{s}_{k+1}, \tilde{s}_k)$. the respective characteristic values are $c_k = z_k + s_k t_k$ and

$\tilde{c}_k = z_k + \tilde{s}_k t_k$.
Then observe the property $c_l = \tilde{c}_l \ \forall l \notin \{k, k+1\}$ and $c_k = \tilde{c}_{k+1}$, $\tilde{c}_{k+1} = c_k$. Let π be the permutation that leads to $c_{\pi(1)} \leq \cdots \leq c_{\pi(N)}$. The permutation σ needed to achieve $\tilde{c}_{\sigma(1)} \leq \cdots \leq \tilde{c}_{\sigma(N)}$ is given by $\sigma = \tau_k \circ \pi$, and it has one collision less than π w.r.t. the indices $\tilde{s}_k$. Inserting this into equation (8.8) yields

$$\begin{aligned}
\psi_{\tilde{s}_1 \ldots \tilde{s}_N} &\overset{(8.8)}{=} e^{i\phi^{\sigma}_{\tilde{s}_1 \ldots \tilde{s}_N}} g_{\tilde{s}_{\sigma(1)} \ldots \tilde{s}_{\sigma(N)}} \left(\tilde{c}_{\sigma(1)}, \ldots, \tilde{c}_{\sigma(N)} \right) \\
&\overset{(8.10)}{=} e^{i\phi^{\pi}_{s_1 \ldots s_N}} e^{-i\varphi^{(k)}} g_{s_{\pi(1)} \ldots s_{\pi(N)}} \left(c_{\pi(1)}, \ldots, c_{\pi(N)} \right) \\
&\overset{(8.8)}{=} e^{-i\varphi^{(k)}} \psi_{s_1 \ldots s_N}. \qquad (8.20)
\end{aligned}$$

This shows that (7.37) is valid.

These four points establish existence: the function given in (8.8) is indeed the solution of the IBVP. □

Remark: Uniqueness of solutions can also be proven in a completely different way by directly showing that every solution of the IVBP has to fulfill equation (8.8). The proof is rather complicated, but still we present it here because it shows a more direct way to assure oneself that (8.8) is the right formula for the solutions. It works by induction over the number of colliding particles, which is a similar concept to the collisions defined above, but not the same.

Definition: We define the set Col at a certain point $(t_1, z_1 \ldots, t_N, z_N)$ for a component $\psi_{\circ_1 \circ_2 \cdots \circ_N}$ as

$$Col := \{k \in \{1, \ldots, N-1\} \,|\exists l > k \text{ with } c_k > c_l\}. \qquad (8.21)$$

A component of ψ is said to have n *colliding particles* at a certain point iff $|Col| = n$.

Remark: The number of *collisions* defined atop lemma 8.2 is larger than or equal to the number of colliding particles. Figuratively speaking, a colliding particle can undergo several collisions. Compare figure 3 where the left particle, at point A, is a colliding particle with two collisions (as apparent from the intersections of characteristic lines).

Proof: In this alternative proof of uniqueness, we want to show that every solution of the IVBP has to fulfill equation (8.8). For $n \in \{1, \ldots, N-1\}$ we define $I(n)$ to be the statement:
"If a component of a solution ψ of the IVBP has at least n colliding particles at a certain point in $\mathscr{S}_1$, then equation (8.8) must hold at this point."
We prove that $I(n)$ is true for all n by induction.

1. $I(0)$ *is true:* If a component ψ_j of a solution of the IVBP satisfies $c_k \leq c_{k+1} \ \forall k$ at a point $(t_1, z_1, \ldots, t_N, z_N) \in \mathscr{S}_1$, then the value of ψ_j is uniquely determined by the initial values: Notice that any multi-time characteristic $S_{\circ_1 \circ_2 \cdots \circ_N}(c_1, \ldots, c_N)$ has a unique intersection point with the initial values surface $\{t_1 = t_2 = \cdots = t_N = 0\}$

given by $(0, c_1, 0, c_2, \ldots, 0, c_N)$. In the case we are considering here, the value of ψ_j at this point is determined by the initial condition (8.2) because $c_1 \leq c_2 \leq \cdots \leq c_N$, hence

$$\psi_j(t_1, z_1, \ldots, t_N, z_N) \overset{\text{characteristics}}{=} \psi_j(0, c_1, \ldots, 0, c_N) \overset{(8.2)}{=} g_j(c_1, \ldots, c_N), \tag{8.22}$$

which is exactly (8.8) (with $\pi = 1$ in this case implying the absence of the phase factor $e^{i\phi}$).
Of course, the constancy along the characteristic surface can only be used provided there is a path on that surface that stays in $\mathscr{S}_1$ and connects the two points. One such path is given by $((1-\tau)t_1, z_1 + s_1\tau t_1, \ldots, (1-\tau)t_N, z_N + s_N\tau t_N))$, $\tau \in [0,1]$. This obviously lies on the relevant characteristic, and it stays in $\mathscr{S}_1$ because for $k < l$,

$$z_k < z_l \wedge c_k \leq c_l \Rightarrow \; s_k t_k \leq s_l t_l \Rightarrow z_k + s_k\tau t_k < z_l + s_l \tau t_l, \tag{8.23}$$

and the points are also space-like related since for $k < l$ and all $\tau \in [0,1]$,

$$\begin{aligned} & ((1-\tau)t_l - (1-\tau)t_k))^2 \\ < \; & (1-\tau)^2 \cdot (z_l - z_k)^2 \leq (z_l - z_k)^2 \\ \leq \; & (z_l - z_k)^2 + 2\tau \underbrace{(z_l - z_k)(s_l t_l - s_k t_k)}_{\geq 0, \text{ see above}} + \tau^2 (s_l t_l - s_k t_k)^2 \\ = \; & \left(z_l + s_l(\tau t_l) - z_k - s_k(\tau t_k)\right)^2 . \end{aligned} \tag{8.24}$$

2. $I(n) \Rightarrow I(n+1)$: Let $\psi_{s_1 \ldots s_N}$ and $(t_1, z_1, \ldots, t_N, z_N) \in \mathscr{S}_1$ such that the component has $n+1$ colliding particles at this point. By the definition of Col (8.21), this means that we have $(c_k \leq c_l \; \forall l > k) \; \forall k \in \{1, \ldots, N-1\} \backslash Col$ with $|Col| = n+1$. Let furthermore $a := \max Col$ and $b + 1 := \min \{k > a | (k \leq N \wedge c_a \leq c_k) \vee k = N+1\}$. In plain words, a is the particle with the highest number that makes problems because its characteristic line crosses others on the way back to the initial data surface, and it crosses all lines from $a+1$ to b. An obvious generalization of the the auxiliary claim in the proof of lemma 8.2 is that in such a setting, we have

$$\begin{cases} \text{either } s_a = +1 \; \wedge \; s_k = -1 \; \wedge \; t_a > 0 \; \wedge \; t_k > 0 \; \; \forall a < k \leq b \\ \quad \text{or} \quad s_a = -1 \; \wedge \; s_k = +1 \; \wedge \; t_a < 0 \; \wedge \; t_k < 0 \; \; \forall a < k \leq b \end{cases} \tag{8.25}$$

Because both cases are analogous, the induction step will only be shown for the first case, so let $s_a = +1, s_k = -1, t_a > 0, t_k > 0 \; \; \forall a < k \leq b$.
The point $(\tilde{t}, \tilde{z})$ with $\tilde{z} = \frac{1}{2}(c_a + c_{a+1}), \tilde{t} = \frac{1}{2}(c_a - c_{a+1})$ satisfies $\tilde{z} + \tilde{t} = c_a$ and $\tilde{z} - \tilde{t} = c_{a+1}$, thus we stay on the same multi-time characteristic if we move along a path on which only the coordinates of the a-th and $a+1$-th particle change to $\tilde{t}$

and $\tilde{z}$. Then we can use the boundary conditions[13]

$$\begin{aligned}
&\psi_{s_1\cdots+-\ldots s_N}\left(t_1,z_1,\ldots,t_N,z_N\right)\\
\overset{\text{char.}}{=}&\,\psi_{s_1\cdots+-\ldots s_N}\left(t_1,z_1,\ldots,t_{a-1},z_{a-1},\tilde{t},\tilde{z},\tilde{t},\tilde{z},t_{a+2},\ldots,t_N,z_N\right)\\
\overset{(8.4)}{=}&\,\psi_{s_1\cdots-+\ldots s_N}\left(t_1,z_1,\ldots,\tilde{t},\tilde{z},\tilde{t},\tilde{z},\ldots,t_N,z_N\right)\cdot e^{i\varphi^{(a)}_{s_1\ldots s_N}}
\end{aligned}\tag{8.26}$$

If we have $b=a+1$, then we are done now. If not, we can iterate this procedure $b-a$ times and thereby switch from one component to another which determines the former. After this iteration, we obtain a wave-function component that only features n colliding particles, because c_a to c_N are now in ascending order, so we can use $I(n)$ to prove[14]

$$\begin{aligned}
&\psi_{s_1\cdots+--\cdots-s_{b+1}\ldots s_N}\left(t_1,z_1,\ldots t_N,z_N\right)\\
=&\,\psi_{s_1\cdots---\cdots-+s_{b+1}\ldots s_N}\left(t_1,z_1\ldots\tilde{t},\tilde{z},\tilde{\tilde{t}},\tilde{\tilde{z}},\ldots t_N,z_N\right)\cdot e^{i\varphi_{ab}}\\
\overset{\text{char.}}{=}&\,\psi_{\text{—"—}}\left(0,c_1,\ldots 0,c_{a-1},0,c_{a+1},0,c_{a+2},\ldots 0,c_b,0,c_a,0,c_{b+1}\ldots 0,c_N\right)\cdot e^{i\varphi_{ab}}\\
=:&\,(*)
\end{aligned}\tag{8.27}$$

We now introduce three permutations: π is the permutation that orders $c_{\pi(1)}\leq\cdots\leq c_{\pi(N)}$, σ is the permutation that maps

$$(1,2,\ldots,N)\mapsto(1,2,\ldots,a-1,a+1,\ldots,b,a,b+1,b+2,\ldots,N)\tag{8.28}$$

[13] As in the base case, one should assure oneself that there really exists a path connecting the two points that does not leave $\mathscr{S}_1$. One can again use the standing to reason path $\left(\ldots,t_a+\tau(\tilde{t}-t_a),z_a+\tau(\tilde{z}-z_a),t_{a+1}+\tau(\tilde{t}-t_{a+1}),z_{a+1}+\tau(\tilde{z}-z_{a+1}),\ldots\right)$, $\tau\in[0,1]$, where all the other components are left untouched. We only show how space-likeness of two points comes about on this path for the example of two particles with labels $k<a$, it is then easy to convince oneself that a similar reasoning is always possible, thus the path indeed stays in $\mathscr{S}_1$ for $\tau\in[0,1)$. Noticing that $\tilde{t}-t_a=-(\tilde{z}-z_a)\leq 0$, we can calculate

$$\begin{aligned}
&\left(t_k-t_a-\tau(\tilde{t}-t_a)\right)^2\\
=&(t_k-t_a)^2-2\tau(t_k-t_a)(\tilde{t}-t_a)+\tau^2(\tilde{t}-t_a)^2\\
=&(t_k-t_a)^2+2\tau(t_k-t_a)(\tilde{z}-z_a)+\tau^2(\tilde{z}-z_a)^2\\
\leq&(t_k-t_a)^2+2\tau(|t_k-t_a|)(|\tilde{z}-z_a|)+\tau^2(\tilde{z}-z_a)^2\\
<&(z_k-z_a)^2+2\tau(|z_k-z_a|)(\tilde{z}-z_a)+\tau^2(\tilde{z}-z_a)^2\\
=&(z_k-z_a)^2-2\tau(z_k-z_a)(\tilde{z}-z_a)+\tau^2(\tilde{z}-z_a)^2\\
=&\left(z_k-z_a-\tau(\tilde{z}-z_a)\right)^2.
\end{aligned}$$

Additionally, one should check that $\left(t_1,z_1,\ldots t_{a-1},z_{a-1},\tilde{t},\tilde{z},\tilde{t},\tilde{z},t_{a+2},\ldots t_N,z_N\right)\in\mathscr{C}_{k,k+1}$ is indeed true so that using (8.4) is allowed. This can be seen by calculations like the one above; all particles stay space-like to each other apart from the a-th and a+1-th which are moved to the same space-time point.

[14] We use a shorthand notation for the phases appearing here and leave away unimportant indices. φ_{ab} just indicates the phase that comes about when iterating the step that transports the + sign one place to the right $b-a$ times, i.e.

$$\varphi_{ab}=\sum_{k=a}^{b-1}\varphi^{(k)}_{s_1\ldots s_{a-1}--\cdots-s_{b+1}\cdots s_N}$$

.

and π' is the permutation that is needed to order the constants one gets after applying σ, i.e.

$$\pi = \pi' \circ \sigma. \tag{8.29}$$

With these definitions, we proceed:

$$\begin{aligned}(*) &= \psi_{s_{\sigma(1)}\dots s_{\sigma(N)}}\left(0, c_{\sigma(1)}, \dots 0, c_{\sigma(N)}\right) \cdot e^{i\varphi_{ab}} \\ &\overset{I(n)}{=} g_{s_{\pi'(\sigma(1))}\dots s_{\pi'(\sigma(N))}}\left(c_{\pi'(\sigma(1))}, \dots c_{\pi'(\sigma(N))}\right) \cdot e^{i(\varphi^{\pi'}+\varphi_{ab})} \\ &= g_{s_{\pi(1)}\dots s_{\pi(N)}}\left(c_{\pi(1)}, \dots c_{\pi(N)}\right) \cdot e^{i\varphi^{\pi}}\end{aligned} \tag{8.30}$$

The equality of the phases follows by repeated use of the condition (8.10), using that σ can be written as a composition of $b-a$ transpositions where always a $+$ sign is on the left, so $\varphi^{\sigma} = \varphi_{ab}$. For such a composition transporting only pluses to the right, condition (8.10) really implies that the phases add up, as used in $\varphi^{\pi} = \varphi^{\pi'} + \varphi_{ab}$. Thus, by reducing the number of colliding particles by 1 with the help of the boundary conditions, we have used the induction hypothesis to show that $I(n+1)$ must hold, too.

Thus, every solution of the IVBP fulfills equation (8.8), and the proof is complete. □

9 Interaction and effective potential

In addition to the mathematical and physical features already established, it remains to show that our model is interacting. Moreover, we demonstrate that the interaction can, at equal times, be described effectively using potentials involving δ-functions of the particle distances.

9.1 The model is interacting

In sec. 2.2, we have discussed the general criterion from [2] that states that a physical model is called *interacting* if it generates entanglement, i.e. if there are wave functions that are initially product states and become entangled under time evolution. Note that for the antisymmetrized wave functions we are considering, a product state means "wedge product". We present a simple argument why our model is interacting in this sense, by effectively reducing the general case to the two-particle-case.

Lemma 9.1 *The model defined by* (6.8) *with* $\Omega = \mathscr{S}_1$ *and boundary conditions* (7.37) *is interacting if there exists* $k \in \{1, \dots, N-1\}$ *with* $\varphi^{(k)} \neq \pi$.

