Spectral Theory for Systems of Ordinary Differential Equations with Distributional Coefficients

Rudi Weikard

University of Alabama at Birmingham

Recent Results on Quantum Many-Body Systems

Herrsching

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I am reporting on joint work with

- Ahmed Ghatasheh (UAB)

Introduction

The Sturm-Liouville equation

- Sturm and Liouville (early 1830s):

$$-(pu')' + qu = \lambda wu,$$
$$A_1 u(0) + B_1(pu')(0) = A_2 u(L) + B_2(pu')(L) = 0$$

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- Eckhardt et al.: These are covered by a system with locally integrable coefficients

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- The DE gives, in general, only relations not operators.

Distributions and Relations

Distributions

- A linear functional u on $C_c^\infty(a,b)$ is called a distribution, if, for each compact set K, there are constants C and k such that

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- Examples: $\phi \mapsto \int f\phi$, $\phi \mapsto \phi(0)$, $\phi \mapsto \int \phi d\mu$ (all have order 0)
- Every distribution has a derivative: $u'(\phi) = -u(\phi')$.
- Distributions also have anti-derivatives, any two differ by a constant: $u_1(\phi) - u_2(\phi) = C\int\phi$ if $u_1' = u_2'$ (Du Bois-Reymond)

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- If r, g are distributions of order 0, we may pose the differential equation $u' = ru + g$ and seek solutions in $\mathrm{BV}_{\mathrm{loc}}((a, b))$.
- Since, in general, distributions cannot be multiplied with another one, this is as far as generalizations of the Sturm-Liouville equation can go.

Integration by parts

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- Therefore we want our $\mathrm{BV}_{\mathrm{loc}}$ functions balanced.

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- Every relation $S \subset \mathcal{H}_1 \times \mathcal{H}_2$ has an inverse and an adjoint;

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- If $\mathcal{H}_1 = \mathcal{H}_2$ and $S \subset S^*$, then S is called symmetric; if $S = S^*$, then S is called self-adjoint.

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- $(R_\lambda)^* = R_{\overline{\lambda}}$

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- These dimension (denoted by $n_\pm$) are called deficiency indices of E.
- If E is a closed symmetric relation in $\mathcal{H} \times \mathcal{H}$, then

$$E^* = E \oplus D_i \oplus D_{-i}.$$

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- Finally, $\ker A$ is self-adjoint if and only if $A\mathcal{J}A^* = 0$, i.e., $m = d/2$.

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- $\mathcal{H}_0 = \mathcal{H} \ominus \mathcal{H}_\infty$
- $T_0 = T \cap (\mathcal{H}_0 \times \mathcal{H}_0)$ is a self-adjoint linear operator, densely defined in $\mathcal{H}_0$.

Spectral theory for systems of ordinary differential equations with distributional coefficients

Existence and uniqueness theorem

- $u' = \alpha\delta_0 u$ is equivalent to $u_r - u_\ell = \alpha u(0)$

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- Our condition: $2 \pm \Delta_r(x)$ invertible is more general.
- Existence and uniqueness of balanced solutions for initial-value problems for $u' = ru + g$ follows.

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- $T^*_{\min} = T_{\max}$

Boundary conditions

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Properties of Green's function I

- The variation of constants formula: if $x > x_0$

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- $M_\pm$ encode the spectral properties of T.

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- M is a Herglotz-Nevanlinna function

$$M(\lambda) = A\lambda + B + \int \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\right)\nu(t)$$

where $\nu = N'$ and N a non-decreasing matrix.

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- $\mathcal{F} \circ \mathcal{G} = \mathbb{1}$ and $\mathcal{G} \circ \mathcal{F}$ is the projection onto $\mathcal{H}_0$.
- $(u, f) \in T$ if and only if $(\mathcal{F}f)(t) = t(\mathcal{F}u)(t)$.

Example and Special Cases

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- $T_{\max} \cong \mathbb{C} \times L^2(w)$ (u_1 constant) and
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- $L^2(w)$ is infinite-dimensional
- General solution:
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- Fourier transform: $(\mathcal{F}f)(\lambda) = \int f_1 \binom{1}{0}$.
- $\mathcal{F}|_{\mathcal{H}_0}$ is unitary and T is represented by multiplication with independent variable in Fourier space $L^2(\nu)$.

Special case: $n = 2$, real coefficients, DC holds, Part I

- J real implies $J = \beta\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with $\beta \in \mathbb{R} \setminus \{0\}$ (choose $\beta = 1$).

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- Following Weyl one says to have the limit-point case or limit-circle case at b, respectively.

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- In the last case there are two types of boundary conditions: separated (one condition on either end) or coupled (two conditions both of which involve both endpoints).

Special case: $n = 2$, real coefficients, DC holds, Part III

- If a regular and b limit-point we obtain self-adjoint restrictions by posing a boundary condition at a: $\cos(\alpha)u_1(a) - \sin(\alpha)u_2(a) = 0$.

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- The Titchmarsh-Weyl m-function is determined by the M-matrix:
$$m(\lambda) = \frac{1}{\beta}(\sin\alpha, \cos\alpha)M(\lambda)\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix}.$$

Thanks for your attention!