On Courant's nodal domain property for linear combinations of eigenfunctions (after P. Bérard and B. Helffer).

May 2018

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Abstract

We revisit Courant's nodal domain property for linear combinations of eigenfunctions (ECP), and propose new, simple and explicit counterexamples to the so-called "Herrmann's statement" for domains in $\mathbb{R}^d$, $\mathbb{S}^2$ or $\mathbb{T}^2$.

Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain or, more generally, a compact Riemannian manifold with boundary.

Consider the eigenvalue problem

$$\begin{cases} -\Delta u &= \lambda\, u \quad \text{in } \Omega\,, \\ B(u) &= 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

where $B(u)$ is some boundary condition on $\partial\Omega$, so that we have a self-adjoint boundary value problem (including the empty condition if Ω is a closed manifold).

For example, $D(u) = u\big|_{\partial\Omega}$ for the Dirichlet boundary condition, or $N(u) = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ for the Neumann boundary condition.

Call $H(\Omega, B)$ the associated self-adjoint extension of $-\Delta$, and list its eigenvalues in nondecreasing order, counting multiplicities,

$$0 \leq \lambda_1(\Omega, B) < \lambda_2(\Omega, B) \leq \lambda_3(\Omega, B) \leq \cdots \tag{2}$$

For any integer $n \geq 1$, define the index

$$\tau(\Omega, B, \lambda_n) = \min\{k \mid \lambda_k(\Omega, B) = \lambda_n(\Omega, B)\}. \tag{3}$$

$\mathcal{E}(\lambda_n)$ will denote the eigenspace associated with λ_n.

The Courant nodal theorem

For a real continuous function v on Ω, we define its *nodal set*

$$\mathfrak{Z}(v) = \overline{\{x \in \Omega \mid v(x) = 0\}}\,, \tag{4}$$

and call $\beta_0(v)$ the number of connected components of $\Omega \setminus \mathfrak{Z}(v)$ i.e., the number of *nodal domains* of v.

Courant's nodal Theorem (1923)

For any nonzero $u \in \mathcal{E}(\lambda_n(\Omega, B))$,

$$\beta_0(u) \leq \tau(\Omega, B, \lambda_n) \leq n\,. \tag{5}$$

Courant's nodal domain theorem can be found in Courant-Hilbert [10].

The extended Courant nodal property

Given $r > 0$, denote by $\mathfrak{L}(\Omega, B, r)$ the space

$$\mathfrak{L}(\Omega, B, r) = \left\{ \sum_{\lambda_j(\Omega,B)\leq r} c_j\, u_j \mid c_j \in \mathbb{R}, u_j \in E_{\lambda_j(\Omega,B)} \right\}. \tag{6}$$

Extended Courant Property:= (ECP)

We say that $v \in \mathfrak{L}\,(\Omega, B, \lambda_n(\Omega, B))$ satisfies (ECP) if

$$\beta_0(v) \leq \tau(\Omega, B, \lambda_n)\,. \tag{7}$$

A footnote in Courant-Hilbert [10] indicates that this property also holds for any linear combination of the n first eigenfunctions, and refers to the PhD thesis of Horst Herrmann (Göttingen, 1932) [16].

Historical remarks : Sturm (1836), Pleijel (1956).

1. (ECP) is true for Sturm-Liouville equations. This was first announced by C. Sturm in 1833, [31] and proved in a long Memoir in 1836 [32]. Other proofs were later on given by J. Liouville and Lord Rayleigh who both cite Sturm explicitly. This is proven in a stronger form:

Strong ECP Property in (1D)

If the sum of the eigenfunctions corresponds to eigenspaces associated to eigenvalues λ_j ($k \leq j \leq n$) then the sum has at least $k-1$ zeroes and at most $n-1$ zeroes.