Proof: W.l.o.g. $k = 1$. Let the initial conditions be such that $\psi|_{\mathcal{I}}$ is a product wave function. In particular, this means that there exist functions $\alpha, \beta, \gamma, \delta \in C^m(\mathbb{R}, \mathbb{C})$ and $\zeta \in C^m(\mathbb{R}^{N-2}, \mathbb{C})$ with

$$\begin{aligned} g_{+-+\cdots+}(z_1, ..., z_N) &= \alpha(z_1)\beta(z_2)\zeta(z_3, ..., z_N) \\ g_{-++\cdots+}(z_1, ..., z_N) &= \gamma(z_1)\delta(z_2)\zeta(z_3, ..., z_N) \end{aligned} \tag{9.1}$$

for $z_1 \leq \cdots \leq z_N$.
Antisymmetry (6.10) implies

$$\alpha(z_1)\beta(z_2) \;=\; -\gamma(z_2)\delta(z_1). \tag{9.2}$$

Consider the solution at a point $p = (t, z_1, ..., t, z_N) \in \mathscr{S}_1$ with common time $t > 0$. The auxiliary claim in the proof of lemma 8.2 implies that the characteristic values at p with respect to the component $\psi_{+-+\cdots+}$ are in ascending order iff $z_1 + t \leq z_2 - t$. Thus we can use formula (8.8) to obtain $\psi_{+-+\cdots+}(p)$, with the permutation π being the identity if $z_1 \leq z_2 - 2t$ and the transposition τ_1 if $z_1 > z_2 - 2t$. Written via the Heaviside function Θ, this yields

$$\begin{aligned}
\psi_{+-+\cdots+}(p) \;&=\; g_{+-+\cdots+}(c_1, ...c_N)\,\Theta\left(z_2 - z_1 - 2t\right) + \\
&\quad + e^{i\varphi^{(1)}} g_{-++\cdots+}(c_2, c_1, c_3, ..., c_N)\,\Theta\left(2t + z_1 - z_2\right) \\
&=\; \alpha(c_1)\beta(c_2)\zeta(c_3, ..., c_N)\,\Theta\left(z_2 - z_1 - 2t\right) + \\
&\quad + e^{i\varphi^{(1)}} \gamma(c_2)\delta(c_1)\zeta(c_3, ..., c_N)\,\Theta\left(2t + z_1 - z_2\right).
\end{aligned} \tag{9.3}$$

Using (9.2), this simplifies to

$$\psi_{+-+\cdots+}(p) = \alpha(c_1)\beta(c_2)\zeta(c_3, \ldots, c_N) \cdot \Big(\Theta(c_2 - c_1) - e^{i\varphi^{(1)}}\Theta(c_1 - c_2)\Big). \tag{9.4}$$

This function contains the Heaviside function of a combination of t, z_1 and z_2 in a non-factorizable form. The Θ-function cannot be left away for general initial values (as might be the case if they were zero in some regions). Furthermore, because of the nontrivial prefactor $e^{i\varphi^{(1)}}$ of the second summand, we cannot write it as a product as long as $\varphi^{(1)} \neq \pi$. □

9.2 Effective single-time model, Potentials

A single-time model can be obtained from the multi-time model by setting $t_1 = t_2 = t$. Of course, the main virtue of our model, the manifest Lorentz invariance, is lost by such a restriction. We just consider this helpful because it connects the formalism of multi-time wave functions with a more familiar setting.
Let for simplicity $N = 2$. The single-time wave function is denoted by $\chi(z_1, z_2, t) = \psi(t, z_1, t, z_2)$. Then the single-time model is given by the domain $\{(z_1, z_2, t) \in \mathbb{R}^3 : z_1 \neq z_2\}$, initial data at $t = 0$, boundary conditions (7.37) (with t_1, t_2 replaced by t in all the constructions) and the wave equation

$$i\frac{\partial \chi}{\partial t} \;=\; -i\left(\sigma_3 \otimes \mathbb{1}\frac{\partial}{\partial z_1} + \mathbb{1} \otimes \sigma_3 \frac{\partial}{\partial z_2}\right)\chi \;\equiv\; \hat{H}\chi. \tag{9.5}$$

Note that eq. (9.5) is obtained from the multi-time equations (6.2) by the chain rule.
We introduce new coordinates $u = \frac{1}{2}(z_1 - z_2)$ and $v = \frac{1}{2}(z_1 + z_2)$, so the Hamiltonian becomes

$$\hat{H}_0 = -i\ \mathrm{diag}\left(\partial_v, \partial_u, -\partial_u, -\partial_v\right). \tag{9.6}$$

Then the boundary conditions (7.37) can be reformulated by antisymmetry $\psi_2(u,v) = -\psi_3(-u,v)$ as

$$\begin{aligned} \lim_{u \nearrow 0} \chi_2(u,v) &= \lim_{u \searrow 0} -e^{-i\varphi}\chi_2(u,v), \\ \lim_{u \nearrow 0} \chi_3(u,v) &= \lim_{u \searrow 0} -e^{i\varphi}\chi_3(u,v). \end{aligned} \tag{9.7}$$

The components χ_1 and χ_4 evolve freely and have to be continuous and zero at $u = 0$ because of antisymmetry. For $\varphi = \pi$, eq. (9.7) also reduces to the condition of continuity. In that case, the model becomes free – in agreement with lemma 9.1.
If one wants to implement the interaction by a potential added to the Hamiltonian, this should be done in the manner

$$\hat{H}_0 + \operatorname{diag}\left(0, V(u), -V(u), 0\right), \tag{9.8}$$

because the components χ_1 and χ_4 do not participate in the interaction, and the relative minus sign is dictated by antisymmetry. Seen this way, one can forget about the components 1 and 4. Then the reduced spinor $\tilde{\chi} = (\chi_2, \chi_3)$ satisfies a single-particle Dirac equation, i.e.

$$i\partial_t \tilde{\chi}(u,v) = \left[-i\sigma_3\partial_u + \sigma_3 V(u)\right]\tilde{\chi}(u,v). \tag{9.9}$$

In the physics literature, one does not hesitate to write δ-distributions in the Hamiltonian and calculate the boundary conditions this would give. The main result of [52, 53] is – translated to our case – that the boundary conditions (9.7) correspond to a potential term

$$V(u) = (\pi - \varphi)\,\delta(u). \tag{9.10}$$

Just writing a δ-function in a Hamiltonian is a risky business, though. Ambiguities like those stated in [54] may occur. Mathematically, it is clear that a distribution like $\delta(u)$ is not an operator on Hilbert space. Of course, one can approximate the δ-function by appropriate smooth potentials that provide us with well-defined Hamiltonians, which is the strategy in [52, 53]. However, the mathematically rigorous way to implement a δ-potential is by self-adjoint extensions of $\hat{H}_0$ on a suitable domain (compare [47]), which is done in the following subsection. There we also rederive the approximation by smooth potentials in a, hopefully, more understandable way.

9.3 Self-adjoint extensions of the two-particle Dirac Hamiltonian

Let us collect a number of mathematical facts that are important for the treatment of self-adjoint extensions. In order not to digress too much from our main topic, we will present the functional analytical material very tersely. We are interested in operators on the Hilbert space $\mathscr{H}$ with scalar product $\langle\cdot,\cdot\rangle$. At the base of this discussion lies the insight that a *symmetric* operator $\hat{A} : \mathscr{H} \to \mathscr{H}$, which fulfills

$$\left\langle \alpha, \hat{A}\beta \right\rangle = \left\langle \hat{A}\alpha, \beta \right\rangle \;\; \forall \alpha, \beta \in \operatorname{dom}(\hat{A}) \tag{9.11}$$

need not be a *self-adjoint* operator (defined by $\hat{A}^* = \hat{A}$) because one might have $\mathrm{dom}(\hat{A}) \subsetneq \mathrm{dom}(\hat{A}^*)$. Consequently, the general strategy to obtain a self-adjoint operator from a symmetric one is by enlargement of its domain. In other words, one looks for *self-adjoint extensions*. There are possibly many self-adjoint extensions of a given symmetric operator which correspond to different physical situations, e.g. boundary conditions [43, p. 255–259]. An operator is called *essentially self-adjoint* on its domain if it possesses a unique self-adjoint extension given by its closure.
A handy mathematical tool for studying self-adjoint extensions are the von Neumann deficiency subspaces (compare [55]). We will give their definition and state the central theorem [56, p. 138–140] in the form needed here.

Definition: Let $\hat{A}$ be a symmetric operator. The *deficiency subspaces* of $\hat{A}$ are

$$\begin{aligned} \mathscr{K}_+ &:= \ker(\hat{A}^* - i\mathbb{1}) \\ \mathscr{K}_- &:= \ker(\hat{A}^* + i\mathbb{1}). \end{aligned} \tag{9.12}$$

The numbers (n_+, n_-) given by $n_+ := \dim(\mathscr{K}_+)$, $n_- := \dim(\mathscr{K}_-)$ are called the *deficiency indices* of $\hat{A}$.

Theorem 9.2 *Let $\hat{A}$ be a closed symmetric operator with deficiency indices $n_+ = n_-$ (possibly both infinite). Then there is a n_+^2-parameter family of self-adjoint extensions and every self-adjoint extension $\tilde{A}$ of $\hat{A}$ corresponds to an isometric surjective map $\tilde{f} : \mathscr{K}_+ \to \mathscr{K}_-$ where*

$$\mathrm{dom}(\tilde{A}) = \left\{ \tilde{\psi} = \psi + \phi^+ + \tilde{f}\phi^+ \,\middle|\, \psi \in \mathrm{dom}(\hat{A}), \phi^+ \in \mathscr{K}_+ \right\} \tag{9.13}$$

and

$$\tilde{A}\tilde{\psi} = \hat{A}\psi + i\phi^+ - i\tilde{f}\phi^+. \tag{9.14}$$

Now we apply this machinery to the free Dirac Hamiltonian for two particles, that can be written in our choice of basis and coordinates as

$$\hat{H}_0 = -i \begin{pmatrix} \partial_v & & & \\ & \partial_u & & \\ & & -\partial_u & \\ & & & -\partial_v \end{pmatrix}. \tag{9.15}$$

As already alluded to, the specification of the domain is of high importance. It is known [25, thm. 1.1] that the free Dirac Hamiltonian $\hat{H}_1$ for one particle is essentially self-adjoint on the domain $C_c^\infty(\mathbb{R}) \otimes \mathbb{C}^2$ and self-adjoint on the Sobolev space $H^1(\mathbb{R}) \otimes \mathbb{C}^2$. By a standard theorem of functional analysis [43, p. 301], it follows that $\hat{H}_0 = \hat{H}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_1$ is essentially self-adjoint on the domain $\mathrm{dom}(\hat{H}_0) = \left(C_c^\infty(\mathbb{R}) \otimes \mathbb{C}^2\right)^{\otimes 2}$. But this does not hold for the domain of self-adjointness, because notorious features of tensor products of Hilbert spaces prevent a naive approach. It was observed in [57] that the domain of self-adjointness of $\hat{H}_0$ is larger than $(H^1(\mathbb{R}) \otimes \mathbb{C}^2)^{\otimes 2}$. Nevertheless, the essential

self-adjointness secures that there is no freedom of choice, there is only one self-adjoint extension and this of course describes the case of two non-interacting Dirac particles. The Hamiltonian of interest here, which will be called $\hat{H}$, acts on another Hilbert space $\mathscr{H} = L^2(M) \otimes \mathbb{C}^4$ and has a smaller domain

$$\mathrm{dom}(\hat{H}) = C_c^\infty(M) \otimes \mathbb{C}^4, \quad M := \left\{ (u,v) \in \mathbb{R}^2 \middle| \, u \neq 0 \right\}, \tag{9.16}$$

but its action on functions is the same as the action of $\hat{H}_0$ defined in eq. (9.15). The set M appears because we consider only the space-like configurations, where the relative coordinate u must not be zero.

Lemma 9.3 *The operator $\hat{H}$ defined above is symmetric on its domain and has deficiency indices $n_+ = n_- = \infty$, and thus infinitely many self-adjoint extensions.*

Remark:

- Theorem 9.2 only applies for closed operators, but this is not an obstacle because every densely defined symmetric operator has a closure, for which we can apply the theorem.
- Even an operator with smaller deficiency indices, e.g. $n_+ = n_- = 1$, has infinitely many self-adjoint extensions, because it has a one-parameter family of self-adjoint extensions and this parameter can of course take on infinitely many different values. But in the case of $\hat{H}$, there are even infinitely many parameters, so the set of self-adjoint extensions is tremendously larger.

Proof: For symmetry, we need to check condition (9.11). Let $\alpha, \beta \in \mathrm{dom}(\hat{H})$, then

$$\left\langle \alpha, \hat{H}\beta \right\rangle = -i \int_M \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \alpha_3^* \\ \alpha_4^* \end{pmatrix} \cdot \begin{pmatrix} \partial_v \beta_1 \\ \partial_u \beta_2 \\ -\partial_u \beta_3 \\ -\partial_v \beta_4 \end{pmatrix} = i \int_M \begin{pmatrix} \partial_v \alpha_1^* \\ \partial_u \alpha_2^* \\ -\partial_u \alpha_3^* \\ -\partial_v \alpha_4^* \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \left\langle \hat{H}\alpha, \beta \right\rangle, \tag{9.17}$$

and there are no boundary terms appearing when we integrate by parts because the functions are compactly supported in M.
Next, we determine the deficiency indices $n_\pm$. Because of the symmetry of $\hat{H}$, the action of its adjoint $\hat{H}^*$ is the same as of the operator itself (see also [55, thm. 4.9]). Therefore, the elements of $\mathscr{K}_\pm$ are exactly the functions $\psi \in \mathscr{H}$ satisfying

$$\psi_1(u,v) = \mp \partial_v \psi_1(u,v), \tag{9.18}$$
$$\psi_2(u,v) = \mp \partial_u \psi_2(u,v), \tag{9.19}$$
$$\psi_3(u,v) = \pm \partial_u \psi_3(u,v), \tag{9.20}$$
$$\psi_4(u,v) = \pm \partial_v \psi_4(u,v). \tag{9.21}$$

The equations (9.18) and (9.21) imply that ψ_1, ψ_4 have to be weakly differentiable with respect to v, so they are in fact continuous for almost every fixed u (because every

function in $H^1(\mathbb{R})$ has a continuous representative [58, thm. 8.2]). As the functions $ce^{\pm v}$ (which are the only solutions for the ordinary differential equation $\partial_v f(v) = \pm f(v)$) are not integrable for any $c \in \mathbb{R} \setminus \{0\}$, these two equations have no solutions in $\mathscr{H}$ apart from $\psi_1 = \psi_4 = 0$.
A similar argument works for eqs. (9.19) and (9.20), so the functions ψ_2 and ψ_3 have to be continuous for a.e. fixed v, but with the difference that $u \in \mathbb{R} \setminus \{0\}$. This allows to define the functions piece-wise for $u > 0$ and $u < 0$ and set them equal to zero on the part that would spoil integrability. Therefore, there are non-trivial solutions given by

$$\begin{aligned} \psi_2(u,v) &= f(v)e^{\mp u}\Theta(\pm u)\,, \\ \psi_3(u,v) &= g(v)e^{\pm u}\Theta(\mp u) \end{aligned} \tag{9.22}$$

for arbitrary $f, g \in L^2(\mathbb{R})$. This proves that the deficiency subspaces are

$$\mathscr{K}_\pm = \left\{ \psi = \begin{pmatrix} 0 \\ f(v)e^{\mp u}\Theta(\pm u) \\ g(v)e^{\pm u}\Theta(\mp u) \\ 0 \end{pmatrix} \middle| f, g \in L^2(\mathbb{R}) \right\}, \tag{9.23}$$

so $n_+ = n_- = \infty$, so the operator $\hat{H}$ has infinitely many self-adjoint extensions. □

Most of the self-adjoint extensions are not relevant for us because they yield boundary conditions depending on v, which could be spoken of as δ-functions with strength depending on the absolute coordinate $z_1 + z_2$. It seems that there is only one family of self-adjoint extensions that is compatible with both antisymmetry of the wave function and Poincaré invariance. Define for some $\varphi \in (-\pi, \pi]$ the onto, isometric map

$$\begin{gathered} \tilde{f} : \mathscr{K}_+ \to \mathscr{K}_- \\ \begin{pmatrix} 0 \\ f(v)e^{-u}\Theta(+u) \\ g(v)e^{+u}\Theta(-u) \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ e^{i(\pi-\varphi)}f(v)e^{+u}\Theta(-u) \\ e^{i(\pi-\varphi)}g(v)e^{-u}\Theta(+u) \\ 0 \end{pmatrix}. \end{gathered} \tag{9.24}$$

