Two dimensional Schrödinger operators with point interactions

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Herrsching

Conference on the occasion of Heinz' retirement from LMU for wishing his happy life during his long coming years

Thank you for the invitation

Abstract

We consider 2-dim Schrödinger operators H with multi-center local point interactions and show that

1) The threshold behavior of $(H-\zeta)^{-1}$ is either

a) of regular type, viz. $(H-\zeta)^{-1}$ is continuous up to 0, or
b) it has singularity of type s-wave resonance, or
c) it has singularity of type p-wave resonances, or
d) it has singularity of type zero-energy eigenvalue .

2) Characterize zero modes of H.

3) $W_{\pm}$ are bounded in L^p for all $1 < p < \infty$ if H is of regular type.

For the single center case H is always of regular type.

Joint work with Horia Cornean and Alessandro Michelangeli

Let $Y = \{y_1, \dots, y_N\} \subset \mathbb{R}^2$, $N \geq 1$ and

$$T_0 \colon = -\Delta\Big|_{C_0^\infty(\mathbb{R}^2 \setminus Y)}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Schrödinger operator on $\mathbb{R}^2$ with point interactions at Y is any selfadjoint extension of T_0.

We are concerned with the ones $H_{\alpha,Y}$ with local point interactions at Y and strengths $\alpha \equiv (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$

$$H_{\alpha,Y} = \text{“} - \Delta + \sum_{j=1}^{N} \alpha_j \delta(x - y_j)\text{”}. \tag{1}$$

Multiplication by $\delta(x - y_j)$ is not $-\Delta$ bounded nor $-\Delta$ form-bounded

We follow Albeverio-Gesztesy-HøeghKrohn-Holden's definition via resolvent equation

$$
\begin{aligned}
&(H_{\alpha,Y} - z^2)^{-1} - (H_0 - z^2)^{-1} \\
&\qquad = \sum_{j,k=1}^{N} \{\Gamma_{\alpha,Y}(z)\}^{-1}_{jk}\, \mathcal{G}_z(\cdot - y_j) \otimes \overline{\mathcal{G}_z(\cdot - y_k)},
\end{aligned}
$$

for $z \in \mathbb{C}^+ \setminus \mathcal{E}$, where

$$
\begin{aligned}
\mathcal{G}_z(x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\xi} d\xi}{\xi^2 - z^2} = \frac{i}{4} H_0^{(1)}(z|x|), \\
\Gamma_{\alpha,Y}(z)_{jk} &= \left(\alpha_j + \frac{1}{2\pi} \log\left(\frac{z}{2i}\right) + \frac{\gamma}{2\pi}\right)\delta_{jk} - \mathcal{G}_z(y_j - y_k)\widehat{\delta}_{jk}
\end{aligned}
$$

and where $\mathcal{E}$ is the set of z such that $\det \Gamma_\alpha(z) = 0$, δ_{jk} is the Kronecker delta, $\widehat{\delta}_{jk} = 1 - \delta_{jk}$ and γ the Euler number.

Some properties of $\mathcal{G}_z(x)$. For small $z|x|$, we use

$$\begin{aligned}\mathcal{G}_z(x) =& \frac{i}{4}H_0^{(1)}(z) = \Big(-\frac{1}{2\pi}\log\left(\frac{z}{2i}\right) - \frac{\gamma}{2\pi}\Big)J_0(z)\\ &-\frac{1}{2\pi}\Big(\frac{\frac{1}{4}z^2}{(1!)^2} - \Big(1+\frac{1}{2}\Big)\frac{(\frac{1}{4}z^2)^2}{(2!)^2} + \dots\Big),\\ J_0(z) =& \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\Big(\frac{z^2}{4}\Big)^k\end{aligned}$$

and for small $\lambda|x|$

$$\frac{i}{4}H_0^{(1)}(\lambda|x|) = g(\lambda) + G_0(x) + O(\lambda^2|x|^2 g(\lambda)),$$

$$g(\lambda) = -\frac{1}{2\pi}\log\left(\frac{\lambda}{2i}\right) - \frac{\gamma}{2\pi}, \quad G_0(x) = -\frac{1}{2\pi}\log|x|$$

For large $z|x|$, use Watson's formula

$$\frac{i}{4}H_0^{(1)}(z|x|) = \frac{e^{iz|x|}}{2^{\frac{3}{2}}\pi}\int_0^\infty e^{-t}t^{-\frac{1}{2}}\left(\frac{t}{2} - iz|x|\right)^{-\frac{1}{2}} dt,$$

which produces for $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$

$$\mathcal{G}_z(x) = e^{iz|x|}\omega(z|x|),\ \ |\partial_\lambda^\alpha \omega(\lambda)| \leq C\langle\lambda\rangle^{-\frac{1}{2}-|\alpha|},$$

viz. $(1-\chi(\lambda))\omega(\lambda)$ is a symbol of order $-1/2$ for a bump function χ around 0.

