Spectral Theory and
Dynamics of
Quantum Systems
GRADUIERTENKOLLEG 1838

Spectral Theory of the Fermi Polaron

Marcel Griesemer
University of Stuttgart

joint work with **Ulrich Linden**

Heinz Siedentop farewell
Herrsching 2018

What is the Fermi polaron?

We consider a system consisting of N **fermions** and 1 additional particle (called **impurity**) in all of $\mathbb{R}^d$ or in a box $[0, L]^d \subset \mathbb{R}^d$. Informally

$$H_N = -\frac{1}{M}\Delta_y - \sum_{j=1}^{N} \Delta_{x_j} - g \sum_{j=1}^{N} \delta(x_j - y)$$

In physics,

- this model is used to study unbalanced spin populations in Fermi gases. Here N **spin up** fermions versus **one spin** down fermion.
- there are conjecture about the form of the ground state for weak and strong coupling (Mora, Chevy 2009 / Punk, Dimutrescu, Zwerger 2009). • These conjectures are based on a second quantised model with UV cutoff.

Some References on δ-Potentials

One particle systems with δ-potential.

- Albeverio, Gesztesy, Høegh-Krohn, Holden: *Solvable Model in Quantum Mechanics*, 1998
- Albeverio, Kurasov: *Singular perturbations of differentiable operators*, 1999.

***N*-particle systems with δ-interactions via TMS extension:**

- Dell'Antonio, Figari, Teta: `AIHP` **60** `1994`
- Correggi, Dell'Antonio, Finco, Michelangeli, Teta: `RMP` **24** `2012`
- Correggi, Finco, Teta: `EPL` **111** `2015`
- Michelangeli, Ottolini: `RMP` **79** `2017`

`TMS = Ter Martirosyan-Skornyakov.`

A different approach:

- Dimock, Rajeev: `J. Phys. A. 2004`

Warm up: $H = -\Delta_x - g\delta(x)$ with eigenvalue $E < 0$

Let G_λ be the Green's function solving $(-\Delta + \lambda)G_\lambda(x) = \delta(x)$, $\lambda > 0$. Then H is given by

$$\boxed{(H+\lambda)^{-1} = (-\Delta+\lambda)^{-1} + \frac{1}{G_E(0) - G_\lambda(0)} |G_\lambda\rangle\langle G_\lambda|}$$

This is the norm resolvent limit of

$$H_n := -\Delta - g_n |\delta_n\rangle\langle\delta_n| \qquad \text{in } L^2(\mathbb{R}^d).$$

where

$$\delta_n(x) := (2\pi)^{-d} \int_{|k|\le n} e^{ikx}\, dk, \qquad x \in \mathbb{R}^d$$

$$g_n^{-1} := \langle \delta_n, (-\Delta + E)^{-1} \delta_n \rangle = (2\pi)^{-d} \int_{|k|\le n} (k^2 + E)^{-1}\, dk.$$

Note that $\langle \delta_n, \varphi \rangle \to \varphi(0)$ and $g_n \to 0$ as $n \to \infty$ (for $d \ge 2$).

Abstract operator-theoretic approach

1. regularized theory
2. approximation theorem

Let $H_0 : D \subset \mathscr{H} \to \mathscr{H}$ be positive, $A : \mathscr{H} \to \tilde{\mathscr{H}}$ bounded, $g \in \mathbb{R}$. Let

$$H = H_0 - gA^*A \qquad \text{in } \mathscr{H}$$

and define a generalized Birman-Schwinger-operator

$$\phi(E) := \frac{1}{g} - A(H_0 - E)^{-1}A^* \qquad \text{in } \tilde{\mathscr{H}}$$

Lemma

Then $E \in \rho(H_0)$ is an eigenvalue of H if and only if 0 is an eigenvalue of $\phi(E)$. Moreover,

$$(H_0 - E)^{-1}A^* \;:\; \ker \phi(E) \to \ker(H - E)$$

is an isomorphism.