This map corresponds to a self-adjoint extension $\tilde{H}$ of $\hat{H}$ with the same action and with domain (according to eq. (9.13))

$$\mathrm{dom}(\tilde{H}) = \left\{ \psi + \begin{pmatrix} 0 \\ f(v)\left[e^{-u}\Theta(+u) - e^{-i\varphi}e^{+u}\Theta(-u)\right] \\ g(v)\left[e^{+u}\Theta(-u) - e^{-i\varphi}e^{-u}\Theta(+u)\right] \\ 0 \end{pmatrix} \middle| \psi \in \mathrm{dom}(\bar{H}), f, g \in L^2(\mathbb{R}) \right\} \tag{9.25}$$

where we write $\bar{H}$ for the closure of $\hat{H}$. The boundary conditions corresponding to this extension can be readily determined from this equation by convincing oneself that

functions in $\mathrm{dom}(\bar{H})$ vanish in the limit $u \to 0$, thus for $\tilde{\psi} \in \mathrm{dom}(\tilde{H})$ it follows that

$$\begin{aligned} \lim_{u \nearrow 0} \tilde{\psi}_2(u,v) &= \lim_{u \searrow 0} -e^{-i\varphi} \tilde{\psi}_2(u,v), \\ \lim_{u \nearrow 0} \tilde{\psi}_3(u,v) &= \lim_{u \searrow 0} -e^{i\varphi} \tilde{\psi}_3(u,v). \end{aligned} \tag{9.26}$$

This is the same boundary condition as in equation (9.7), which means that the self-adjoint Hamiltonian $\tilde{H}$ is indeed the one corresponding to our effective single-time model. Having constructed the Hamiltonian of the single-time model in a rigorous way, we can now investigate how to approximate the dynamics it generates by continuous potentials. This can be achieved in a physicist's way by adapting the results of [52, 53], but it will be derived rigorously here.
We define the sequence of Hamiltonians $(H_n)_{n \in \mathbb{N}}$ by

$$H_n := \bar{H} + \mathrm{diag}(0, -V_n(u), V_n(u), 0), \quad \mathrm{dom}(H_n) = \mathrm{dom}(\bar{H}) \subset L^2(M), \tag{9.27}$$

where for every n, the potential V_n is a step function[15] with $\mathrm{supp}\,(V_n) \subseteq [-a_n, a_n]$ and therefore, we add a bounded operator to $\bar{H}$, which allows to keep the same domain. We require

$$\int_{\mathbb{R}} V_n(u) du = \pi - \varphi =: \lambda \quad \text{and} \quad \lim_{n \to \infty} a_n = 0. \tag{9.28}$$

These two conditions imply that in the sense of distributions,

$$V_n(u) \overset{n \to \infty}{\longrightarrow} \lambda \cdot \delta(u). \tag{9.29}$$

Now the question is in which sense we can prove that the Hamiltonians H_n approximate our Hamiltonian $\tilde{H}$ with contact interaction. Because of the different domains, the usual convergence of operators is not appropriate here. What we are interested in at the end of the day is really the convergence of the time evolution operators, because they determine the dynamics. This is easier to handle because these operators are unitary and can be defined on the whole Hilbert space. There is an extended notion of convergence of operators due to Kurtz [59]. We will state Kurtz's results in a slightly simplified form[16] and then use them to prove convergence of the unitary groups with a method heavily inspired by [60, sec. 1.2].

Definition: Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of operators on a Hilbert space $\mathscr{H}$.
Let $\mathrm{dom}(\mathrm{ex}\lim_{n \to \infty} A_n)$ be the set of all $\psi \in \mathscr{H}$ for which there is a sequence $(\psi_n)_{n \in \mathbb{N}}$ with $\psi_n \in \mathrm{dom}(A_n)$ and an $\eta \in \mathscr{H}$ such that

$$\begin{aligned} \lim_{n \to \infty} \psi_n &= \psi, \\ \lim_{n \to \infty} A_n \psi_n &= \eta. \end{aligned} \tag{9.30}$$

[15] Here, a step function on $I \subset \mathbb{R}$ is understood to be a function such that there is $m \in \mathbb{N}$ and a partition of the interval I in m half-open intervals such that the function is constant on each of those intervals. We conjecture that all constructions below are also possible for continuous V_n that satisfy all other properties, but this would give rise to technicalities.

[16] Since only one Hilbert space appears here, we omit his operators P_n.

The *extended limit* operator of the sequence, abbreviated by $\operatorname{ex}\lim_{n\to\infty} A_n$, is then defined on the domain $\operatorname{dom}(\operatorname{ex}\lim_{n\to\infty} A_n)$ by

$$(\operatorname{ex}\lim_{n\to\infty} A_n)\psi := \lim_{n\to\infty} A_n\psi_n = \eta. \tag{9.31}$$

Theorem 9.4 *(Kurtz.) For each n, let $U_n(t)$ be a strongly continuous contraction semigroup defined on $\mathscr{H}$ with infinitesimal operator A_n. Let $A = \operatorname{ex}\lim_{n\to\infty} A_n$. Then there exists a strongly continuous contraction semigroup $U(t)$ on $\mathscr{H}$ such that*

$$\lim_{n\to\infty} U_n(t)\psi = U(t)\psi \quad \forall\psi \in \mathscr{H}, \quad \forall t \in [0,\infty) \tag{9.32}$$

if and only if the domain $\operatorname{dom}(A)$ is dense in $\mathscr{H}$ and for some $\lambda_0 > 0$, the range $\operatorname{ran}(\lambda_0 - A)$ is dense in $\mathscr{H}$.
If the domain of the extended limit of infinitesimal operators A_n contains a set D such that $\tilde{A} := \operatorname{ex}\lim_{n\to\infty} A_n|_D$ is an infinitesimal operator, then the sequence of semigroups $U_n(t)$ converges strongly to the semigroup corresponding to $\tilde{A}$.

The proof is found in [59], the second statement is a (weakened) adaptation of the remark following thm. 2.1 ibid. and is most helpful for us here. We will now state the theorem about the approximation of $\tilde{H}$ by our general step function potentials, and then first prove a technical lemma necessary for the proof of the theorem.

Theorem 9.5 *The sequence of time evolution operators generated by the Hamiltonians H_n defined in eq.* (9.27) *with the conditions* (9.28) *converges strongly to the time evolution operator generated by $\tilde{H}$, i.e.*

$$\lim_{n\to\infty} e^{itH_n}\psi = e^{it\tilde{H}}\psi \quad \forall\psi \in \mathscr{H}. \tag{9.33}$$

Lemma 9.6 *For a given $c(v) = (0, c^2(v), c^3(v), 0) \in L^2(\mathbb{R}) \otimes \mathbb{C}^4$, there is a zero-energy eigenfunction of H_n in the interval $[-a_n, a_n]$, i.e. a function $\varphi_n \in \operatorname{dom}(H_n)$ with $\varphi_n(-a_n, v) = c(v)$ for all v and*

$$H_n\varphi_n(u,v) = 0 \quad \text{for } u \in [-a_n, a_n]. \tag{9.34}$$

This function has the further property that

$$\varphi_n(a_n, v) = \operatorname{diag}\left(0, e^{-i\lambda}, e^{i\lambda}, 0\right)\varphi_n(-a_n, v) \quad \forall v \in \mathbb{R}. \tag{9.35}$$

Furthermore, there is a constant C such that for all n, $|\varphi_n|^2 \leq C|c|^2$.

Proof: Let $c(v) = (0, c^2(v), c^3(v), 0) \in L^2(\mathbb{R}) \otimes \mathbb{C}^4$. The value of n stays fixed throughout the proof, so we will henceforth omit the index n. The first and fourth component of the function we look for can be set equal to zero everywhere. We will therefore now treat the second component of the functions. By induction over m, we want to prove the following statement:

$A(m)$: For every $b_m > a$ and every step function S_m with m steps and $\operatorname{supp} S_m = [-a, b_m]$ and the property

$$\int_{\mathbb{R}} S_m(x)dx = \lambda, \tag{9.36}$$

there is a zero-energy eigenfunction φ_m^2 (the second component of φ_m) with the properties $\varphi_m^2(-a, v) = c^2(v)$ for all v,

$$(-i\partial_u - S_m(u))\varphi_m^2(u, v) = 0 \quad \text{for } u \in [-a, b_m], \tag{9.37}$$

and also $\varphi_m^2(b_m, v) = e^{-i\lambda}\varphi_m^2(-a, v)$ and for all u, $|\varphi_m^2(u, v)| \leq |c^2(v)|$.
Base case $A(1)$: By (9.36), it follows that $S_1 = \frac{\lambda}{b_1+a} \cdot \mathbb{1}_{[-a,b_1]}$. The differential equation for φ_1^2 (considering only values of $u \in [-a, b_m]$) reads

$$\partial_u \varphi_1^2(u, v) = -\frac{i\lambda}{b_1 + a}\varphi_1^2(u, v), \tag{9.38}$$

which has with boundary value $\varphi_1^2(-a, v) = c^2(v)$ the unique solution

$$\varphi_1^2(u, v) = c^2(v) \cdot \exp\left(-\frac{i\lambda}{b_1 + a}(u + a)\right). \tag{9.39}$$

This function obviously satisfies

$$\varphi_1^2(b_1, v) = e^{-i\lambda}c^2(v) = e^{-i\lambda}\varphi_1^2(-a, v) \quad \text{and} \quad |\varphi_1^2(u, v)| = |c^2(v)| \ \forall u \in \mathbb{R} \setminus \{0\}. \tag{9.40}$$

Induction step $A(m) \Rightarrow A(m+1)$: Let S_{m+1} with support $[-a, b_{m+1}]$ be a step function with $m + 1$ steps. Such a function can naturally be decomposed as $S_{m+1} = \tilde{S}_m + \tilde{S}_1$ with $\operatorname{supp} \tilde{S}_m = [-a, b]$ and $\operatorname{supp} \tilde{S}_1 = (b, b_{m+1}]$ for some $b \in (-a, b_{m+1})$, that means $\tilde{S}_m$ stands for the first m steps and $\tilde{S}_1$ for the last step. One can obtain two functions $\tilde{\varphi}_m^2$ and $\tilde{\varphi}_1^2$ using the induction hypothesis and the statement $A(1)$ (with a replaced by b and boundary value such that $\tilde{\varphi}_m^2(b, v) = \tilde{\varphi}_1^2(b, v)$), which have the properties

$$\begin{aligned}
\left(-i\partial_u - \tilde{S}_m(u)\right)\tilde{\varphi}_m^2(u, v) &= 0 \quad \text{for } u \in [-a, b], \\
\tilde{\varphi}_m^2(b, v) &= \exp\left(\int_{-a}^{b} \tilde{S}_m(x)dx\right)\tilde{\varphi}_m^2(-a, v), \\
\left(-i\partial_u - \tilde{S}_1(u)\right)\tilde{\varphi}_1^2(u, v) &= 0 \quad \text{for } u \in [b, b_{m+1}], \\
\tilde{\varphi}_1^2(b_{m+1}, v) &= \exp\left(\int_{b}^{b+1} \tilde{S}_1(x)dx\right)\tilde{\varphi}_1^2(b, v), \\
|\tilde{\varphi}_m^2(u, v)| \leq |c^2(v)|, &\quad |\tilde{\varphi}_1^2(u, v)| \leq |\tilde{\varphi}_m^2(b, v)|.
\end{aligned} \tag{9.41}$$

By just plugging everything in, it becomes clear now that the function defined by

$$\varphi_{m+1}^2(u, v) = \begin{cases} \tilde{\varphi}_m^2(u, v) & \text{for } u \in [-a, b] \\ \tilde{\varphi}_1^2(u, v) & \text{for } u \in [b, b_{m+1}] \end{cases} \tag{9.42}$$

satisfies all required properties. This finishes the induction.
Now we have shown the statement of the lemma for the second component. The third component is dealt with in an analogous way because only the sign of the derivative changes. The uniform boundedness of all functions φ_n follows with $C = 1$. □

Proof of the Theorem: As main ingredient for the proof, we first show that $\tilde{H} \subseteq \operatorname{ex}\lim_{n\to\infty} H_n$, i.e.

$$\operatorname{dom}(\tilde{H}) \subseteq \operatorname{dom}(\operatorname{ex}\lim_{n\to\infty} H_n) \quad \text{and} \quad (\operatorname{ex}\lim_{n\to\infty} H_n)\psi = \tilde{H}\psi \quad \forall \psi \in \operatorname{dom}(\tilde{H}). \tag{9.43}$$

In the first and fourth component, the action of H_n and $\tilde{H}$ is identical (compare eq. (9.27)) and also the domain is unchanged, implying that our statement is trivial in these components. Therefore, all objects in this proof will be considered in their reduced form by replacing every vector or operator

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix}. \tag{9.44}$$

Abusing, but also simplifying the notation, we nevertheless denote everything with the same symbols as before. So let $\psi \in \operatorname{dom}(\tilde{H})$ be a reduced vector. Define

$$\psi_n(u,v) := \begin{cases} \psi(u+a_n, v) & \text{for } u \in (-\infty, -a_n) \\ \varphi_n(u,v) & \text{for } u \in [-a_n, a_n] \\ \psi(u-a_n, v) & \text{for } u \in (a_n, \infty) \end{cases}, \tag{9.45}$$

where φ_n is a zero-energy eigenfunction of H_n in the interval $[-a_n, a_n]$ with

$$\varphi_n(-a_n, v) = \lim_{u \nearrow 0} \psi(u,v), \tag{9.46}$$

which exists by lemma 9.6. Eq. (9.35) also ensures the further property

$$\varphi_n(a_n, v) = \begin{pmatrix} e^{-i\lambda} \\ e^{i\lambda} \end{pmatrix} \lim_{u \nearrow 0} \psi(u,v) = \lim_{u \searrow 0} \psi(u,v), \tag{9.47}$$

where the second equality follows by inspection from eq. (9.26) using $\lambda = \pi - \varphi$. Hence, the defined functions ψ_n are continuous and elements of $\operatorname{dom}(H_n)$. Now we show the two properties:

1. $\psi_n \to \psi$: We calculate

$$\begin{aligned} \|\psi_n - \psi\|^2 &= \int_{\mathbb{R}} dv \left[\int_{-\infty}^{-a_n} du\ |\psi(u+a_n, v) - \psi(u,v)|^2 + \right. \\ &\qquad \left. + \int_{-a_n}^{a_n} du\ |\varphi_n(u,v) - \psi(u,v)|^2 + \int_{a_n}^{\infty} du\ |\psi(u-a_n, v) - \psi(u,v)|^2 \right] \\ &\leq \int_M dv du |\psi(u+a_n, v) - \psi(u,v)|^2 + \int_{\mathbb{R}} dv \int_{-a_n}^{a_n} du\ |\varphi_n(u,v) - \psi(u,v)|^2, \end{aligned} \tag{9.48}$$

where we used a change of variables to combine the first and third integral. With the shift operator defined for $a \in \mathbb{R}$ as $(T_a\psi)(u,v) := \psi(u+a,v)$, we can rewrite the first integral as

$$\int_M dvdu|(T_{a_n} - \mathbb{1})\psi(u,v)|^2 \leq \|T_{a_n} - \mathbb{1}\|_{op}^2 \|\psi\|_2^2 \to 0 \tag{9.49}$$

because the shift operator T_{a_n} converges in norm to the identity operator $\mathbb{1}$ as $a_n \to 0$ [60]. The second term also vanishes because the integrand goes to zero while on the set $[-a_n, a_n]$, $|\Psi|^2$ is bounded and $|\Phi_n|^2$ is uniformly bounded (see lemma 9.6).