Albeverio and others proved that:

(1) The resolvent equation defines s.a. $H_{\alpha,Y}$ uniquely in $L^2(\mathbb{R}^2)$ with domain

$$D(H_{\alpha,Y}) = \{u = v + \sum [\Gamma_{\alpha,Y}(z)^{-1}]_{jk} v(y_k)\mathcal{G}_z(x-y_j) \colon v \in H^2\} \quad (2)$$

which is independent of $z \in \mathbb{C}^+ \setminus \mathcal{E}$.

(2) $v \in H^2(\mathbb{R}^2)$ of (1) is uniquely determined by $u \in D(H_{\alpha,Y})$ and

$$(H_{\alpha,Y} - z^2)u(x) = (-\Delta - z^2)v(x).$$

(3) $H_{\alpha,Y}$ is a real local operator,

(4) $\sigma(H_{\alpha,Y})$ =AC part $[0,\infty) \cup \{$at most N eigenvalues $\leq 0\}$.

(5) $H_{\alpha,Y}$ is a rank N perturbation of $-\Delta$ and

$$W_\pm = \lim_{t\to\pm\infty} e^{itH_{\alpha,Y}} e^{-itH_0} \tag{3}$$

exist and are complete : Range $W_\pm = L^2_{ac}(H_{\alpha,Y})$,

$$W_\pm^* W_\pm = 1, \quad W_\pm W_\pm^* = P_{ac}(H_{\alpha,Y}),$$

where $P_{ac}(H_{\alpha,Y})$ is the projection onto $L^2_{ac}(H_{\alpha,Y})$. For Borel f

$$f(H_{\alpha,Y})P_{ac}(H_{\alpha,Y}) = W_\pm f(H_0) W_\pm^*. \tag{4}$$

• **LAP.** Let $L^2_\sigma = L^2(\mathbb{R}^2, \langle x\rangle^{2\sigma}dx)$, $\mathbf{B}_\sigma = \mathbf{B}(L^2_\sigma, L^2_{-\sigma})$ for $\sigma \in \mathbb{R}$. Then:

• Agmon-Kuroda theory for the LAP for $(-\Delta - z^2)^{-1}$,

• Explicit formula for the resolvent $(H_{\alpha,Y} - z)^{-1}$,

• Behavior of the kernel $\mathcal{G}_z(x)$ as $|x| \to \infty$

imply that, for $\sigma > 1/2$, $\mathbf{B}_\sigma$-valued analytic function $(H_{\alpha,Y} - z^2)^{-1}$ of $z \in \mathbb{C}^+ \setminus \mathcal{E}$,

$$\mathcal{E} = \{\sqrt{\lambda} \colon \lambda \in \sigma_p(H_{\alpha,Y})\} \subset i[0,\infty)$$

has boundary values on $\mathbb{R}\setminus\{0\}$ which is locally Hölder continuous. However, it can have singularities at $\lambda = 0$ and we first study its behavior near $\lambda = 0$ in $\lambda \in \overline{\mathbb{C}}^+ \setminus \{0\}$. We write λ instead of z when we want to emphasize that λ is in $\overline{\mathbb{C}}^+ \setminus \{0\}$ not only in $\mathbb{C}^+$.

Notation • We use the vector notation:

$$\widehat{\mathcal{G}}_{\lambda,Y}(x) = \begin{pmatrix} \mathcal{G}_\lambda(x-y_1) \\ \vdots \\ \mathcal{G}_\lambda(x-y_N) \end{pmatrix},$$

$$D(\lambda,x,y) = \langle \widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma_{\alpha,Y}(\lambda)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle,$$

where $\langle \mathbf{a},\mathbf{b}\rangle = \sum a_j b_j$. Then,

$$(H_{\alpha,Y}-\lambda^2)^{-1} = (H_0-\lambda^2)^{-1} + D(\lambda,x,y).$$

• Recall $G_0(x) = -\frac{1}{2\pi}\log|x|$. We define

$$\widehat{G}_{0,Y}(x) = \begin{pmatrix} G_0(x-y_1) \\ \vdots \\ G_0(x-y_N) \end{pmatrix}.$$

$$\mathbf{e} = \frac{1}{\sqrt{N}}\hat{\mathbf{1}},\ \ \hat{\mathbf{1}} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},\ \ P = \mathbf{e}\otimes\mathbf{e},\ \ S = 1-P.$$

• $\tilde{\mathcal{D}} = \mathcal{D}(\alpha, Y)$ and $\mathcal{G}_1(Y)$ are $N \times N$ real symmetric matrices:

$$\tilde{\mathcal{D}} = \Big(\delta_{jk}\alpha_j + \frac{\widehat{\delta}_{jk}}{2\pi}\log|y_j - y_k|\Big),$$

$$\mathcal{G}_1 = -\Big(\frac{\widehat{\delta}_{jk}}{4N}|y_j - y_k|^2\Big).$$

• Identify integral operator K and its kernel $K(x,y)$.