Proof. The operators $H - E$ and $\phi(E)$ are the first and second Schur complements, respectively, of the auxiliary block operator

$$\begin{pmatrix} H_0 - E & A^* \\ A & g^{-1} \end{pmatrix} \quad \text{in } \mathscr{H} \oplus \tilde{\mathscr{H}}.$$

Approximation Theorem

Let $H_0 \geq 0$, and for $n \in \mathbb{N}$ let $A_n \in \mathscr{L}(\mathscr{H}, \tilde{\mathscr{H}})$, $g_n \in \mathbb{R}_+$ and

$$H_n := H_0 - g_n A_n^* A_n$$
$$\phi_n(z) := g_n^{-1} - A_n(H_0 - z)^{-1} A_n^*.$$

Suppose there exists $\mu < 0$ such that

(a) $R_\mu := \lim_{n\to\infty} A_n(H_0 - \mu)^{-1}$ exists,

(b) $\phi_n(\mu)\psi \to \phi(\mu)\psi$ for all $\psi \in D$ where $\phi(\mu) \restriction D$ is essentially s.a.

(c) $\phi_n(\mu) \geq c > 0$ for some c and all $n \in \mathbb{N}$ large.

Then, there exists $H = H^*$ such that, in the strong sense

$$(H_n - \mu)^{-1} \to (H - \mu)^{-1} = (H_0 - \mu)^{-1} + R_\mu^* \phi(\mu)^{-1} R_\mu.$$

Remark: in applications g_n is determined by (b) and a spectral condition.

Proof

Relation between resolvents of the Schur complements $H_n - \mu$ and $\phi(\mu)$:

$$(H_n - \mu)^{-1} = (H_0 - \mu)^{-1} + (H_0 - \mu)^{-1} A_n^* \cdot \phi_n(\mu)^{-1} \cdot A_n (H_0 - \mu)^{-1}$$

where, as $n \to \infty$,

$$\begin{aligned} A_n (H_0 - \mu)^{-1} &\to R_\mu \\ (H_0 - \mu)^{-1} A_n^* &\to R_\mu^* \\ \phi_n(\mu)^{-1} &\to \phi(\mu)^{-1}, \qquad \text{strongly.} \end{aligned}$$

It follows that

$$(H_n - \mu)^{-1} \to (H_0 - \mu)^{-1} + R_\mu^* \phi(\mu)^{-1} R_\mu =: (H - \mu)^{-1}$$

Theorem (domain of H and lower bound)

A vector $\varphi \in \mathscr{H}$ belongs to $D(H)$ if and only if there exists a vector $w_\varphi \in D(\phi(z))$ such that for some (and hence all) $z \in \rho(H_0)$,

$$\varphi - R_{\bar{z}}^* w_\varphi \in D(H_0) \quad \text{and} \quad A(\varphi - R_{\bar{z}}^* w_\varphi) = \phi(z) w_\varphi. \tag{1}$$

$A\psi := \lim_{n\to\infty} A_n \psi$, $\psi \in D(H_0)$. Then

$$(H - z)\varphi = (H_0 - z)(\varphi - R_{\bar{z}}^* w_\varphi). \tag{2}$$

It follows that for $E < \inf \sigma(H_0)$,

$$\langle \varphi, (H - E)\varphi \rangle = \langle (\varphi - R_E^* w_\varphi), (H_0 - E)(\varphi - R_E^* w_\varphi) \rangle + \langle w_\varphi, \phi(E) w_\varphi \rangle.$$

Hence

$$\boxed{\phi(E) \geq 0 \quad \Rightarrow \quad H \geq E.}$$

- basis for all lower bounds on H (due to explicit formula for $\phi(E)$)
- Equation in (1) is the generalized TMS-condition.
- For the FP described by the TMS Hamiltonian analog results are known.

Abstract theory continued

Suppose the assumptions (a)-(c) of the abstract theory hold, and, in addition,

(d) H_0 has a compact resolvent, (`e.g. FP in a box`)

(e) $\ker R_z^* = \{0\}$ for all $z \in \rho(H_0)$.

Then,

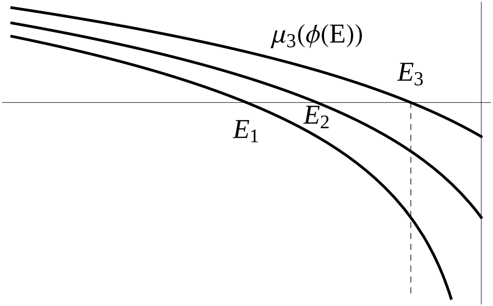
1. $\sigma(H)$ is purely discrete, and $\sigma(\phi(z))$ in $\mathbb{C}\backslash[c, \infty)$ is purely discrete for each $z \in \rho(H_0)$,
2. $(H - z)^{-1} = (H_0 - z)^{-1} + R_{\bar{z}}^* \phi(z)^{-1} R_z$ for $z \in \rho(H_0) \cap \rho(H)$,
3. The map $\tau \mapsto \mu_\ell(\phi(\tau))$ is <u>continuous</u>, and, if $\mu_\ell(\phi(\tau)) < c$, it is <u>strictly decreasing</u>.
4. For any $E < \min \sigma(H_0)$, and $\ell \in \mathbb{N}$,