2. $H_n\psi_n \to \tilde{H}\psi$: Using that H_n and $\tilde{H}$ differ only in the interval $[-a_n, a_n]$, we obtain

$$\begin{aligned}\|H_n\psi_n - \tilde{H}\psi\|^2 = &\int dv \int_{-\infty}^{a_n} du\ |\tilde{H}\psi(u+a_n,v) - \tilde{H}\psi(u,v)|^2 + \\ &+ \int dv \int_{-a_n}^{a_n} du\ |\underbrace{H_n\varphi_n(u,v)}_{=0} - \tilde{H}\psi(u,v)|^2 \\ &+ \int dv \int_{a_n}^{\infty} du\ |\tilde{H}\psi(u-a_n,v) - \tilde{H}\psi(u,v)|^2 \end{aligned} \tag{9.50}$$

We continue by combining the first and third integral with the same change of variables as before and again use the shift operator T_a:

$$\begin{aligned}\|H_n\psi_n - \tilde{H}\psi\|^2 \leq &\int_M dvdu|\tilde{H}(\psi(u+a_n,v) - \psi(u,v))|^2 \\ &+ \int_{\mathbb{R}} dv \int_{-a_n}^{a_n} du\ |\tilde{H}\psi(u,v)|^2 \\ =&\|\tilde{H}(T_{a_n} - \mathbb{1})\psi\|_2^2 + \int_{\mathbb{R}} dv \int_{-a_n}^{a_n} du\ |\tilde{H}\psi(u,v)|^2 \end{aligned} \tag{9.51}$$

Because the shift operator T_{a_n} which acts on the u-coordinate commutes with the operator $\tilde{H}$ that contains only derivatives with respect to v in the second and third components, the first integral vanishes by the same reasoning as above. The function in the second integral is square-integrable and the domain of integration vanishes for $n \to \infty$, therefore the integral converges to zero.

Now let us recall what we have proven so far, returning to 4-component vectors and operators from now on:

$$\mathrm{dom}(\tilde{H}) \subseteq \mathrm{dom}(\mathrm{ex}\lim_{n\to\infty} H_n) \ \text{ and } \ (\mathrm{ex}\lim_{n\to\infty} H_n)\psi = \tilde{H}\psi \ \ \forall\psi \in \mathrm{dom}(\tilde{H}). \tag{9.52}$$

This obviously also holds if we put an i in front of every operator. In addition, we know that H_n is self-adjoint for every n, so it generates a strongly continuous unitary group $U_n(t) = e^{iH_nt}$, and the same for $\tilde{H}$ which generates $\tilde{U}(t) = e^{i\tilde{H}t}$. These groups are special cases of strongly continuous contraction semigroups (see [56, p. 235]). The infinitesimal operators of these groups are given by iH_n and $i\tilde{H}$ respectively. And now this is exactly the setting of the second part of thm. 9.4. Application of the theorem directly yields that the unitary group e^{iH_nt} converges strongly to the group $e^{i\tilde{H}t}$. □

Theorem 9.5 tells us the precise mathematical content of the intuitive statement that our single-time model corresponds to a two-particle Dirac Hamiltonian with a spin-dependent potential term $V(u) = (\pi - \varphi)\delta(u)$. The multi-time model describes a contact interaction of N particles in one dimension in a fully relativistic way.

10 Possible extensions of the model

10.1 Higher dimensions

The one-dimensional model has the advantage that the multi-time system of Dirac equations (6.2) is simultaneously diagonalizable (impossible for $d > 1$), which enables us to give the explicit solution. This strengthens the power of our model as a toy model with the help of which one can understand certain mechanisms of relativistic interaction and probability conservation. Aiming at a description of realistic physics, the obvious question that arises now is: Can we generalize the model to higher dimensions, especially to $d = 3$?

Our geometric analysis of probability conservation is possible in general dimensions. The derivation with the help of Stokes' theorem in the proof of lemma 7.2 can be repeated in a similar way, but we come across two differences. First, the set $\mathscr{S}$ is connected in $d > 1$ dimensions, so the dissection in $N!$ parts is not possible. This makes the treatment of antisymmetry much more intricate. Second, if one defines in analogy to eq. (7.16) a set

$$W_R := \left\{ (t_1, \mathbf{x}_1, \ldots, t_N, \mathbf{x}_N) \in \overline{\mathscr{S}} \;\middle|\; \begin{array}{c} \exists \tau \in [0,1] : \forall k : t_k = t_\Sigma(\mathbf{x}_k) + \tau \left(t_{\Sigma'}(\mathbf{x}_k) - t_\Sigma(\mathbf{x}_k) \right) \\ \text{and } |\mathbf{x}_k| \leq R \end{array} \right\} \tag{10.1}$$

then its boundary in the sense of manifolds with boundary is given by

$$\partial W_R = \Sigma \cup \Sigma' \cup M_R, \tag{10.2}$$

where M_R is the set of points where there is some k with $|\mathbf{x}_k| = R$. The striking point is that the set $M_1 = \partial S \cap W_R$ cannot appear here because it is of dimension $Nd + 1 - d$ which is smaller than Nd. It is only a zero measure set, so in any case, the integral vanishes,

$$\int_{M_1} \omega_j = 0, \tag{10.3}$$

as ω_j is an Nd-form. This shows directly that no boundary conditions at all are necessary to ensure probability conservation. Even more, we can apply theorem 7.1 and see that there is a unique solution in $d > 1$ which must of course be the free solution. Boundary conditions would either fit to this free solution (and then have no impact) or they would lead to contradictions. Therefore, a direct generalization to higher dimensions only gives us a free model. This can also be seen by the functional analytical treatment in [61], which is applicable only to the effective single-time model, but shows that relativistic point interactions in dimensions $d > 1$ have no effect on the dynamics.

10.2 Non-existence of solutions for configurations with a minimal space-like distance

In this section, we leave the setting of the model (6.8), just retaining the multi-time Dirac equation (6.2), and focus on a different aspect: the question if a consistent Lorentz-invariant and probability-conserving dynamics exists on the domain $\mathscr{S}_\alpha$ of space-like configurations with a minimum space-like distance α.
The relevance of the question is clear from the previous subsection, where it was explained that for $d > 1$, the set of coincidence points in space-time is too low-dimensional to have impact on the dynamics. Therefore, it suggests itself to ask if a change in the domain (and thereby its boundary) can be made such that the dimension of the set across which the probability flux could leave the boundary is increased. The first idea one might have is the set of *α-space-like configurations* (here for $N = 2$):

$$\mathscr{S}_\alpha \;=\; \left\{(t_1, \mathbf{x}_1, t_2, \mathbf{x}_2) \in \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} : (t_1 - t_2)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2 < -\alpha^2\right\}. \tag{10.4}$$

$\partial\mathscr{S}_\alpha$ has dimension $2d+1$ and its intersection with $\Sigma \times \Sigma$, the set appearing in the proof for probability conservation [2], has dimension $2d$ which is sufficient to have impact on the dynamics (the reason being that ω_j is a $2d$-form). Compare with the dimension $1 + d$ of the set of coincidence points. However, because $\partial\mathscr{S}_\alpha$ itself has a dimension greater than $2d$, the dimension of the initial data surface, the question arises if there is a consistent dynamics on it at all. In the following, we approach this question for the simplest case, $d = 1$ for which the necessary mathematical tools are available, and show that the answer is negative.
First we show that there can only be one kind of boundary conditions with the desired properties. In a second step we then prove that the corresponding IBVP on $\mathscr{S}_\alpha$ does not possess any non-trivial solutions. We make use of the following definitions:

$$\begin{aligned} \mathscr{S}_\alpha^+ &= \{(t_1, z_1, t_2, z_2) \in \mathscr{S}_\alpha : z_1 - z_2 > 0\}, \\ \mathscr{S}_\alpha^- &= \{(t_1, z_1, t_2, z_2) \in \mathscr{S}_\alpha : z_1 - z_2 < 0\}. \end{aligned} \tag{10.5}$$

We have $\mathscr{S}_\alpha = \mathscr{S}_\alpha^+ \cup \mathscr{S}_\alpha^-$.

Lemma 10.1 *Let $\alpha > 0$. For the multi-time Dirac equations (6.2) on the domain $\mathscr{S}_\alpha$, there exist no other Poincaré invariant boundary conditions which can lead to probability conservation on every space-like hypersurface and which are compatible with antisymmetry besides the ones given by:*

$$\psi_{+-}(p) \;=\; e^{\pm i\varphi}\psi_{-+}(p) \quad \forall p \in \partial\mathscr{S}_\alpha^\pm \tag{10.6}$$

with a fixed $\varphi \in (-\pi, \pi]$.

Proof: Let $p = (t_1, z_1, t_2, z_2) \in \partial\mathscr{S}_\alpha$. Because the two points (t_1, z_1) and (t_2, z_2) are space-like to each other, there is a Lorentz frame with $t_1 = t_2$. We work in this frame, so we can write either $p = (t_p, z, t_p, z + \alpha)$ or $p = (t_p, z, t_p, z - \alpha)$. The idea is to use

Stokes' theorem in a similar way as in the proof of lemma 7.2 to obtain a condition for probability conservation on equal-time hypersurfaces $\Sigma_{\tau_1}, \Sigma_{\tau_2}$ $(\tau_1 < \tau_2)$ in the considered Lorentz frame. Using the set

$$V \;:=\; \left\{(t, z_1, t, z_2) \in \overline{\mathscr{S}}_\alpha | \tau_1 \leq t \leq \tau_2\right\}, \tag{10.7}$$

(which may replace a V_R like in eq. (7.16) for R large enough because of compactly supported wave functions), one deduces

$$0 \;=\; \int_V d\omega_j = \int_{\partial V} \omega_j. \tag{10.8}$$

Note, however, that like in [2, proof of thm. 4.4], there are two connected components of the domain $\mathscr{S}_\alpha$, so probability conservation in the form

$$\int_{(\Sigma_{\tau_1} \times \Sigma_{\tau_1}) \cap \mathscr{S}_\alpha} \omega_j = \int_{(\Sigma_{\tau_2} \times \Sigma_{\tau_2}) \cap \mathscr{S}_\alpha} \omega_j \tag{10.9}$$

is equivalent to

$$\int_{M^{(1)}} \omega_j \;=\; \int_{M^{(2)}} \omega_j, \tag{10.10}$$

where $M^{(j)} = \{(t, z_1, t, z_2) \in \partial\mathscr{S}_\alpha : \max\{z_1, z_2\} = z_j \;\wedge \tau_1 < t < \tau_2\}$ for $j = 1, 2$. Observe that from $(t, z_1, t, z_2) \in M^{(j)}$ it follows that $z_j = z_{3-j} + \alpha$. Furthermore, antisymmetry implies:

$$\omega_j(t, z_1, t, z_1 + \alpha) \;=\; -\omega_j(t, z_1 + \alpha, t, z_1). \tag{10.11}$$

We insert this relation into eq. (10.10) which allows us conclude that

$$\int_{M^{(1)}} \omega_j \;=\; -\int_{M^{(1)}} \omega_j \;=\; 0. \tag{10.12}$$

As this relation must hold for every τ_1, τ_2, we must have $\omega_j(p) = 0$. In components:

$$|\psi_{+-}(p)|^2 - |\psi_{-+}(p)|^2 \;=\; 0 \quad \Leftrightarrow \quad \psi_{+-}(p) \;=\; e^{i\varphi(p)}\psi_{-+}(p), \tag{10.13}$$

where $\varphi : \partial\mathscr{S}_\alpha \to (-\pi, \pi]$ could in principle be a function that is not constant.
Because p was an arbitrary boundary point, this equation must hold on the whole of $\partial\mathscr{S}_\alpha$. Moreover, the requirement of Poincaré invariance has the consequence that $\varphi(p)$ has to be locally constant (see the example preceding lemma (7.5)). The domain $\mathscr{S}_\alpha$ has the two connected components $\mathscr{S}_\alpha^\pm$ and by antisymmetry one obtains:

$$\varphi|_{\mathscr{S}_\alpha^+} \;=\; -\,\varphi|_{\mathscr{S}_\alpha^-}\,. \tag{10.14}$$

Thus, indeed no other boundary conditions than (10.6) are permitted. □

Remark:

1. A similar proof for distinguishable particles shows that in this case another possibility appears: the two contributions in eq. (10.10) could also cancel instead of vanishing individually. However, this cancelling is not physically sensible because it would imply a non-vanishing current from $\mathscr{S}_\alpha^+$ to $\mathscr{S}_\alpha^-$ and vice versa. Provided the Born rule holds, the particles could then swap place instantaneously.

2. The boundary conditions (10.6) do indeed imply Poincaré invariance and probability conservation. However, this was not shown explicitly as they do not lead to the existence of dynamics (see the following lemma).

Lemma 10.2 *Let $\alpha > 0$ and consider the IBVP given by*

$$\begin{cases} i\gamma_k^\mu \partial_{k,\mu} \psi(t_1, z_1, t_2, z_2) &= 0 \quad \text{for} \quad k = 1, 2, \\ \psi(0, z_1, 0, z_2) &= g(z_1, z_2), \\ \psi_{+-} &= e^{i\varphi}\, \psi_{-+} \text{ on } \partial\mathscr{S}_\alpha \end{cases} \tag{10.15}$$

on the domain $\mathscr{S}_\alpha$. Here, $\varphi \in (-\pi, \pi]$ and $g : \{(z_1, z_2) \in \mathbb{R}^2 : |z_1 - z_2| > \alpha\} \to \mathbb{C}^4$ is supposed to be a C^1-function.
Then, if there exist real numbers $a_1 < b_1 < a_2 < b_2$ with $g_{+-}(a_1, a_2) \neq g_{+-}(b_1, b_2)$ or $g_{-+}(a_1, a_2) \neq g_{-+}(b_1, b_2)$ the IBVP (10.15) does not have any C^1-solution.

Proof: Assume that there exist real numbers $a_1 < b_1 < a_2 < b_2$ with $g_{-+}(a_1, a_2) \neq g_{-+}(b_1, b_2)$. The case for g_{+-} is similar and will not be shown explicitly. Suppose that ψ is a solution of (10.15). We obtain a contradiction by constructing points (t_1, y_1, t_2, y_2) and $(s_1, x_1, s_2, x_2) \in \mathscr{S}_\alpha$ which lie on the same multi-time characteristic with respect to the component ψ_{+-} (see fig. 4). The construction proceeds as follows:

1. Choose a point (t_1, y_1, t_2, y_2) on the same multi-time characteristic as $(0, a_1, 0, a_2)$ and on the boundary of $\mathscr{S}_\alpha$, i.e.

$$\begin{cases} a_1 &= y_1 - t_1 \\ a_2 &= y_2 + t_2 \\ (t_1 - t_2)^2 &= (y_1 - y_2)^2 - \alpha^2 \end{cases} \tag{10.16}$$

   This especially implies that

$$\psi_{-+}(t_1, y_1, t_2, y_2) \;=\; g_{-+}(a_1, a_2). \tag{10.17}$$

2. Consider the set of points (s_1, x_1, s_2, x_2) on the same multi-time characteristic as $(0, b_1, 0, b_2)$ and on the boundary of $\mathscr{S}_\alpha$, i.e.

$$\begin{cases} b_1 &= x_1 - s_1 \\ b_2 &= x_2 + s_2 \\ (s_1 - s_2)^2 &= (x_1 - x_2)^2 - \alpha^2 \end{cases} \tag{10.18}$$

   This means

$$\psi_{-+}(s_1, x_1, s_2, x_2) \;=\; g_{-+}(b_1, b_2). \tag{10.19}$$

3. Then select a point (s_1, x_1, s_2, x_2) on the same multi-time characteristic with respect to the component ψ_{+-}, i.e.

$$\begin{cases} x_1 + s_1 &= y_1 + t_1 \\ x_2 - s_2 &= y_2 - t_2 \end{cases} \tag{10.20}$$

This implies that the value at (t_1, y_1, t_2, y_2) can be obtained in two different ways: firstly by using the boundary condition at that point and secondly by going along the characteristic surface[17] to (s_1, x_1, s_2, x_2) and using the value from there. In formulas:

$$\begin{aligned} \psi_{+-}(t_1, y_1, t_2, y_2) &\overset{\text{b.c.}}{=} e^{i\varphi}\, \psi_{-+}(t_1, y_1, t_2, y_2) \\ &\overset{(10.17)}{=} e^{i\varphi}\, g_{-+}(a_1, a_2). \end{aligned} \tag{10.21}$$

$$\begin{aligned} \psi_{+-}(t_1, y_1, t_2, y_2) &\overset{\text{char.}}{=} \psi_{+-}(s_1, x_1, s_2, x_2) \\ &\overset{\text{b.c.}}{=} e^{i\varphi}\, \psi_{-+}(s_1, x_1, s_2, x_2) \\ &\overset{(10.19)}{=} e^{i\varphi}\, g_{-+}(b_1, b_2). \end{aligned} \tag{10.22}$$

Thus

$$g_{-+}(b_1, b_2) = g_{-+}(a_1, a_2), \tag{10.23}$$

in contradiction to the assumption.