Definition 1. *$\varphi \in L^2_{-\sigma}(\mathbb{R}^2)$ is a threshold resonance of $H_{\alpha,Y}$ of s-wave or p-wave types if φ satisfies*

$$-\Delta\varphi(x) + \sum f_j \delta(x - y_j)\varphi = 0$$

for constants $f_1, \ldots, f_N$ and

$$\varphi(x) = b + \frac{a_1 x_1 + a_2 x_2}{|x|^2} + O\Big(\frac{1}{|x|^2}\Big), \quad |x| \to \infty$$

respectively with $b \neq 0$ or $b = 0$ where a_1, a_2 are real constants.

Results. Matrices $S\tilde{\mathcal{D}}S$, $T\tilde{\mathcal{D}}^2T$ and $\mathcal{G}_1$, where T is the proj. onto $\mathrm{Ker}_{S\mathbb{C}^N} S\tilde{\mathcal{D}}S$, control the behavior as $\lambda \to 0$ of

$$\overline{\mathbb{C}}^+ \setminus \{0\} \ni \lambda \mapsto (H_{\alpha,Y} - \lambda^2)^{-1} \in \mathbf{B}_\sigma, \quad \sigma > 1.$$

Theorem 2. *$(H_{\alpha,Y} - \lambda^2)^{-1}$ can be continuously extended to $\overline{\mathbb{C}}^+$ if and only if $S\tilde{\mathcal{D}}S|_{S\mathbb{C}^N}$ is non-singular. In this case*

$$(H_{\alpha,Y} - \lambda^2)^{-1}(x,y) = G_0(x-y) - N^{-1}\sum(\log|x-y_j| + \log|y-y_j|)$$
$$- N^{-2}\langle \hat{\mathbf{1}}, \tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle$$
$$+ \left\langle [S\tilde{\mathcal{D}}S]^{-1} S\Big(\hat{G}_{0,Y}(x) - N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}}\Big), S\Big(\hat{G}_{0,Y}(y) - N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}}\Big)\right\rangle + O(g(\lambda)^{-1})$$

where $\|O(g(\lambda)^{-1})\|_{\mathbf{B}_\sigma} \le C|g(\lambda)^{-1}|$ as $\lambda \to 0$. The leading behavior as $|x| + |y| \to \infty$ of $O(1)$ term is like

$$-\frac{1}{2\pi}(\log|x-y| - \log|x| - \log|y|) + C + O(|x|^{-1}).$$

In this case we say $H_{\alpha,Y}$ is of regular type.

2) If $S\tilde{\mathcal{D}}S|_{S\mathbb{C}^N}$ is singular and T is proj. in $S\mathbb{C}^n$ onto $\mathrm{Ker}_{S\mathbb{C}^N} S\tilde{\mathcal{D}}S$,

$$\mathrm{rank}\, T\tilde{\mathcal{D}}^2 T \leq 1.$$

Theorem 3. *Suppose $S\tilde{\mathcal{D}}S|_{S\mathbb{C}^N}$ is singular and $T\tilde{\mathcal{D}}^2 T|_{T\mathbb{C}^N}$ is non-singular. Then*

- $\mathrm{rank}\, T = 1$. *If $T = \mathbf{f} \otimes \mathbf{f}$ for $\mathbf{f} = {}^t(f_1, \ldots, f_N)$ then*

$$\varphi(x) = -\left\langle \mathbf{f}, \frac{\tilde{\mathcal{D}}\hat{\mathbf{1}}}{N} \right\rangle - \sum_{j=1}^{N} \frac{f_j}{2\pi} \log(x - y_j)$$

is the resonance of s-wave type.

- *The resolvent has leading $\log\lambda$ singularity as $\lambda \to 0$:*

$$(H_{\alpha,Y} - \lambda^2)^{-1} = a^{-2} g(\lambda) \varphi \otimes \varphi + O(1)$$

with positive $a^{-2} = \langle \mathbf{f}, \tilde{\mathcal{D}}^2 \mathbf{f} \rangle > 0$.