2. Å. Pleijel mentions (ECP) in his well-known paper of 1956 [27] on the asymptotic behaviour of the number of nodal domains of a Dirichlet eigenfunction associated with the n-th eigenvalue in a plane domain. He starts his paper by recalling Sturm's theorem (lower bound and upper bound) and at the end of the paper, he writes:

"In order to treat, for instance the case of the free three-dimensional membrane $]0,\pi[^3$*, it would be necessary to use, in a special case, the theorem quoted in [CH1931].... However, as far as I have been able to find there is no proof of this assertion in the literature."*

Historical remarks: V. Arnold (1973-1979)

3. As pointed out by V. Arnold [1], when $\Omega = \mathbb{S}^d$, (ECP) is related to Hilbert's $16-$th problem. Arnold [2] mentions that he actually discussed the footnote with R. Courant, that (ECP) cannot be true, and that O. Viro produced in 1979 counter-examples for the 3-sphere $\mathbb{S}^3$, and any degree larger than or equal to 6, [33].

More precisely V. Arnold wrote:
Having read all this, I wrote a letter to Courant:
"Where can I find this proof now, 40 years after Courant announced the theorem?".
Courant answered that one can never trust one's students: to any question they answer either that the problem is too easy to waste time on, or that it is beyond their weak powers.

And V. Arnold continues:

The point is that for the sphere $\mathbb{S}^2$ *(with the standard Riemannian metric) the eigenfunctions (spherical functions) are polynomials. Therefore, their linear combinations are also polynomials, and the zeros of these polynomials are algebraic curves (whose degree is bounded by the number n of the eigenvalue). Therefore, from the generalized Courant theorem one can, in particular, derive estimates for topological invariants of the complements of projective real algebraic curves (in terms of the degrees of these curves).*

... And then it turned out that the results of the topology of algebraic curves that I had derived from the generalized Courant theorem contradict the results of quantum field theory. Nevertheless, I knew that both my results and the results of quantum field theory were true. Hence, the statement of the generalized Courant theorem is not true.

Historical remarks: Gladwell-Zhu (2003)

4. In [12], Gladwell and Zhu refer to (ECP) as the *Courant-Herrmann conjecture*.
They observe that this extension of Courant's theorem is not stated, let alone proved, in Herrmann's thesis or subsequent publications. They consider the case in which Ω is a rectangle in $\mathbb{R}^2$, stating that they were not able to find a counter-example to (ECP) in this case.
They also provide numerical evidence that there are counter-examples for more complicated (non convex) domains.

They suggest that may be the conjecture could be true in the convex case.

Our goal

The purpose in this talk is to provide simple counter-examples to the *Extended Courant Property* for domains in $\mathbb{R}^d$, $\mathbb{S}^2$ or $\mathbb{R}^3$, including convex domains. No algebraic topology will be involved.
In this talk I choose to present four types of examples:

- Domains with cracks.
- Cuboids
- Polygones
- C^∞ domains

The results have been obtained in collaboration with Pierre Bérard.

Rectangle with a crack

Let $\mathfrak{R}_0$ be the rectangle $]0, 4\pi[\times]0, 2\pi[$. For $0 < a \leq 1$, let $C_a :=]0, a] \times \{\pi\}$ and $\mathfrak{R}_a := \mathfrak{R}_0 \setminus C_a$ and consider the Neumann condition. The setting is described in Dauge-Helffer [11].

We call

$$\{ \; 0 = \nu_1(0) < \nu_2(0) < \nu_3(0) = \nu_4(0) \leq \cdots \tag{8}$$

the Neumann eigenvalues of $-\Delta$ in $\mathfrak{R}_0$.
They are given by the $\frac{m^2}{16} + \frac{n^2}{4}$ for pairs (m, n) of non-negative integers.
Corresponding eigenfunctions are products of cosines.
Similarly, the Neumann eigenvalues of $-\Delta$ in $\mathfrak{R}_a$ are denoted by

$$\{ \; 0 = \nu_1(a) < \nu_2(a) \leq \nu_3(a) \leq \cdots \quad . \tag{9}$$

The first three Neumann eigenvalues for the rectangle $\mathfrak{R}_0$ are as follows.