This proves the claim, provided the points we use do exist. Indeed, the combination of the eight equations (10.16), (10.18) and (10.20) with eight unknowns leads to rather

[17] One may wonder how it is possible to have a path connecting the two points which neither leaves the characteristic nor the domain. This is achieved as follows. Concatenate the two linear paths from (t_1, y_1, t_2, y_2) to (t_1, y_1, s_2, x_2) and from (t_1, y_1, s_2, x_2) to (s_1, x_1, s_2, x_2), so first move the right point from Y_2 to X_2 and afterwards the left from Y_1 to X_1. One can see from the hyperbolas in figure 4 that this path only leaves $\mathscr{S}_\alpha$ at its endpoints.

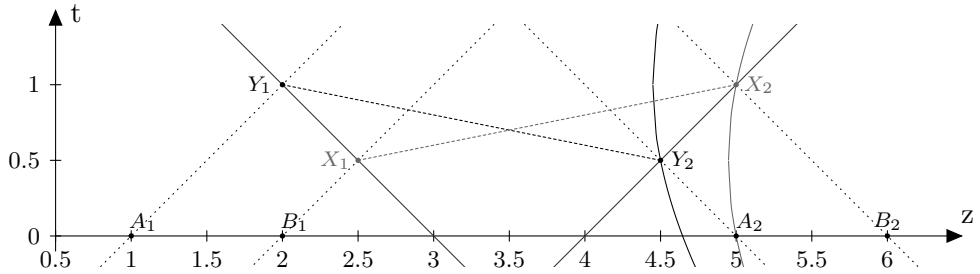

Figure 4: Construction in the proof for values $a_1 = 1$, $b_1 = 2$, $a_2 = 5$, $b_2 = 6$ and $\alpha = \sqrt{6}$. The points are $A_j = (a_j, 0)$, $B_j = (b_j, 0)$, and $Y_j = (y_j, t_j)$, $X_j = (x_j, s_j)$ for $j = 1, 2$. The black hyperbola consists of points that have a space-like distance α to Y_1 and the grey one of thoses which have a space-like distance of α to X_1. The configurations (X_1, X_2) and (Y_1, Y_2) lie on the same multi-time-characteristic, comprised of the two solid black lines.

lengthy quadratic equations, whose general solution is given by

$$
\begin{aligned}
y_1 &= a_1 + \frac{1}{2}\left(-a_1 + b_1 + \frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right), \\
2t_1 &= -a_1 + b_1 + \frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi, \\
y_2 &= a_1 + \frac{1}{2}\left(-a_1 + b_1 + \frac{1}{2}(a_2 - 2b_1 + b_2) + \frac{1}{2}\xi\right), \\
t_2 &= \frac{a_2 - b_2 + 2\left(\alpha^2 - b_1^2 + 2b_1b_2 - b_2^2 + (b_2 - b_1)\left(\frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right)\right)}{(4b_1 - 4b_2 + 2(a_2 - 2b_1 + b_2) - 2\xi)}, \\
x_1 &= b_1 + \frac{1}{4}(a_2 - 2b_1 + b_2) + \frac{1}{4}\xi, \\
s_1 &= \frac{1}{4}(a_2 - 2b_1 + b_2) + \frac{1}{4}\xi, \\
x_2 &= \frac{b_2 - \alpha^2 - b_1^2 + 2b_1b_2 - b_2^2 + (b_2 - b_1)\left(\frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right)}{(2b_1 - 2b_2 + (a_2 - 2b_1 + b_2) + \xi)}, \\
s_2 &= \frac{\alpha^2 - b_1^2 + 2b_1b_2 - b_2^2 + (b_2 - b_1)\left(\frac{1}{2}(a_2 - 2b_1 + b_2) - \frac{1}{2}\xi\right)}{(2b_1 - 2b_2 + (a_2 - 2b_1 + b_2) + \xi)},
\end{aligned}
\tag{10.24}
$$

where

$$
\xi = \sqrt{\frac{(b_2 - a_2)^2(b_1 - a_1) + 4\alpha^2(b_2 - a_2)}{b_1 - a_1}}. \tag{10.25}
$$

The radicand is positive since $a_1 < b_1$ and $a_2 < b_2$. □

Remark: The lemma demonstrates that the most general Lorentz invariant and probability-conserving IBVP (10.15) on $\mathscr{S}_\alpha$ leads to over-determination. Eq. (10.23) shows that the only admissible initial data are those for which g_{-+} is constant (and thus also

g_{+-}). Due to normalization, this constant has to be zero. The two other components are exactly those which are not affected by boundary conditions. Moreover, it becomes clear from the proof that the problem originates from the too high dimension of $\partial\mathscr{S}_\alpha$ which implies (regardless of initial conditions) that certain components of the wave function have to be constant sets like the initial data surface. One cannot avoid this problem by simply prescribing boundary conditions only on a part of the boundary due to the requirement of Lorentz invariance.

10.3 Outlook

The result of the two previous sections is negative in the sense that a direct generalization of our model to the physically most relevant case of three dimensions is not feasible. In higher dimensions, the boundary would have to be "fattened" in order to have any impact on the dynamics, but the natural way of achieving this, using the only Poincaré invariant restriction of $\mathscr{S}$, namely $\mathscr{S}_\alpha$, leads to a contradiction.
We remark that the assumption of mass-less particles is, in contrast to $d = 1$, only due to technical simplicity and we conjecture that it can be dropped and we can obtain an interacting model also for $m \neq 0$. The only detriment of this case is that the system (6.6) with an additional $+m_k\sigma_{1,k}$ in each Dirac equation is not simultaneously diagonalizable and therefore our method of finding an explicit solution is not applicable. However, we can suspect already from our treatment of the effective single-time model in sec. 9.3 that an interacting dynamics still exists. Since the addition of a bounded operator like the mass-term in the Hamiltonian does not change any of the self-adjointness properties, we would obtain the same form of the interaction potential.
In the end, must we bury the idea of relativistic interactions by boundary conditions? As mentioned in sec. 5.2, there is a new concept called *interior boundary conditions* by Teufel and Tumulka [45] which connects pair creation with boundary conditions. So far, they have been successfully applied to non-relativistic and single-time models of pair creation, where the construction of Hamiltonians without UV-divergences was achieved. But the reasoning in sec. 10.1 about uniqueness of solutions even without boundary conditions also applies to models with IBCs, hence no direct generalization of the non-relativistic models to the Dirac case in three dimensions is feasible. However, tricky alterations of the models that save the promising idea might be possible. Not yet should we who enter the realm of interaction in relativistic quantum theories by boundary conditions abandon all hope.

Part III
The Bethe-Salpeter equation

In this part, integral equations for multi-time wave functions are studied at the example of the Bethe-Salpeter equation, which is a two-particle equation first presented in 1951 [62] and intended for the description of bound states in QED. Since a rigorous treatment of the problems in this part is not possible, we focus on pointing out physically important observations and possible starting points for further research. In section 11, we show how the Bethe-Salpeter equation is treated normally, how it can in some sense be derived from QED and how it is tried to obtain explicit solutions of a simplified equation without spin, the Wick-Cutkosky model. We then place the Bethe-Salpeter equation in a more general context of multi-time integral equations (sec. 12) and discuss the appropriate domain of integration. Subsequently, in sec. 13, several problems of the Bethe-Salpeter equation are exemplified. Before concluding this thesis, we also present in sec. 14 how the non-relativistic limit of the Bethe-Salpeter equation may be taken in position space.

11 Standard treatment

11.1 Different forms of the Bethe-Salpeter equation

In this section, we follow Greiner [46, ch. 6]. We will not repeat all details of the derivation of the Bethe-Salpeter equation here, but just rephrase the essential ideas. If one wishes to describe two-particle bound states in QED, one is led to use a two-time wave function $\psi_{ab}(x_1, x_2)$, where $a, b \in \{1, ..., 4\}$ are Dirac spinor indices (that are occasionally omitted). The connection to the Heisenberg field operators $\hat{\Phi}(x)$ of QED is given by [7]

$$\psi(x_1, x_2) = \frac{1}{\sqrt{2}} \langle 0| \hat{\Phi}(x_1)\hat{\Phi}(x_2) |\Psi\rangle . \tag{11.1}$$

Here, $|\Psi\rangle$ is the Heisenberg state, i.e. a fixed vector in the Fock space.
The starting point for the derivation is the observation that for free Dirac particles, there exists a two-particle propagator which is just the product of two one-particle propagators. The Feynman propagator of the free one-particle Dirac equation is known to be

$$S_F(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\not{p} + m}{p^2 - m^2 + i\varepsilon} \tag{11.2}$$

and as usual, the limit $\varepsilon \to 0$ is implied. If we integrate over a three-dimensional hypersurface that includes the points x_1 and x_2 with normal vector field n, the free solution $\varphi_{ab}(x_1, x_2)$ can be expressed as (see [46])

$$\varphi_{ab}(x_1, x_2) = \int d\sigma(x_3) d\sigma(x_4) i S_F^a(x_1, x_3) i S_F^a(x_2, x_4) \not{n}(x_1) \not{n}(x_2) \varphi_{ab}(x_3, x_4). \tag{11.3}$$

Interaction now leads to corrections to this formula, which are computed with the help of Feynman diagrams in a perturbation series. But the usual expansion in orders of the

fine structure constant α is not appropriate for bound state situations – a bound state is supposed to exist forever, so there might be infinitely many photon exchanges between the particles. One may still use a perturbation series by recognizing that the probability of N photons being present *at the same time* shrinks with growing N. One usually rearranges the Feynman diagrams in a clever way and writes down a self-consistent equation for the propagator to account for this. The Bethe-Salpeter equation therefore features an interaction kernel $K^{ab}(x_5, x_6; x_3, x_4)$ which stands for the sum of all *two-particle irreducible diagrams*. These are all Feynman diagrams which cannot be cut into two non-connected diagrams by splitting two fermion lines. One should not overlook that this step is not only performed on a very formal level, one even rearranges terms in an infinite sum that does not converge.
With these ingredients at hand, we can state the Bethe-Salpeter equation in position space:

$$\begin{aligned}\psi_{ab}(x_1, x_2) \;&=\; \varphi_{ab}(x_1, x_2) + \\ &\int d^4x_3 d^4x_4 d^4x_5 d^4x_6 \; iS_F^a(x_1, x_5) \; iS_F^b(x_2, x_6) K^{ab}(x_5, x_6; x_3, x_4) \psi_{ab}(x_3, x_4)\end{aligned} \tag{11.4}$$

As an exception from our usual convention, summation over the indices a, b is never implied in these equations. We can reformulate the equation (11.4) by acting with the free Dirac operator from the left, using that the propagator S_F is a Green's Function, i.e.

$$(i\partial\!\!\!/_1 - m_1) S_F(x_1', x_1) = \delta^{(4)}(x_1' - x_1). \tag{11.5}$$

It follows that

$$(i\partial\!\!\!/_1 - m_1)\psi_{ab}(x_1, x_2) = i \int d^4x_3 d^4x_4 d^4x_6 \; iS_F^b(x_2, x_6) K^{ab}(x_1, x_6; x_3, x_4) \psi_{ab}(x_3, x_4). \tag{11.6}$$

The form one will find nearly all the time in applications and physics papers is the Fourier transformed equation. In momentum space, one writes

$$(p\!\!\!/_1 - m_1)(p\!\!\!/_2 - m_2)\psi_{ab}(p_1, p_2) = -\int d^4p_1' d^4p_2' K^{ab}(p_1, p_2; p_1', p_2')\psi_{ab}(p_1', p_2'). \tag{11.7}$$

We will comment on this Fourier transform, which has a dubious mathematical status, in section 12.2, and use the position space equation for our considerations.

Ladder approximation
In many applications, the Bethe-Salpeter equation is only used up to first order, where the interaction kernel becomes [46, p. 392]

$$K^{ab}(x_5, x_6; x_3, x_4) = \delta^{(4)}(x_5 - x_3)\delta^{(4)}(x_6 - x_4) \; (-ie_1)\gamma_\mu^a \; iD^{\mu\nu}(x_3, x_4) \; (-ie_2)\gamma_\nu^b, \tag{11.8}$$

with the free photon propagator being denoted as $D^{\mu\nu}$ and the charge of particle j as e_j. In this *ladder approximation*, the Bethe-Salpeter equation reads

$$\begin{aligned}\psi_{ab}(x_1,x_2) &= \varphi_{ab}(x_1,x_2)+ \\ &ie_1e_2\int d^4x_1'd^4x_2'\; S_F^a(x_1,x_1')\; S_F^b(x_2,x_2')\gamma_{1,\mu}^a D^{\mu\nu}(x_1',x_2')\gamma_{2,\nu}^b\psi_{ab}(x_1',x_2').\end{aligned} \tag{11.9}$$

Now, the most pressing question if we want to take it seriously as a fundamental equation is: What does the object $\psi(x_1,x_2)$ in the Bethe-Salpeter equation mean? To call it a wave function and use the usual letter ψ suggests that, as always, $|\psi|^2$ should describe a probability density. This question, however, has to be postponed until section 13 where we discuss the problems of such a straightforward understanding of the function appearing in the Bethe-Salpeter equation. We first focus on the question if there are even solutions to eq. (11.9).

11.2 The spin-less model and its solutions

Since the Bethe-Salpeter equation, even in ladder approximation (11.9), is highly complicated, a further simplification which comes about when all spin effects are neglected might be beneficial. This leads to the so-called Wick-Cutkosky model which is the analog of the Bethe-Salpeter equation for two spin-less particles that interact via exchange of a spin-less mass-less boson. We will only consider it in ladder approximation:

$$\begin{aligned}\psi(x_1,x_2) &= \varphi(x_1,x_2)+ \\ &ie_1e_2\int d^4x_1'd^4x_2'\; G_{m_1}(x_1,x_1')\; G_{m_2}(x_2,x_2')\; G_{m=0}(x_1',x_2')\psi(x_1',x_2')\end{aligned} \tag{11.10}$$

Here, G_m denotes the propagator (Green's function) of the Klein-Gordon equation

$$(\Box + m^2)\psi = 0. \tag{11.11}$$

Interestingly, the model was mainly introduced because Wick and Cutkosky recognized that "explicit solutions" could be obtained by intricate and complicated methods [63, 64, 65, 66]. We now explain their way of solving eq. (11.10) in order to elucidate why "explicit solutions" is put in quotation marks. The equation is always considered in momentum space where it is supposed to read

$$\phi(p) = \left[m_1 - \left(\frac{P}{2}+p\right)^2 - i\epsilon\right]^{-1}\left[m_2 - \left(\frac{P}{2}+p\right)^2 - i\epsilon\right]^{-1}\frac{\lambda}{\pi^2}\int d^4p'\frac{\phi(p')}{-(p-p')^2-i\epsilon}. \tag{11.12}$$

This is the first formula in the review of the Wick-Cutkosky model by Nakanishi [66] although the steps leading here include a dubious Fourier transform, which will be discussed in more detail in 12.2. Here, P is the total and p the relative momentum, $\phi(p)$ is the wave function in momentum space and $\lambda := \frac{e_1e_2}{4\pi^2}$ the coupling constant. Momentum

conservation implies that P cannot change in a physical process, so this quantity is not considered an argument of $\phi(p)$, but a fixed parameter. The idea is that eq. (11.12) can have solutions only for a discrete set of values of P which determine the spectrum of bound state energies for the system [46, p. 335]. One may read the equation as

$$\phi(p) = \lambda \cdot I_P[\phi(p)], \tag{11.13}$$

where I_P abbreviates the whole operator on the right hand side of (11.12), which obviously depends on the total momentum P. The idea that was explicated in [67] is now to consider (11.13) as an eigenvalue equation: for every P, one looks for values of $\lambda(P)$ such that it can be fulfilled. This is not very helpful of course, because there is only one true value of the coupling constant. Therefore, one inverts the relation between P and λ afterwards, and should thereby in principle be able to obtain the values of $P(\lambda)$ for the given λ for which the Bethe-Salpeter equation is solvable. This method combined with a number of other tricks allows to find equations for the eigenvalues and eigenfunctions of I_P as in (11.12), which are ordinary differential equations. The solutions cannot be given analytically, but one can prove that they exist and that there is a discrete set of eigenvalues $\lambda(P)$.
At this point, one would have to first invert the function $\lambda(P)$ and then give all the eigenfunctions for the true λ and investigate whether they are orthogonal and span a suitable space of functions. None of this is done in the literature, so the solutions are everything but "explicit". And one might ask: is it even justified to call them "solutions"? This also needs further clarification because the functions one obtains are, in Nakanishi's words, "badly infrared divergent on the mass shell" [66]. As long as these basic questions remain unsolved, the solvability of the Wick-Cutkosky model cannot give many hints on how to understand existence and uniqueness of solutions to the Bethe-Salpeter equation. Further unpleasant properties of some of the model's solutions will be discussed in section 13.