$$\mu_\ell(H) \leq E \quad \Leftrightarrow \quad \mu_\ell(\phi(E)) \leq 0.$$

and therefore

$$\langle w, \phi(E) w \rangle \leq 0 \quad \Rightarrow \quad \min \sigma(H) \leq E.$$

$$\mu_\ell(H) \le E \quad \Leftrightarrow \quad \mu_\ell(\phi(E)) \le 0$$



The zeros $E_1, E_2, E_3, \ldots$ of the black curves are the eigenvalues of H.

The Fermi-polaron in $\mathbb{R}^2$

Let

$$H_n = \frac{1}{M}(-\Delta_y) + \sum_{i=1}^{N}(-\Delta_{x_i}) - g_n W_n$$

where

$$\langle \psi, W_n \psi \rangle = \sum_{i=1}^{N} \int dx_1 \ldots \widehat{dx_i} \ldots dx_N dy \left| \int dx_i \, \eta_n(x_i - y)\psi(y, \boldsymbol{x}) \right|^2$$

$$g_n^{-1} = \int dk \, \frac{\hat{\eta}_n(k)^2}{(1+\frac{1}{M})k^2 - E_B}.$$

Then, after passing to the **center-of-mass frame and F.T.**,

$$H_n = \int_{\mathbb{R}^2}^{\oplus} \left(\frac{1}{M+N} P^2 + H_{\mathrm{rel}}(n, P) \right) \, dP.$$

where

$$H_{\mathrm{rel}}(n, P) = \frac{1}{M} P_f^2 + H_f - g_n a^*(\hat{\eta}_{n,P}) a(\hat{\eta}_{n,P})$$

$P_f = \int k \, a_k^* a_k \, dk$, $H_f = \int k^2 \, a_k^* a_k \, dk$.

The assumptions (a)-(c) of the abstract theory are satisfied and

$$\phi(E) = \alpha + \phi^0(E) + \phi^I(E)$$

where

$$\begin{aligned}
\alpha &= \frac{\pi}{1+\frac{1}{M}} \log(-E_B) \\
\phi^0(E) &= \frac{\pi}{1+\frac{1}{M}} \log\left(\tfrac{1}{M+1} P_f^2 + H_f - E\right) \\
\phi^I(E) &= \int dk\, dl\, a_k^* \frac{1}{\frac{1}{M}(P_f + k + l)^2 + H_f + k^2 + l^2 - E}\, a_l .
\end{aligned}$$

It follows that $H_n \to H$ in the strong resolvent sense and that

$$\boxed{\phi(E) \geq 0 \quad \Rightarrow \quad H \geq E.}$$

In particular, H is a TMS extension for the $N+1$ fermion system.

Stability

Theorem (U. Linden, M.G.)

Let $E_B < 0$ and $M > 1.225$, then there exists $\mu < E_B$ such that,

$$H_N \geq \mu \quad \textit{for all } N \in \mathbb{N}.$$

Remarks:

- *H_N is bounded from below for all $M > 0$ and $N \in \mathbb{N}$, but the lower bound may depend on N (and M).*
- *It is an open question whether or not some condition $M > c$ is necessary.*
- *In 3D an analog results holds under the condition $M > 0.36$ (Seiringer, Moser), while for $M < 0.0735$ one has instability.*

The Fermi-polaron in a box

$$d = 2$$

Regularized model. N fermions and one impurity particle in a box $\Omega = [0, L]^2 \subset \mathbb{R}^2$ with periodic boundary conditions.

$$\mathscr{H}_N := L^2(\Omega) \otimes \bigwedge^N L^2(\Omega) \subset \mathcal{F} \otimes \mathcal{F}.$$

Let a_k, b_k, a_k^*, b_k^* be the annihilation and creation operators associated with the ONB of $L^2(\Omega)$ given by

$$\varphi_k(x) := e^{ikx}/L \quad \text{for } k \in \tfrac{2\pi}{L}\mathbb{Z}^2.$$