$\nu_1(0)$	0	$(0,0)$	$\psi_1(x,y) = 1$
$\nu_2(0)$	$\frac{1}{16}$	$(1,0)$	$\psi_2(x,y) = \cos(\frac{x}{4})$
$\nu_3(0)$		$(0,1)$	$\psi_3(x,y) = \cos(\frac{y}{2})$
$\nu_4(0)$	$\frac{1}{4}$	$(2,0)$	$\psi_4(x,y) = \cos(\frac{x}{2})$

(10)

Dauge-Helffer (1993) prove:

Theorem

For $i \geq 1$,

1. $[0,1] \ni a \mapsto \nu_i(a)$ is non-increasing.
2. $]0,1[\ni a \mapsto \nu_i(a)$, is continuous.
3. $\lim_{a\to 0+} \nu_i(a) = \nu_i(0)$.

It follows that for $0 < a$, small enough, we have

$$0 = \nu_1(a) = \nu_1(0) < \nu_2(a) \leq \nu_2(0) < \nu_3(a) \leq \nu_4(a) \leq \nu_3(0) = \nu_4(0)\,. \tag{11}$$

Observe that for $i = 1$ and 2, $\frac{\partial \psi_i}{\partial y}(x, y) = 0$. Hence for a small enough, ψ_1 and ψ_2 are the first two eigenfunctions for $\mathfrak{R}_a$ with the Neumann condition with associated eigenvalues

$$\nu_1(a) = 0$$

and

$$\nu_2(a) = \frac{1}{4} < \nu_3(a)\,.$$

We have

$$\alpha \psi_1(x, y) + \psi_2(x, y) = \alpha + \cos(\frac{x}{4}).$$

We can choose the coefficient $\alpha \in]-1, +1[$ in such a way that these linear combinations of the first two eigenfunctions have two or three nodal domains in $\mathfrak{R}_a$.

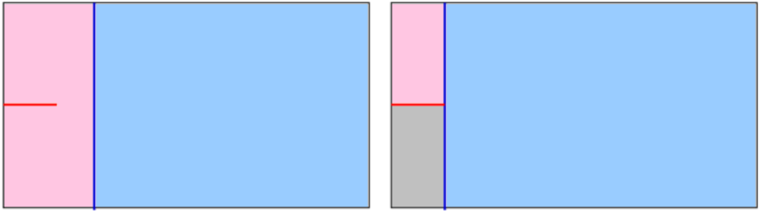


Figure: Rectangle with a crack (Neumann condition)

This proves that (ECP) is false in $\mathfrak{R}_a$ with Neumann condition.

Notice that we can introduce several cracks

$$\{(x, b_j) \mid 0 < x < a_j\}_{j=1}^{k}$$

so that for any $d \in \{2, 3, \ldots k+2\}$ there exists a linear combination of 1 and $\cos(\frac{x}{4})$ with d nodal domains.

Sphere $\mathbb{S}^2$ with cracks

On the round sphere $\mathbb{S}^2$, we consider the geodesic lines $(x,y,z) \mapsto (\sqrt{1-z^2}\cos\theta_i, \sqrt{1-z^2}\sin\theta_i, z)$ through the north pole $(0,0,1)$, with distinct $\theta_i \in [0,\pi[$.
Removing the geodesic segments $\theta_0 = 0$ and $\theta_2 = \frac{\pi}{2}$ with $1 - z \leq a \leq 1$, we obtain a sphere $\mathbb{S}^2_a$ with a crack in the form of a cross.
We consider the Neumann condition on the crack.
We then easily produce a function in the space generated by the two first eigenspaces of the sphere with a crack having five nodal domains.

The function z is also an eigenfunction of $\mathbb{S}^2_a$ with eigenvalue 2.
For a small enough, $\lambda_4(a) = 2$, with eigenfunction z.
For $0 < b < a$, the linear combination $z - b$ has five nodal domains in $\mathbb{S}^2_a$, see Figure below in spherical coordinates.