12 Alternative approach to the Bethe-Salpeter equation as multi-time evolution equation

12.1 Motivation of generalized integral equations

We have discussed in part I that integral equations may be a general possibility to obtain interaction in relativistic QM. The direct way of generalizing the differential equations from non-relativistic QM leads to consistency problems (see sec. 5.1), so we try to generalize an integral version of those equations. As in [68], the non-relativistic stationary Schödinger equation for one particle,

$$E\psi(\mathbf{x}) = \left(-\frac{1}{2m}\triangle + V(\mathbf{x})\right)\psi(\mathbf{x}), \tag{12.1}$$

can be rewritten as

$$\psi(\mathbf{x}) = \varphi(\mathbf{x}) + \int d^3\mathbf{x}'\; G_S(\mathbf{x}-\mathbf{x}')V(\mathbf{x}')\psi(\mathbf{x}') \tag{12.2}$$

with the propagator of the Schrödinger equation

$$G_S(\mathbf{x}-\mathbf{x}') := -\frac{2m}{4\pi}\frac{\exp(i\sqrt{2mE}|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}. \tag{12.3}$$

Again, φ is the free solution, i.e. the solution of eq. (12.1) with $V \equiv 0$. When we generalize this to a relativistic theory of two interacting particles, a multi-time wave function is needed and the potential V is replaced by some "interaction kernel" $K(x_1, x_2)$. This leads to the following general form of the integral equation:

$$\psi(x_1, x_2) = \varphi(x_1, x_2) + \int d^4x_1' d^4x_2'\; G(x_1 - x_1')G(x_2 - x_2')K(x_1', x_2')\psi(x_1', x_2') \tag{12.4}$$

If one considers a more general form of interaction kernels K that even depend on four space-time points, one arrives at the form of eq. (11.4). The ladder approximation of the Bethe-Salpeter equation is recovered by choosing K as in (11.8). This approach shows several interesting points:

1. From the comparison with the eigenvalue equation (12.1), one may already expect that a multi-time integral equation is only solvable for a discrete set of values for the energy (in bound state situations). This expectation is known to be fulfilled in the case of the Wick-Cutkosky model [66]. Therefore, we have replaced the issue of consistency that arises for systems of differential equations by the question for which energies the one integral equation can be solved.

2. The propagators G only have to be Green's functions of the relevant single-particle equation, here the Dirac equation. It is not clear *a priori* which boundary conditions shall be imposed, so they need not necessarily be the Feynman propagators, but could also be retarded propagators or others. If one wants to find a quantum version of Wheeler-Feynman electrodynamics, one could also think of taking the time-symmetric propagators in an equation resembling the Bethe-Salpeter equation.

3. The interaction kernel of the Bethe-Salpeter equation, which is problematic because it consists of a series of divergent diagrams, might be replaced by a more suitable interaction term in a general multi-time equation of the form (12.4) in order to describe relativistic interactions. This promising idea, however, lies beyond the scope of this thesis.

12.2 Domain of integration and Fourier transform

If the function $\psi(x_1, x_2)$ really is a multi-time wave function for a two-particle system, we have observed several times that its domain should only be the set of space-like configurations $\mathscr{S}$. But in the usual formulation like in eq. (11.9), the range of integration is $\mathbb{R}^4 \times \mathbb{R}^4$, so also time-like configurations appear as arguments of ψ. In our alternative approach, it seems natural to impose from the beginning that an integral as in eq. (12.4)

be restricted to $\mathscr{S}$. This, however, has striking consequences for the momentum space description.
Let us see how the standard derivation of the Fourier transformed Bethe-Salpeter equation from the position space equation (11.9) is performed in Greiner's book [46]. First, he defines the wave function in momentum space, which we will also call ψ in slight abuse of notation, by

$$\psi_{ab}(p_1, p_2) = \frac{1}{(2\pi)^4} \int d^4x_1 \; d^4x_2 \; e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \psi_{ab}(x_1, x_2). \tag{12.5}$$

A mathematician will shake his head at this point, and he will be quite right: There is no reason for this expression to be well-defined. While a mere spatial Fourier transform of the wave function is reasonable because one expects that $\psi(t_1, \cdot, t_2, \cdot) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, it is clear that the integration along the time direction in (12.5) need not be well-defined. One can possibly make mathematical sense of this Fourier transform in some way, but that needs to be clarified. When transforming the Bethe-Salpeter equation (11.4) to momentum space, Greiner obtains the formula

$$\begin{aligned} &\int d^4x_1 \; d^4x_2 \; e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} (i\not{\partial}_1 - m_1)(i\not{\partial}_2 - m_2) \psi_{ab}(x_1, x_2) \\ = &\int d^4x_1 \; d^4x_2 \; d^4x_3 \; d^4x_4 \; e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} K^{ab}(x_1, x_2; x_3, x_4) \psi_{ab}(x_3, x_4). \end{aligned} \tag{12.6}$$

At this point, Greiner integrates by parts on the left hand side in order to move the derivatives to the exponential function. In this step, boundary terms do not appear. This makes sense for the spatial directions where the wave function can be assumed to drop off when $|\mathbf{x}| \to \infty$, but this is unreasonable for $|t| \to \infty$ if some probability conserving property like

$$\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \; |\psi(t, \mathbf{x}_1, t, \mathbf{x}_2)|^2 = 1 \quad \forall t \tag{12.7}$$

is assumed to hold. Therefore, the steps which lead to the momentum space version of the Bethe-Salpeter equation (11.7) at least need further justification.
In order to Fourier transform the Bethe-Salpeter equation while taking seriously the restriction of the domain to $\mathscr{S}$, one possible method is to restrict the integral e.g. by using the indicator function. The result is denoted as χ to distinguish it from the $\psi(p_1, p_2)$ from eq. (12.5):

$$\chi_{ab}(p_1, p_2) = \frac{1}{(2\pi)^4} \int d^4x_1 \; d^4x_2 \; e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \psi_{ab}(x_1, x_2) \cdot \mathbb{1}_{\mathscr{S}}. \tag{12.8}$$

One can rewrite

$$\mathbb{1}_{\mathscr{S}} = \Theta(|\mathbf{x}_1 - \mathbf{x}_2|^2 - (t_1 - t_2)^2), \tag{12.9}$$

so it is evident that in the general case, we have

$$\chi_{ab}(p_1, p_2) \neq \psi_{ab}(p_1, p_2). \tag{12.10}$$

We cannot derive a simple equation in momentum space that χ fulfills. The integration by parts will surely give boundary terms at $\partial\mathscr{S}$, apart from the problems mentioned so far. Indeed, we see that the restriction to $\mathscr{S}$ leads to a severe consequence: The equivalence between the description in momentum and position space-time does not persist.
Since we do not really know what the function $\psi(x_1, x_2)$ should mean, it is also not so obvious whether we really need to restrict it to space-like configurations. We only see that if we do, the usual way of treating the equation does not work anymore.

13 Issues with the Bethe-Salpeter equation

13.1 Gauge invariance

The Bethe-Salpeter equation in ladder approximation (11.9) contains the photon propagator $D^{\mu\nu}$. As this object is connected to the evolution equation of the electromagnetic 4-potential A^μ, it depends on the chosen gauge. It is well-known that this gauge dependence drops out in any physical amplitude that one computes in QED, so the formalism is gauge invariant in the end. But is this also true for the Bethe-Salpeter equation? The surprising answer is: No.
We will see this clearly when we derive the non-relativistic limit of the Bethe-Salpeter equation in section 14. If this is done in the Feynman gauge that is mostly used in QED, another effective potential comes out than in Coulomb gauge. Therefore, gauge invariance does not hold for the Bethe-Salpeter equation in ladder approximation. There is an article by Barbieri et. al. [69] where it is derived that the gauge invariance becomes restored up to order $\frac{v}{c}$ if one considers one order more than the ladder approximation. Therefore, we can expect that the gauge invariance is lost due to the approximation of considering only finitely many Feynman diagrams.

13.2 Current conservation

It was exemplified in sec. 7.1 how the meaning of a wave function relates to a conserved tensor current providing a probability density. So it is a natural question to ask if there is a conserved current for the Bethe-Salpeter equation (11.4). In a paper by Böhm [70], current conservation for the Bethe-Salpeter equation is considered to be synonymous to a certain kind of Ward identity. This is a well-known equation in QED which connects the vertex function to the Feynman propagator, compare [23, ch. 7.4]. While this identity is the translation of gauge invariance and current conservation to Feynman diagrams and amplitudes, it is not so clear how the wave function $\psi(x_1, x_2)$ can gain probabilistic meaning if only a Ward identity holds. Furthermore, there is no real answer to the question whether current conservation holds for equations (11.9) or (11.4) in [70]. Therefore, we will investigate now if a conserved current $j^{\alpha\beta}$ exists. The first guess is, of course, the tensor current of the free Dirac equation,

$$j^{\alpha\beta} = \bar{\psi}\gamma_1^\alpha\gamma_2^\beta\psi. \tag{13.1}$$

Let us compute its 4-divergence. Recall eq. (11.6) which comes about by acting with the free Dirac operator on the Bethe-Salpeter equation,

$$(i\not{\partial}_1 - m_1)\psi_{ab}(x_1,x_2) = i\int d^4x_1' d^4x_2' d^4x_2''\; iS_F^b(x_2,x_2'')K^{ab}(x_1,x_2'';x_1',x_2')\psi_{ab}(x_1',x_2'). \tag{13.2}$$

We also need the adjoint equation, which reads

$$\bar\psi_{ab}(x_1,x_2)(i\overleftarrow{\not{\partial}}_1 + m_1) = \int d^4x_1' d^4x_2' d^4x_2''\; \bar\psi_{ab}(x_1',x_2')\bar K^{ab}(x_1,x_2'';x_1',x_2')\bar S_F^b(x_2,x_2''), \tag{13.3}$$

where the barred objects are defined in analogy to $\bar\psi = \psi^\dagger\gamma_1^0\gamma_2^0$ as $\bar S = \gamma_1^0\gamma_2^0 S^\dagger\gamma_1^0\gamma_2^0$ and $\bar K = \gamma_1^0\gamma_2^0 K^\dagger\gamma_1^0\gamma_2^0$. Now we add up the equations multiplied by some factors:

$$(*) := \bar\psi\gamma_2^\beta\cdot(13.2) + (13.3)\cdot\gamma_2^\beta\psi$$

The left hand side of $(*)$ is then, omitting the indices ab and the argument (x_1,x_2) in all functions,

$$\begin{aligned}
&\bar\psi\gamma_2^\beta(i\overrightarrow{\not{\partial}}_1 - m_1)\psi + \bar\psi(i\overleftarrow{\not{\partial}}_1 - m_1)\gamma_2^\beta\psi \\
&= i\left(\bar\psi\gamma_2^\beta(\not{\partial}_1\psi) + (\not{\partial}_1\bar\psi)\gamma_2^\beta\psi\right) \\
&= i\partial_\alpha j^{\alpha\beta}.
\end{aligned} \tag{13.4}$$

The right hand side of $(*)$ gives

$$\begin{aligned}
&-\bar\psi(x_1,x_2)\gamma_2^\beta\int d^4x_1' d^4x_2' d^4x_2''\; S_F(x_2,x_2'')K(x_1,x_2'';x_1'x_2')\psi(x_1',x_2') \\
&+\int d^4x_1' d^4x_2' d^4x_2''\bar\psi(x_1',x_2')\bar K(x_1,x_2'';x_1',x_2')\bar S_F(x_2,x_2'')\gamma_2^\beta\psi(x_1,x_2).
\end{aligned} \tag{13.5}$$

Defining a function

$$\zeta^\beta(x_1,x_2,x_1',x_2',x_2'') := \bar\psi(x_1,x_2)\gamma_2^\beta S_F(x_2,x_2'')K(x_1,x_2'';x_1'x_2')\psi(x_1',x_2'), \tag{13.6}$$

we can combine (13.4) and (13.5) to obtain the following expression:

$$\partial_\alpha j^{\alpha\beta}(x_1,x_2) = -2\cdot\int d^4x_1' d^4x_2' d^4x_2''\;\mathrm{Im}\,\zeta^\beta(x_1,x_2,x_1',x_2',x_2''). \tag{13.7}$$

Although we cannot prove it rigorously, this equation strongly suggests that the free Dirac current is not conserved in the Bethe-Salpeter equation. The imaginary part of ζ^β contains very different combinations of the components of ψ for different β, but current conservation would mean that $\partial_\alpha j^{\alpha\beta} = 0$ for all β. It is very unlikely that this holds. In the ladder approximation, the divergence of the current simplifies to

$$\partial_\alpha j^{\alpha\beta}(x_1,x_2) = -2\cdot\int d^4x_2'\;\mathrm{Im}\,\xi^\beta(x_1,x_2,x_2'), \tag{13.8}$$

where

$$\xi^{\beta}(x_1, x_2, x_2') := -ie_1e_2\bar{\psi}(x_1, x_2)\gamma_2^{\beta}S_F(x_2, x_2')\gamma_1^{\mu}D_{\mu\nu}(x_1 - x_2')\gamma_2^{\nu}\psi(x_1, x_2'). \qquad (13.9)$$

Since we conjecture that the free Dirac current is neither conserved in the full Bethe Salpeter equation nor in the ladder approximation, the meaning of the wave function $\psi(x_1, x_2)$ is very unclear. One could now try to identify other tensor currents that are conserved, similarly to the treatment of the two-body Dirac equations, see [44]. But it is likely that such a modified current $j_{mod}(x_1, x_2)$ will not be of a simple form, in particular that it will not be a local object but depend also on the values of ψ at other points than (x_1, x_2). In the normal Schrödinger equation, this is exactly what happens if one adds "non-local potentials", i.e. integral operators: the usual current j is not conserved and a non-local modification is necessary [71]. It is disputable how much sense such a non-local probability density would make.

13.3 Problematic solutions of the spin-less model

It was described in sec. 11.2 how some proclaimed explicit solutions to the Wick-Cutkosky model can be obtained. There was a long-standing discussion about these solutions in physics publications of the 1960s and 70s summarized by Nakanishi [66] which ended without a real conclusion. The discussion was mainly about three unpleasant features of the solutions: divergences, negative-norm states and so-called abnormal solutions. We will comment on all three problems from our multi-time perspective now, where their fundamental meaning is more apparent.

1. **Divergences:** It is not too surprising that an equation that is closely related to QED, an agglomeration of divergent expressions, features divergent solutions. We have already remarked that a divergent expression can not directly be considered to be a solution of an equation and some mathematical work has to be done to see in which weakened sense, if in any, one can consider such functions as solutions. We can not solve the problem of divergences in the Bethe-Salpeter equation here but just say that the "right" multi-time integral equation that might describe relativistic interaction of quantum particles would be one without such problems.