Let $H_n := H_0 - g_n W_n$, where

$$H_0 := \sum_k k^2 \left(\tfrac{1}{M} b_k^* b_k + a_k^* a_k\right),$$

$$W_n := \sum_{k,l,q}^{n} a_k^* \, b_{q-k}^* \, b_{q-l} \, a_l$$

$$g_n^{-1} := \sum_k^n \frac{1}{(1 + \frac{1}{M})k^2 - E_B}, \quad E_B < 0.$$

Let

$$V_n = \sum_{k,q}^{n} m_q^* b_{q-k} a_k, \qquad (m_q^* = b_q^*)$$

then $m_p m_q^* = \delta_{p,q}$ on the vacuum and hence

$$W_n = V_n^* V_n, \qquad \text{in } \mathscr{H}_N.$$

Theorem (Linden, M.G.)

$H_n = H_0 - g_n V_n^* V_n$ satisfies all hypotheses of the abstract theory and hence there exists $H = H^*$ such that

$$H_n \to H, \qquad \textit{in strong resolvent sense.}$$

For all $z \in \rho(H_0) \cap \rho(H)$,

$$(H - z)^{-1} = (H_0 - z)^{-1} + R_{\bar{z}}^* \phi(z)^{-1} R_z.$$

Polaron and Molecule states

Fix $\mu > 0$, let $|\mathrm{FS}_\mu\rangle = \wedge_{|k|^2 \le \mu} \varphi_k$ be the fermi sea, and let

$$N_\mu = \{k \in \tfrac{2\pi}{L}\mathbb{Z}^2 \mid k^2 \le \mu\}.$$

Polaron states are of the form

$$|\mathrm{P}\rangle = \boxed{\alpha_0 b_0^* \,|\mathrm{FS}_\mu\rangle} + \sum_{\substack{K^2 > \mu \\ q^2 \le \mu}} \alpha_{K,q} b_{q-K}^* a_K^* a_q \,|\mathrm{FS}_\mu\rangle$$

$$\alpha_0, \alpha_{K,q} \in \mathbb{C} \quad \text{are variational parameters.}$$

Molecule states are of the form:

$$|\mathrm{M}\rangle = \boxed{\sum_{K^2 > \mu} \beta_K b_{-K}^* a_K^* \,|\mathrm{FS}_\mu\rangle} + \sum_{\substack{K^2, L^2 > \mu \\ q^2 \le \mu}} \beta_{K,L,q} b_{q-K-L}^* a_K^* a_L^* a_q \,|\mathrm{FS}_\mu\rangle$$

$$\beta_k, \beta_{K,L,q} \in \mathbb{C} \quad \text{are variational parameters.}$$

Our polaron Ansatz for $\phi(E)$

$$|\widetilde{\mathrm{P}}\rangle = \sum_{q^2 \leq \mu} \tilde{\alpha}_q m_q^* a_q \, |\mathrm{FS}_\mu\rangle \, ,$$

If E is a solution to

$$\min_{\|\widetilde{P}\|=1} \langle \widetilde{\mathrm{P}}, \phi(E) \widetilde{\mathrm{P}} \rangle = 0$$

then

$$\inf \sigma(H) \leq E$$

Theorem (M.G., Linden)

Let $E_\mu := \sum_{k^2 \leq \mu} k^2$. Any solution E to the ***polaron equation***

$$E_\mu - E = \sum_{q^2 \leq \mu} \frac{1}{G_E(E_\mu - q^2, q)} \tag{3}$$

$$G_E(\lambda, q) := \sum_k \left(\frac{1}{(1 + \frac{1}{M})k^2 - E_B} - \frac{\chi(k^2 > \mu)}{\frac{1}{M}(q-k)^2 + k^2 + \lambda - E} \right)$$

is an upper bound to the ground state energy of H on $\mathscr{H}_{N_\mu}$. The equation (3) *has at least one solution $E < E_\mu$.*

In a similar way the solutions to the **molecule equation** can be shown to be upper bounds to the ground state energy of H in $\mathscr{H}_{N_\mu+1}$. If E_P and E_M denote the lowest solutions to the polaron and the molecule equations, respectively, then

$$E_M - \mu < E_P$$

for M, L and $|E_B|$ large enough. (Linden, 2017).

Enjoy your eternal sabbatical, Heinz!