It follows that (ECP) does not hold on the sphere with cracks.

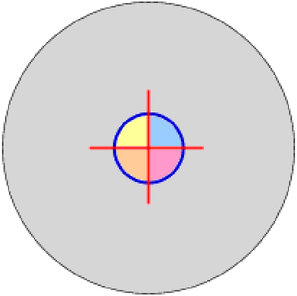


Figure: Sphere with crack, five nodal domains

Remark. Removing more geodesic segments around the north pole, we can obtain a linear combination $z - b$ with as many nodal domains as we want.

The hypercube with Dirichlet boundary condition

We can adapt the method of Gladwell-Zhu in any dimension.

Let $\mathcal{C}_n(\pi) :=]0,\pi[^n$ be the *hypercube* of dimension n, with either the Dirichlet or Neumann boundary condition on $\partial\mathcal{C}_n(\pi)$. A point in $\mathcal{C}_n(\pi)$ is denoted by $x = (x_1,\ldots,x_n)$.

A complete set of eigenfunctions of $-\Delta$ for $(\mathcal{C}_n(\pi), \mathfrak{d})$ is given by the functions

$$\prod_{j=1}^{n} \sin(k_j\, x_j) \ \text{ with eigenvalue } \sum_{j=1}^{n} k_j^2, \quad \text{for } k_j \in \mathbb{N}\backslash\{0\}\,, \tag{12}$$

for $x = (x_1,\ldots,x_n) \in]0,\pi[^n$.

A complete set of eigenfunctions of $-\Delta$ for $(\mathcal{C}_n(\pi), \mathfrak{n})$ is given by the functions

$$\prod_{j=1}^{n} \cos(k_j\, x_j) \ \text{ with eigenvalue } \sum_{j=1}^{n} k_j^2, \quad \text{for } k_j \in \mathbb{N}\,. \tag{13}$$

We make use of the classical Chebyshev polynomials $U_k(t), k \in \mathbb{N}$, defined by the relation,

$$\sin\left((k+1)t\right) = \sin(t)\, U_k\left(\cos(t)\right)\,,$$

and such that

$$U_0(t) = 1\,, \quad U_1(t) = 2t\,, \quad U_2(t) = 4t^2 - 1\,.$$

The first Dirichlet eigenvalues (as points in the spectrum) are listed in the following table, together with their multiplicities, and eigenfunctions.

Table: First Dirichlet eigenvalues of $\mathcal{C}_n(\pi)$

Eigenv.	Mult.	Eigenfunctions
n	1	$\phi_1(x) := \prod_{j=1}^n \sin(x_j)$
$n+3$	n	$\phi_1(x)\, U_1\left(\cos(x_i)\right)$ for $1 \leq i \leq n$
$n+6$	$\frac{n(n-1)}{2}$	$\phi_1(x)\, U_1\left(\cos(x_i)\right)\; U_1\left(\cos(x_j)\right)$ for $1 \leq i < j \leq n$
$n+8$	n	$\phi_1(x)\, U_2\left(\cos(x_i)\right)$ for $1 \leq i \leq n$

For the above eigenvalues, the index is given by,

$$\tau(n+3) = 2, \quad \tau(n+6) = n+2, \quad \tau(n+8) = \frac{n(n+1)}{2} + 2\,. \quad (14)$$

In order to study the nodal set of the above eigenfunctions or linear combinations thereof, we use the diffeomorphism

$$(x_1, \ldots, x_n) \mapsto (\xi_1 = \cos(x_1), \ldots, \xi_n = \cos(x_n)) \;, \qquad (15)$$

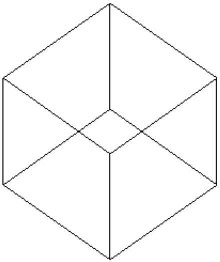
from $]0, \pi[$ onto $] -1, 1[$, and factor out the function ϕ_1 which does not vanish in the open hypercube. We consider the function

$$\Psi_a(\xi_1, \ldots, \xi_n) = \xi_1^2 + \cdots + \xi_n^2 - a$$

which corresponds to a linear combination Φ in $\mathcal{E}(n) \oplus \mathcal{E}(n+8)$. Given some a, $(n-1) < a < n$, this function has $2^n + 1$ nodal domains, see Figure 3 in dimension 3. For $n \geq 3$, we have $2^n + 1 > \tau(n+8)$. The function Φ therefore provides a counterexample to ECP for the hypercube of dimension at least 3, with Dirichlet boundary condition.