2. **Negative norm states:** It is reported in [65] and [66] that "the norm" of every second eigenfunction of the Wick-Cutkosky model is negative. The reader might wonder now: Which norm? Since there is no known conserved current for the equation which could provide a norm by its zero-component, it is not clear how to obtain a norm whatsoever. The usual normalization condition for the wave function in the Bethe-Salpeter equation is obtained either by a formal calculation using poles of Green's functions [65] or by comparison with amplitudes in QED [72]. But these normalization conditions yield no positive definite density in the Wick Cutkosky model. This, at first sight, would not be too problematic because the model correspond to spin-less particles that in the free case satisfy the Klein-Gordon equation. It is well-known that the Klein-Gordon equation has the conserved current given

in (7.8) whose zero-component can become negative. However, the problems of negative norm states are expected to remain in the usual Bethe-Salpeter equation (as can be seen by calculations in simplified models with spin) [65], so the problem lies deeper [18] and cannot be overcome as easily as changing from the Klein-Gordon to the Dirac equation.

3. **Abnormal solutions:** The third issue with the Wick-Cutkosky model is the occurrence of solutions that "have no counterparts in the non-relativistic potential theory" [65, p. 48]. This means, when we take the non-relativistic limit of the Wick-Cutkosky equation, some of the eigenfunctions do not correspond to any familiar physical states. Therefore, they are called abnormal solutions, whose physical status is unclear. It is believed that this problem can neither be cured by considering higher orders in the approximation nor by including spin. But it is reported by Nakanishi [65] that "abnormal solutions seem to come out mainly from the large values of the relative time". Now we recall that in sec. 12 it was explained why the Bethe-Salpeter should most likely be limited to the space-like configurations $\mathscr{S}$. Doing this would exclude very large values of the relative time compared to the spatial distance, so it seems possible that the problem of abnormal solutions arises only because one considers an inappropriate domain for the multi-time wave function. Unfortunately, we cannot give a definite account of this since the lost symmetry when confining the equations to $\mathscr{S}$ makes an explicit solution more difficult than in the original Wick-Cutkosky model.

Although we cannot obtain a final answer to the question how the problematic solutions of the spin-less model arise, we see that the widened perspective, obtained by putting the Bethe-Salpeter equation into the multi-time framework and by really asking what the objects it contains mean physically, helps us to better understand how some of the issues come about and might be resolved. Further research should start from this point.

14 Non-relativistic limit of the Bethe-Salpeter equation

Since the Bethe-Salpeter equation is too complicated for direct application and most situations in which bound states occur can be approximated non-relativistically, the problems we discussed do rarely appear in practical calculations. One mostly uses non-relativistic limiting cases of the equation such as the *Breit equation* [73]. We will now derive this equation in position space and thereby show an example of how a multi-time integral equation which cannot directly be reduced to differential equations with interaction potentials can nevertheless yield such equations in a non-relativistic limit. The position space calculation is also shorter than the usual momentum space derivation given e.g. in [74, ex. 6.4].

[18] I thank Matthias Lienert for pointing this out to me.

This derivation is based on an idea by Matthias Lienert. It needs the following assumptions, which we will motivate and remark on below.

1. *Instantaneous limit:* The wave function behaves such that one may replace the photon propagator by
$$D^{\mu\nu}(t,\mathbf{x}) \to \tilde{D}^{\mu\nu}(\mathbf{x}) \cdot \delta(t) \tag{14.1}$$

2. *Replacing the propagator:* One may replace the Feynman propagators S_F by retarded propagators S_R.

With these assumptions, we want to obtain a single-time equation from the multi-time equation (11.9) in ladder approximation. We first note that

$$\frac{\partial}{\partial t}\psi(t,\mathbf{x}_1,t,\mathbf{x}_2) = \left(\frac{\partial}{\partial t_1}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2) + \frac{\partial}{\partial t_2}\psi(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2)\right)\Big|_{t_1=t_2=t}. \tag{14.2}$$

Now we consider eq. (11.9) at equal times and apply the sum of the two free Dirac operators (times the respective γ^0) from the left, which makes φ vanish because it is a solution of the free Dirac equation:

$$\begin{aligned}\left(i\gamma_1^0\not{\partial}_1 - \gamma_1^0 m_1 + i\gamma_2^0\not{\partial}_2 - \gamma_2^0 m_2\right)\psi_{ab}(t,\vec{x}_1,t,\vec{x}_2) &= ie_1e_2\cdot\left(\gamma_1^0 I_1 + \gamma_2^0 I_2\right),\\ I_1 &:= \int d^4x_2'\ S_F^b(x_2,x_2')\gamma_{1,\mu}^a D^{\mu\nu}(x_1-x_2')\gamma_{2,\nu}^b\psi_{ab}(x_1,x_2'),\\ I_2 &:= \int d^4x_1'\ S_F^a(x_1,x_1')\gamma_{1,\mu}^a D^{\mu\nu}(x_1'-x_2)\gamma_{2,\nu}^b\psi_{ab}(x_1',x_2).\end{aligned} \tag{14.3}$$

Here, we directly used eq. (11.5) on the right hand side to reduce the number of integrations. Now we use assumption 1, the instantaneous approximation, and obtain for the first integral

$$I_1 = \int d^3\mathbf{x}_2'\ S_F^b(t,\mathbf{x}_2,t,\mathbf{x}_2')\gamma_{1,\mu}^a\gamma_{2,\nu}^b\tilde{D}^{\mu\nu}(\mathbf{x}_1-\mathbf{x}_2')\psi_{ab}(t,\mathbf{x}_1,t,\mathbf{x}_2'). \tag{14.4}$$

In order to proceed from eq. (14.4), we have to evaluate the Dirac propagator S_F at equal times. The standard formulas for propagators found in textbooks are only valid for different time arguments, and at equal times, the propagator behaves discontinuously. By integrating eq. (11.5) over a small time interval, one can derive that [75, p. 267]

$$\lim_{\varepsilon\searrow 0}\left(S_F(\varepsilon,\mathbf{x}) - S_F(-\varepsilon,\mathbf{x})\right) = i\gamma^0\delta^{(3)}(\mathbf{x}), \tag{14.5}$$

which stays true, interestingly, if one uses another propagator instead of the Feynman propagator, for example the retarded one S_R. But in contrast to the Feynman propagator, one can assign a sensible value to $S_R(0,\mathbf{x})$ because S_R becomes zero for negative time arguments. Therefore, with assumption 2, we may set

$$S_F(t,\mathbf{x}_2,t,\mathbf{x}_2') \to i\gamma^0\delta^{(3)}(\mathbf{x}_2-\mathbf{x}_2'). \tag{14.6}$$

Inserting this in eq. (14.4), one obtains

$$I_1 = i\gamma_2^0\gamma_{1,\mu}^a\gamma_{2,\nu}^b\tilde{D}^{\mu\nu}(\mathbf{x}_1-\mathbf{x}_2)\psi_{ab}(t,\mathbf{x}_1,t,\mathbf{x}_2). \tag{14.7}$$

We now have to insert the photon propagator. In Feynman gauge and in momentum space, it is given by [75]

$$D^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2+i\varepsilon}. \tag{14.8}$$

Because of assumption 1, we only have to take the spatial Fourier transform and set k^0 to 0, which automatically gives a $\delta(t)$. The Fourier transform gives us [46]

$$\tilde{D}^{\mu\nu}(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\frac{-ig^{\mu\nu}}{|\mathbf{k}|^2} = \frac{-C}{2|\mathbf{x}|}g^{\mu\nu}, \tag{14.9}$$

with some constant prefactor C, which leads to

$$I_1 = -iC\gamma_2^0\gamma_{1,\mu}^a\gamma_2^{\mu,b}\frac{1}{2|\mathbf{x}_1-\mathbf{x}_2|}\psi_{ab}(t,\mathbf{x}_1,t,\mathbf{x}_2). \tag{14.10}$$

Doing the same steps for I_2 and adding up, we obtain

$$\left(i\gamma_1^0\partial\!\!\!/_1 - \gamma_1^0 m_1 + i\gamma_2^0\partial\!\!\!/_2 - \gamma_2^0 m_2\right)\psi_{ab}(t,\mathbf{x}_1,t,\mathbf{x}_2) = Ce_1e_2\gamma_1^0\gamma_2^0\frac{\gamma_{1,\mu}^a\gamma_2^{\mu,b}}{|\mathbf{x}_1-\mathbf{x}_2|}\psi_{ab}(t,\mathbf{x}_1,t,\mathbf{x}_2). \tag{14.11}$$

Using that $(\gamma^0)^2 = \mathbb{1}$ and $\gamma^0\gamma^j =: \alpha^j$ as well as (14.2), we can rewrite this as

$$i\partial_t\psi(t,\mathbf{x}_1,t,\mathbf{x}_2) = \left(H_1(\mathbf{x}_1)+H_2(\mathbf{x}_2)+V(\mathbf{x}_1-\mathbf{x}_2)\right)\psi(t,\mathbf{x}_1,t,\mathbf{x}_2), \tag{14.12}$$

where H_j denotes the free Dirac Hamiltonian for particle j and the potential term is given by

$$V(\mathbf{x}) = g\left(\frac{1}{|\mathbf{x}|}+\frac{\alpha_1^j\alpha_{j,2}}{|\mathbf{x}|}\right), \tag{14.13}$$

with the coupling constant $g = Ce_1e_2$. Thus, we have derived a single-time Dirac equation for two particles with an effective potential that features a Coulomb term and a relativistic correction term that is well-known from the Breit equation. We now briefly comment on our result and assumptions.

Remark:

- The full potential in the Breit equation is given by [73, 46]

$$V(\mathbf{x}) = g\left(\frac{1}{|\mathbf{x}|}+\frac{\alpha_1^j\alpha_{j,2}}{|\mathbf{x}|}+\frac{\alpha_1^i x_i\alpha_2^j x_j}{|\mathbf{x}|^2}\right). \tag{14.14}$$

  So why did we not obtain the third term in our derivation? This comes from the gauge dependence of the Bethe-Salpeter equation. We have used the Feynman

gauge for the photon propagator. In Greiner's book, there is a hand-waving argument why one should rather use the Coulomb gauge, in which the propagator (14.9) becomes

$$\tilde{D}^{ij}(\mathbf{x}) = \frac{-C}{2|\mathbf{x}|}\left(\frac{x_i x_j}{|\mathbf{x}|^2} - \delta_{ij}\right) \tag{14.15}$$

and the other components do not change. But with this propagator, one indeed obtains the potential (14.14), in agreement with the statement in [69] that the lacking gauge invariance of the Bethe-Salpeter equation is less severe in the Coulomb gauge. We expect that the missing term may be restored also in the Feynman gauge by considering higher order irreducible Feynman diagrams, but this calculation is very difficult and needs renormalization procedures.

- Assumption 1 is motivated by the connection between the instantaneous and the non-relativistic case. To give a simple example, a propagator might contain a δ-function of the type $\delta(|\mathbf{x}| - t)$. Since we set $c = 1$, a non-relativistic limit would mean $|\mathbf{x}| \ll |t|$, hence replacing

$$\delta(|\mathbf{x}| - t) \to \delta(t) \tag{14.16}$$

might be justified (see also [37]). The seemingly ad hoc procedure of leaving away the spatial parts in the delta functions should be motivated more rigorously by an assumption about the *wave function* and its behavior. We expect that in a non-relativistic limit, the wave function should behave such that retardation effects become negligible (so it should change slowly with time) and the above approximation is valid.

- In Greiner's derivation in momentum space, the assumption of instantaneous interaction is implemented by saying that the propagator should not be frequency dependent, i.e. $D^{\mu\nu}(k^0, \mathbf{k}) \to D^{\mu\nu}(0, \mathbf{k})$. This is in fact equivalent to our approximation of the position space-time propagator because the Fourier transform gives

$$\begin{aligned} D^{\mu\nu}(t, \mathbf{x}) =& \frac{1}{(2\pi)^4} \int dk^0 \int d\mathbf{k}\; e^{-ik^0 t} \cdot e^{i\mathbf{kx}} D^{\mu\nu}(k^0, \mathbf{k}) \\ =& \; \delta(t) \cdot \int d\mathbf{k} \cdot \frac{1}{(2\pi)^3} e^{i\mathbf{kx}} D^{\mu\nu}(0, \mathbf{k}). \end{aligned} \tag{14.17}$$

- Assumption 2 may be motivated in two different ways. Greiner considers only positive energy solutions for the derivation of the Breit equation. Although it is not explicitly stated, the significance of this assumption is that it allows him to replace the Feynman propagator S_F by the retarded propagator S_R. This works because it is known that they agree on positive energy solutions, compare [75]. From our more general perspective as explained in sec. 12, we could impose right from the start that the Bethe-Salpeter equation contains retarded propagators, which would make assumption 2 unnecessary.

15 Conclusion

In the end, what have we learned? The most obvious lesson is: It is a very difficult task to construct a relativistically interacting quantum theory.
In the first part of this thesis, we clarified the relevant concepts and recognized that an interacting theory must at least possess some dynamics. Therefore, usual quantum field theories cannot be considered as interacting theories, as explained in section 3. They only provide a good effective description and we need to look for alternatives in order to construct a fundamental theory. The most promising object for building relativistic quantum mechanics of N particles is the multi-time wave function $\psi(x_1, ..., x_N)$, the most natural generalization of the non-relativistic wave function from quantum mechanics. There are several possibilities how to formulate a conceptually clear quantum theory with the help of such an object, for example with a Hypersurface-Bohm-Dirac model or a relativistic collapse model. There are numerous open problems on this fundamental level as well, but we expect that the introduction of interaction has to be done on the level of the wave function. So how should the dynamical equations for $\psi(x_1, ..., x_N)$ look like? In the non-interacting case, a system of free Dirac equations

$$(i\partial\!\!\!/_k - m_k)\psi(x_1, ..., x_N) = 0, \quad k = 1, ..., N \tag{15.1}$$

is appropriate. But one must not add potential terms $V(x_j - x_k)$ to these equations because they violate the consistency condition that is necessary for solutions to exist.
The second insight: Relativistically interacting quantum theories *can* be constructed; there are several methods to do so. In section 5.2, four promising possibilities were discussed. One can think of using generalized potentials that depend on the momentum, or one can try to get around the divergences that normally occur in models with pair creation. Two important mechanisms of interaction were then discussed extensively.
The model presented in part II implements interaction by boundary conditions. Since the multi-time wave function $\psi(x_1, ..., x_N)$ is naturally defined only on the set of space-like configurations $\mathscr{S}$, one can impose boundary conditions on $\partial\mathscr{S}$. This was done in such a way that our model of N mass-less particles in one dimension is interacting. The interaction is such that in an effective single-time picture, we obtain a δ-function potential of the particle distances. Alas, a generalization of the model to higher dimensions is not feasible, one only obtains the free model there. Nevertheless, there is a lot to be learned from the model. One can see how probability conservation is connected with a conserved tensor current $j^{\mu_1 \cdots \mu_N}$ in a geometric way and how the boundary conditions are able to ensure that no probability "flows out" of the domain. Furthermore, we recognized that functional analytic methods are of limited utility in multi-time theories and general methods from the theory of partial differential equations such as the (generalized) method of characteristics used in sec. 6.4 may help more.
A different type of equations that might be the right choice for relativistic quantum dynamics are integral or integro-differential equations. In part III, we discussed the Bethe-Salpeter equation as a prominent example of multi-time integral equations. Although very promising at first glance because it is an equation that is thought to describe

interacting systems and bound states in QED, the Bethe-Salpeter equation has multiple problems that are yet unsolved, or even unsolvable. Some of the properties of the Bethe-Salpeter equation can be understood better from the multi-time perspective, but many remain unclear. We discussed that the meaning of the wave function it contains is unclear due to the lack of a conserved current with positive density. Furthermore, solutions of the equation have divergences, it is unknown how to define a (positive) norm and gauge invariance cannot be fully restored. It is typical for the current state of affairs in relativistic quantum physics that despite all those issues, physicists repeatedly succeed in extracting meaningful predictions like energy levels from the Bethe-Salpeter equation.