Proposition

For $n \geq 3$, the hypercube of dimension n, with Dirichlet boundary condition, provides a counterexample to ECP.



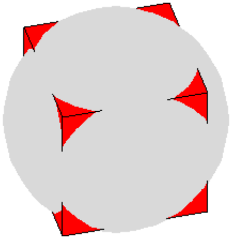


Figure: 3-dimensional cube

Remark 1. An interesting feature of this example is that we get counterexamples to ECP for linear combinations which involve eigenvalues with high energy while the other examples only involve eigenvalues with low energy.
Remark 2. Similar results can be obtained for the hypercube $n \geq 4$ with Neumann boundary condition.

The equilateral triangle (Dirichlet or Neumann)

Let $\mathcal{T}_e$ denote the equilateral triangle with sides 1, see Figure 4.
The eigenvalues and eigenfunctions of $\mathcal{T}_e$, with either Dirichlet or Neumann condition on the boundary $\partial\mathcal{T}_e$, can be completely described.
We show that the equilateral triangle provides a counterexample to the *Extended Courant Property* for both the Dirichlet and the Neumann boundary condition.

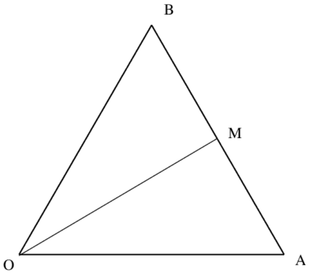


Figure: Equilateral triangle $\mathcal{T}_e = [OAB]$

Neumann boundary condition

The sequence of Neumann eigenvalues of the equilateral triangle $\mathcal{T}_e$ begins as follows,

$$0 = \lambda_1(\mathcal{T}_e, N) < \frac{16\pi^2}{9} = \lambda_2(\mathcal{T}_e, N) = \lambda_3(\mathcal{T}_e, N) < \lambda_4(\mathcal{T}_e, N)\,. \tag{16}$$

The second eigenspace has dimension 2, and contains one invariant eigenfunction φ_2^N under the mirror symmetry w.r.t OM, and another anti-invariant eigenfunction φ_3^N.
φ_2^N is given by

$$\varphi_2^N(x,y) = 2\cos\left(\frac{2\pi x}{3}\right)\left(\cos\left(\frac{2\pi x}{3}\right) + \cos\left(\frac{2\pi y}{\sqrt{3}}\right)\right) - 1\,. \tag{17}$$

The set $\{\varphi_2^N + 1 = 0\}$ consists of the two line segments $\{x = \frac{3}{4}\} \cap \mathcal{T}_e$ and $\{x + \sqrt{3}y = \frac{3}{2}\} \cap \mathcal{T}_e$, which meet at the point $(\frac{3}{4}, \frac{\sqrt{3}}{4})$ on $\partial\mathcal{T}_e$. The sets $\{\varphi_2 + a = 0\}$, with $a \in \{0, 1-\varepsilon, 1, 1+\varepsilon\}$, and small positive ε, are shown in Figure 6. When a varies from $1-\varepsilon$ to $1+\varepsilon$, the number of nodal domains of $\varphi_2 + a$ in $\mathcal{T}_e$ jumps from 2 to 3, with the jump occurring for $a = 1$.

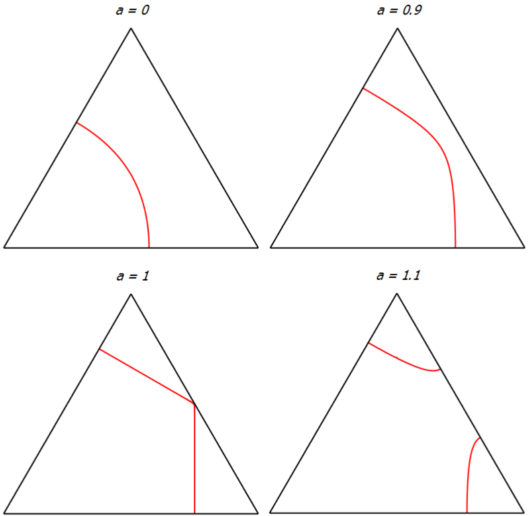


Figure: Levels sets $\{\varphi_2^N + a = 0\}$ for $a \in \{0\,;\,0.9\,;\,1\,;\,1.1\}$

If we take the coordinates $X = \cos \frac{2\pi}{3} x$ and $Y = \cos \frac{2\pi}{\sqrt{3}} y$ we are reduced to the level sets of $(X, Y) \mapsto X(X+Y)$:

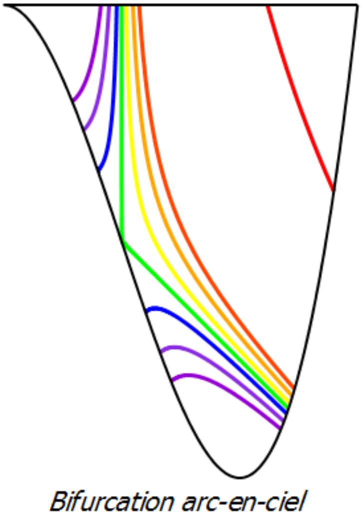


Figure: Levels sets in the X, Y variables

It follows that $\varphi_2^N + a = 0$, for $1 \leq a \leq 1.2$, provides a counterexample to the Extended Courant Property for the equilateral triangle with Neumann boundary condition.

Dirichlet boundary condition

The sequence of Dirichlet eigenvalues of the equilateral triangle $\mathcal{T}_e$ begins as follows,

$$\delta_1(\mathcal{T}_e) = \frac{16\pi^2}{3} < \delta_2(\mathcal{T}_e) = \delta_3(\mathcal{T}_e) = \frac{112\pi^2}{9} < \delta_4(\mathcal{T}_e). \tag{18}$$

Up to scaling, the first eigenfunction φ_1^D is given by

$$\varphi_1^D(x,y) = -8\,\sin\frac{2\pi y}{\sqrt{3}}\,\sin\frac{\pi(\sqrt{3}x+y)}{\sqrt{3}}\,\sin\frac{\pi(\sqrt{3}x-y)}{\sqrt{3}}\,, \tag{19}$$

which shows that it does not vanish inside $\mathcal{T}_e$.

A surprising (new ?) formula.

The second eigenvalue has multiplicity 2, with one eigenfunction φ_2^D symmetric with respect to the median OM, and the other φ_3^D anti-symmetric. Up to scaling, φ_2^D is given by

$$\begin{aligned}\varphi_2^D(x,y) = \;& \sin\left(\tfrac{2\pi}{3}(5x+\sqrt{3}y)\right) - \sin\left(\tfrac{2\pi}{3}(5x-\sqrt{3}y)\right)\\ & + \sin\left(\tfrac{2\pi}{3}(x-3\sqrt{3}y)\right) - \sin\left(\tfrac{2\pi}{3}(x+3\sqrt{3}y)\right)\\ & + \sin\left(\tfrac{4\pi}{3}(2x+\sqrt{3}y)\right) - \sin\left(\tfrac{4\pi}{3}(2x-\sqrt{3}y)\right)\,.\end{aligned} \tag{20}$$

First astonished by some numerics, we arrive to the conclusion that the following surprising result could be true:

Lemma

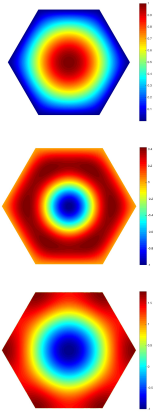
$$\varphi_2^D = -\varphi_1^D \varphi_2^N \,.$$

Proof Express everything in terms of $X = \cos \frac{2\pi}{3} x$ and $Y = \cos \frac{2\pi}{\sqrt{3}} y$. We have then to verify an equality between two polynomials of the variables X and Y.