But a stable basis for all the effective theories is still missing, which is the reason for our search. We only treated one example of multi-time integral equations, so it might be rewarding to consider alternatives to the Bethe-Salpeter equation which have less problems and investigate whether they can be the foundation of a new relativistic quantum theory. It may well be that this forces us to abandon familiar methods of dealing with physics equations – similar to the situation in the divergence-free theory of classical electromagnetism by Wheeler and Feynman [76] where there is no simple initial value problem.

After all, this thesis shows several possible ways one can pursue to finally come up with a theory of relativistic quantum mechanics that is a consistent, understandable theory and has all the formalisms one knows so far as effective descriptions. Considering the amount of possibilities that were not discussed thoroughly in the research so far, we are led to believe that there is still lots of space for such a theory to be developed, although, as Maudlin writes in [24, p. 222], the way will not be easy:

> One way or another, God has played us a nasty trick. The voice of Nature has always been faint, but in this case it speaks in riddles and mumbles as well. Quantum theory and Relativity seem not to directly contradict one another, but neither can they be easily reconciled. – Tim Maudlin

References

[1] Matthias Lienert and Lukas Nickel. A simple explicitly solvable interacting relativistic N–particle model. *J. Phys. A.*, 48(32):5301, 2015. Preprint: arXiv:1502.00917.

[2] Matthias Lienert. A relativistically interacting exactly solvable multi-time model for two mass-less Dirac particles in 1 + 1 dimensions. *J. Math. Phys.*, 56(04):2301, 2015.

[3] Albert Einstein. Zur Elektrodynamik bewegter Körper. *Ann. Phys.*, 322(10):891–921, 1905.

[4] Albert Einstein. *Grundzüge der Relativitätstheorie.* Springer-Verlag, Berlin, Heidelberg, 6th edition, 2002.

[5] Tim Maudlin. Space-Time in the Quantum World. In James T. Cushing, Arthur Fine, and Sheldon Goldstein, editors, *Bohmian Mechanics and Quantum Theory: An Appraisal*, pages 285–307. Kluwer Academic Press, 1996.

[6] Detlef Dürr, Sheldon Goldstein, Karin Münch-Berndl, and Nino Zanghì. Hypersurface Bohm-Dirac models. *Phys. Rev. A*, 60:2729–2736, 1999.

[7] Detlef Dürr, Sheldon Goldstein, Travis Norsen, Ward Struyve, and Nino Zanghì. Can Bohmian mechanics be made relativistic? *Proc. R. Soc. A*, 470(2162):20130699, 2014.

[8] Max Born. *Die Relativitätstheorie Einsteins.* Springer-Verlag, Berlin, Heidelberg, 6th edition, 2001.

[9] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation.* W.H.Freeman and Company, 21st edition, 1998.

[10] Lev D. Landau and Evgeny M. Lifschitz. *Lehrbuch der Theoretischen Physik, Band III. Quantenmechanik.* Akademie-Verlag, Berlin, 8th edition, 1988.

[11] John Stewart Bell. Against measurement. *Phys. World*, 8:33–40, 1990.

[12] Detlef Dürr, Sheldon Goldstein, and Nino Zanghì. Quantum Equilibrium and the Role of Operators as Observables in Quantum Theory. In *Quantum Physics without Quantum Philosophy*, pages 79–161. Springer-Verlag, 2013.

[13] Karin Münch-Berndl, Detlef Dürr, Sheldon Goldstein, G. Peruzzi, and Nino Zanghì. On the Global Existence of Bohmian Mechanics. *Commun. Math. Phys.*, 173:647–673, 1995.

[14] Tim Maudlin. What Bell Did. *J. Phys. A*, 47(42):4010, 2014.

[15] John Stewart Bell. Bertlmann's Socks and the Nature of Reality. In *Speakable and Unspeakable in Quantum Mechanics*. Cambridge University Press, Cambridge, 1987.

[16] Sin-Itiro Tomonaga. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. *Prog. Theor. Phys.*, 1(2):27–42, 1946.

[17] Julian Schwinger. Quantum electrodynamics. I. A covariant formulation. *Phys. Rev.*, 74(10):1439–1461, 1948.

[18] Mario Bacelar Valente. *Conceptual problems in quantum electrodynamics: a contemporary historical-philosophical approach.* Phd thesis, Universidad de Sevilla, 2011.

[19] J. C. Le Guillo and Jean Zinn-Justin. *Large-Order Behaviour of Perturbation Theory.* Elsevier Science Publishers, 1990.

[20] Lev D. Landau and Evgeny M. Lifschitz. *Lehrbuch der Theoretischen Physik, Band IV. Quantenelektrodynamik.* Akademie-Verlag, Berlin, 7th edition, 1991.

[21] G. V. Dunne. New strong-field QED effects at extreme light infrastructure: Non-perturbative vacuum pair production. *Eur. Phys. J. D*, 55(2):327–340, 2009.

[22] Edwin T. Jaynes. Probability In Quantum Theory. In W. H. Zurek (ed.), editor, *Complexity, Entropy, and the Physics of Information*, page 381. Addison-Wesley, Redwood City, 1990.

[23] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to Quantum Field Theory.* Westview Press, 1995.

[24] Tim Maudlin. *Quantum Non-Locality and Relativity.* Wiley-Blackwell, 3rd edition, 2011.

[25] Bernd Thaller. *The Dirac Equation.* Springer-Verlag, Berlin, Heidelberg, 1992.

[26] Sören Petrat and Roderich Tumulka. Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials. *J. Math. Phys.*, 55:032302, 2014.

[27] David B. Malament. In defense of dogma: Why there cannot be a relativistic quantum mechanics of (localizable) particles. In R. Clifton, editor, *Perspectives on Quantum Reality*, pages 1–10. 1996.

[28] Christian Beck. *Localization.* Phd thesis, Ludwig-Maximilians-Universität. In preparation.

[29] Tobias Jung and Lukas Nickel. Messung und Unschärfe in der klassischen Physik. *Philosophia Naturalis*, 50(2), 2013.

[30] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan. Relativistic Invariance and Hamiltonian Theories of Interacting Particles. *Rev. Mod. Phys.*, 35(2):350–375, 1963.

[31] Federico Laudisa. Against the 'no-go' philosophy of quantum mechanics. *Eur. J. Philos. Sci.*, 4:1–17, 2014.

[32] Paul. A. M. Dirac. Quantised Singularities in the Electromagnetic Field. *Proc. R. Soc. A*, 133(821):60–72, 1931.

[33] Paul A. M. Dirac. Relativistic Quantum Mechanics. *Proc. R. Soc. A*, 136(829):453–464, 1932.

[34] Paul. A. M. Dirac, Vladimir A. Fock, and Boris Podolsky. On Quantum Electrodynamics. In Julian Schwinger, editor, *Selected Papers on Quantum Electrodynamics*. 1958.

[35] Sören Petrat and Roderich Tumulka. Multi-Time Wave Functions for Quantum Field Theory. *Ann. Phys.*, 345:17–54, 2013.

[36] Felix Bloch. Die physikalische Bedeutung mehrerer Zeiten in der Quantenelektrodynamik. *Phys. Zeitsch. d. Sowjetunion*, 5:301–315, 1934.

[37] Matthias Lienert. *Lorentz invariant quantum dynamics in the multi-time formalism*. Phd thesis, Ludwigs-Maximilians-Universität München, 2015.

[38] Dirk-A. Deckert and Franz Merkl. Dirac equation with external potential and initial data on Cauchy surfaces. *J. Math. Phys.*, 55(12):122305, 2014.

[39] Roderich Tumulka. A Relativistic Version of the Ghirardi-Rimini-Weber Model. *Phys. Rev. D*, 87:1–19, 2006.

[40] Daniel Bedingham, Detlef Dürr, Giancarlo C. Ghirardi, Sheldon Goldstein, Roderich Tumulka, and Nino Zanghì. Matter Density and Relativistic Models of Wave Function Collapse. *J. Stat. Phys.*, 154:623–631, 2014.

[41] William Feldmann and Roderich Tumulka. Parameter diagrams of the GRW and CSL theories of wavefunction collapse. *J. Phys. A Math. Theor.*, 45(6):065304, 2012.

[42] Angelo Bassi and Giancarlo Ghirardi. Dynamical Reduction Models. *Phys. Rep.*, 379:257, 2003.

[43] Michael Reed and Barry Simon. *Methods of modern mathematical physics I: Functional Analysis*. Academic Press, 2005.

[44] Matthias Lienert. On the question of current conservation for the two-body Dirac equations of constraint theory. *Preprint: arXiv:1501.07027*, 2015.

[45] Stefan Teufel and Roderich Tumulka. New Type of Hamiltonians Without Ultraviolet Divergence for Quantum Field Theories. *Preprint: arXiv:1505.04847v1*, 2015.

[46] Walter Greiner and Joachim Reinhardt. *Quantum Electrodynamics.* Springer-Verlag, Berlin, Heidelberg, 4th edition, 1992.

[47] Sergio Albeverio, Fritz Gesztesy, Raphael Hoegh-Krohn, and Helge Holden. *Solvable models in quantum mechanics.* Springer-Verlag, New York, 1988.

[48] Lev D. Landau and Evgeny M. Lifschitz. *Lehrbuch der Theoretischen Physik, Band IVa. Relativistische Quantentheorie.* Akademie-Verlag, Berlin, 1970.

[49] Lawrence C. Evans. *Partial Differential Equations.* American Mathematical Society, 2nd edition, 2010.

[50] Konrad Königsberger. *Analysis 2.* Springer-Verlag, 5th edition, 2004.

[51] Detlef Dürr and Matthias Lienert. On the description of subsystems in relativistic hypersurface Bohmian mechanics. *Proc. R. Soc. A*, 470:2169, 2014.

[52] Bruce H. J. McKellar and G. J. Stephenson Jr. Klein Paradox and the Dirac-Kronig-Penney model. *Phys. Rev. A*, 36(6):2566–2569, 1987.

[53] M. G. Calkin. Proper treatment of the delta function potential in the one-dimensional Dirac equation. *Am. J. Phys.*, 55(1987):737–739, 1987.

[54] Bill Sutherland and Daniel C. Mattis. Amibiguities with the relativistic delta-function potential. *Phys. Rev. A*, 24(3):1194–1197, 1981.

[55] D. M. Gitman, I. V. Tyutin, and B. L. Voronov. *Self-adjoint Extensions in Quantum Mechanics.* Springer Science+Business Media, New York, 2012.

[56] Michael Reed and Barry Simon. *Methods of modern mathematical physics II: Fourier Analysis, Self-adjointness.* Academic Press, 1975.

[57] Martin Oelker. *On the Self-Adjointness of the Dirac Operator for Two Interacting Particles.* Master thesis, Ludig-Maximilians-Universität München, 2013.

[58] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer Science+Business Media, 2011.

[59] Thomas G. Kurtz. Extensions of Trotter's Operator Semigroup Approximation Theorems. *J. Funct. Anal.*, 3:354–375, 1969.

[60] Lara Beraha. *Über die Selbstadjungiertheit von Hamiltonoperatoren.* Diploma thesis, Ludwig-Maximilians-Universität München, 1994.

[61] E. C. Svendsen. The effect of submanifolds upon essential self-adjointness and deficiency indices. *J. Math. Anal. Appl.*, 80:551–565, 1981.

[62] Edwin. E. Salpeter and Hans. A. Bethe. A relativistic equation for bound-state problems. *Phys. Rev.*, 84(6):1232–1242, 1951.

[63] G. C. Wick. Properties of Bethe-Salpeter wave functions. *Phys. Rev.*, 96(4):1124–1134, 1954.

[64] R. E. Cutkosky. Solutions of a Bethe-Salpeter Equation. *Phys. Rev.*, 96(4), 1954.

[65] Noboru Nakanishi. A General Survey of the Theory of the Bethe-Salpeter Equation. *Prog. Theor. Phys. Suppl.*, 43, 1969.

[66] Noboru Nakanishi. Review of the Wick-Cutkosky Model. *Prog. Theor. Phys. Suppl.*, 95:1–23, 1988.

[67] Jack S. Goldstein. Properties of the Salpeter-Bethe Two-Nucleon Equation. *Phys. Rev.*, 91(6):1516–1524, 1953.

[68] Charles Schwartz and Charles Zemach. Theory and Calculation of Scattering with the Bethe-Salpeter-Equation. *Phys. Rev.*, 141(4), 1966.

[69] R. Barbieri, M. Ciafaloni, and P. Menotti. Retardation effects and gauge transformations in the Bethe-Salpeter equations. *Nuovo Cim. A*, 55(4):701–723, 1968.

[70] M. Böhm. Bethe-Salpeter Equation and Current Conservation. *Nucl. Phys. B*, 91:494–508, 1975.

[71] Changsheng Li, Langhui Wan, Yadong Wei, and Jian Wang. Definition of current density in the presence of a non-local potential. *Nanotechnology*, 19(15):5401, 2008.

[72] Stanley Mandelstam. Dynamical Variables in the Bethe-Salpeter Formalism. *Proc. R. Soc. A*, 233(1193):248–266, 1955.

[73] Gregory Breit. Dirac's equation and the spin-spin interactions of two electrons. *Phys. Rev.*, 39(4):616–624, 1932.

[74] Walter Greiner. *Theoretische Physik Band 6: Relativistische Quantenmechanik*. Verlag Harri Deutsch, Thun und Frankfurt am Main, 1981.

[75] Iwo Bialynicki-Birula and Zofia Bialynicka-Birula. *Quantum Electrodynamics*. Pergamon Press and PWN, Warszawa, 1975.

[76] John Archibald Wheeler and Richard Phillips Feynman. Classical Electrodynamics in Terms of Direct Interparticle Action. *Rev. Mod. Phys.*, 21(3), 1949.

Danksagung

Zuvorderst gilt mein Dank meinem Betreuer Detlef Dürr, nicht nur für die Unterstützung während der Masterarbeit, sondern auch in der Zeit davor: für sein Engagement für meine Mitstudierenden und mich; für das Aufzeigen, was wichtig ist. Ich bin dankbar, dass mich ein solch persönlich und fachlich einzigartiger Mensch mit seinen Erkenntnissen bereichert hat.
Weiter möchte ich mich bei Matthias bedanken, für die intensive und erfolgreiche Zusammenarbeit und den guten Tee. Von seiner Erfahrung habe ich ebenso profitiert wie von seiner Gründlichkeit, und ich danke für die viele Zeit, die er sich für mich genommen hat.
Für hilfreiche inhaltliche Diskussionen möchte ich mich bei Sören Petrat, Martin Oelker und Christian Beck bedanken.
Danke an das Max-Weber-Programm für eine finanzielle und ideelle Unterstützung während des gesamten Studiums.
Ein besonderes Dankeschön geht an Simone, Basti und Max für das Korrekturlesen dieser Arbeit. Ich danke aber allen meinen Freunden dafür, dass sie (für mich) da sind. Ich danke auch, aber nicht ausschließlich, allen folgenden Personen (und wenn sie nicht wissen warum, sollen sie mich fragen): Christina, Marie Sophie, Robin, Markus, Felix, Lea, Roman, Andreas, Simone, Basti, Patrick, Philipp, Florian, Simon, Vera, Alessa, Tommy, Tobias sowie Dirk und Peter.
Zuletzt die wichtigsten Menschen außerhalb des Wissenschaftskontextes: Meinen Eltern danke ich für die durchgehende Unterstützung von allem, was ich tue. Und Marcel danke ich, einfach für alles.

Selbstständigkeitserklärung

Hiermit versichere ich, Lukas Nickel, die vorliegende Arbeit selbstständig und nur unter Zuhilfenahme der genannten Quellen verfasst zu haben. Die Inhalte in Teil II abgesehen von Abschnitt 9.3 sind, wie im Text gekennzeichnet, in Zusammenarbeit mit Matthias Lienert entstanden.

München, den

Lukas Nickel