We deduce from the lemma that the counterexample for Neumann is identical to the counterexample for Dirichlet ! The level sets of φ_2^N and φ_2^D/φ_1^D are the same.

Numerical simulations for Regular polygons (Virginie Bonnaillie-Noël).

In (2D) Gladwell-Zhu were not successful for the square. One can be successfull for the hexagone for Neumann and for Dirichlet (Numerics).



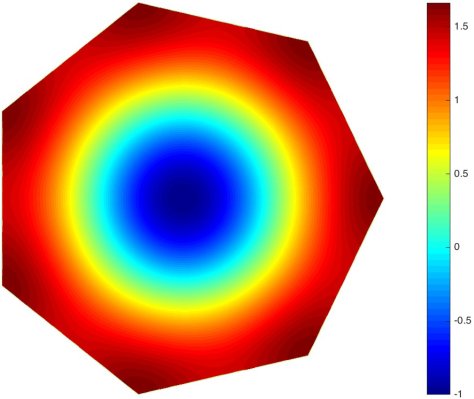


Figure: Level lines of $\frac{w_{6,D}}{w_{1,D}}$ for the Dirichlet problem in the regular heptagon

Regular examples

All the previous examples are example with cracks, corners,...
Hence one could think that the failing of (ECP) could be the effect of non smooth boundaries. Numerical simulations suggest that it was not the case.
Starting from our result on the equilateral triangle $\mathcal{T}_e$, we have recently obtained with P. Bérard

Theorem

There exists a one parameter family of C^∞ domains $\{\Omega_t, 0 < t < t_0\} \subset \mathbb{R}^2$, with the symmetry of $\mathcal{T}_e$, such that:

1. The family is strictly increasing, and Ω_t tends to $\mathcal{T}_e$, in the sense of the Hausdorff distance, as $t \to 0$.
2. $\forall t \in]0, t_0[$, the $\mathrm{ECP}(\Omega_t)$ property is false.
   More precisely, $\forall t$, $\exists$ a linear combination of a symmetric 2nd Neumann eigenfunction and a 1st Neumann eigenfunction of Ω_t, with precisely three nodal domains.

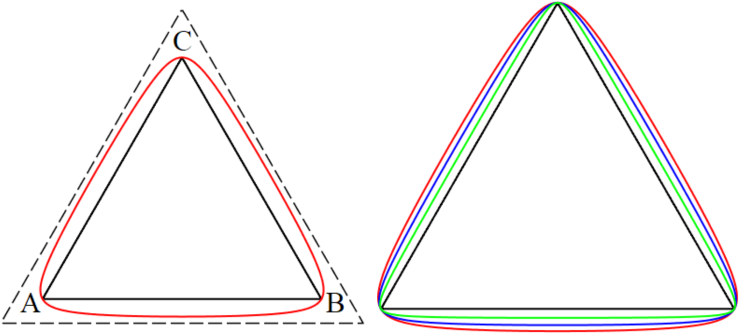


Figure: Domains Ω_t

As we have shown in [6], for $\mathcal{T}_e$, the $ECP(\mathcal{T}_e)$ is false for both the Dirichlet, and the Neumann boundary conditions. The idea is to find a deformation of $\mathcal{T}_e$ s. t. the level sets of the symmetric second Neumann eigenfunction deform nicely. To this end, we revisit a deformation argument given by Jerison and Nadirashsvili (2000) in the framework of the "hot spots" conjecture.

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