Theorem 4. *If $S\tilde{\mathcal{D}}S|_{S\mathbb{C}^N}$ and $T\tilde{\mathcal{D}}^2T|_{T\mathbb{C}^N}$ are both singular, then $T\mathcal{D} = \mathcal{D}T = 0$. Suppose $T\mathcal{G}_1T|_{T\mathbb{C}^N}$ is non-singular. Then:*

- $(H_{\alpha,Y} - \lambda^2)^{-1}$ *has leading* $\lambda^{-2}(\log\lambda)^{-1}$ *singularity:*

$$(H_{\alpha,Y} - \lambda^2)^{-1} = \frac{-1}{Ng\lambda^2}\langle T\widehat{G}_{0,Y}(x), [T\mathcal{G}_1T]^{-1}T\widehat{G}_{0,Y}(y)\rangle + O(\lambda^{-2}).$$
$$= \frac{-1}{Ng\lambda^2}\sum_{j=1}^{n} a_j\varphi_j(x)\varphi_j(y) + O(\lambda^{-2}) \quad (\lambda \to 0),$$

where $n = \operatorname{rank} T$ *and*

$$\varphi_j(x) = \sum_{k=1}^{N} f_{jk}\log|x - y_k| = \frac{a_{j1}x_1 + a_{j,2}x_2}{|x|^2} + O(|x|^{-2})$$

are resonances of p*-wave type.*

Theorem 5. *$S\tilde{\mathcal{D}}S|_{S\mathbb{C}^N}$, $T\tilde{\mathcal{D}}^2T|_{T\mathbb{C}^N}$ and $T\mathcal{G}_1T|_{T\mathbb{C}^N}$ are all singular iff $H_{\alpha,Y}$ has an eigenvalue at zero. Let T_1 be proj. onto* $\mathrm{Ker}\, T\mathcal{G}_1T|_{T\mathbb{C}^N}$ *and*

$$\mathcal{G}_2 = -\left(\frac{\widehat{\delta}_{jk}}{8\pi N}|y_j - y_k|^2 \log\left(\frac{e}{|y_j - y_k|}\right)\right) .$$

Then, $T_1\mathcal{G}_2T_1$ is non-singular and with $Q = [T_1\mathcal{G}_2T_1]^{-1}$

$$\begin{aligned}&(H_{\alpha,Y} - \lambda^2)^{-1}(x,y)\\ &= -\frac{1}{N\lambda^2}\langle T_1\widehat{G}_{0,Y}(x), QT_1\widehat{G}_{0,Y}(y)\rangle + O(\lambda^{-2}g(\lambda)^{-1}).\end{aligned}$$

$$T_1\widehat{G}_{0,Y}(x) = O(|x|^{-2})$$

is an eigenfunction of $H_{\alpha,Y}$ with zero energy.

(•) $H_{\alpha,Y}$ is always of regular type if $N = 1$.

• Zero modes can indeed exist if $N \geq 3$.

Proposition 6. *Let $N \geq 3$. Assume $\mathbf{a} = {}^t(a_1, \dots, a_N) \in \mathbb{R}^N \setminus \{0\}$ satisfies*

$$\sum_{j=1}^{N} a_j = 0, \quad \sum_{j=1}^{N} a_j y_j = 0 \quad \text{and } \tilde{\mathcal{D}}\mathbf{a} = 0. \tag{5}$$

Then, the function

$$\varphi(x) = \sum_{j=1}^{N} a_j \log|x - y_j| \tag{6}$$

belongs to the domain of $H_{\alpha,Y}$ and $H_{\alpha,Y}\varphi(x) = 0$. Moreover, the converse is also true: any zero energy eigenfunction must be of the form (6) with $\mathbf{a} \in \mathbb{R}^N \setminus \{0\}$ of (5).

For $N = 3$ and $y_1, y_2, y_3 \in \mathbb{R}^2$ which are collinear or for $N \geq 4$ and arbitrary $y_1, \dots, y_N \in \mathbb{R}^2$, there exists $\mathbf{a} \in \mathbb{R}^N \setminus \{0\}$ which satisfies the first two of (5) and, we can always find α such that

$$\tilde{\mathcal{D}}\mathbf{a} = \left(\delta_{jk}\alpha_j + \frac{\widehat{\delta}_{jk}}{2\pi}\log|y_j - y_k|\right)\mathbf{a} = 0$$

and, hence, $H_{\alpha,Y}$ has an eigevalue at 0. Thus the statement on the absence of zero eigenvalue in the book by Albeverio et. al. is not totally correct.

In the equation (5), we remark

$$\sum_{j=1}^{N} a_j = 0 \Leftrightarrow \mathbf{a} \in \mathrm{Range}\, S, \quad \sum_{j=1}^{N} a_j y_j = 0 \Leftrightarrow \langle \mathbf{a}, T\mathcal{G}_1 T\mathbf{a}\rangle = 0$$

$$\tilde{\mathcal{D}}\mathbf{a} = 0 \Leftrightarrow \mathbf{a} \in \mathrm{Ker} S\tilde{\mathcal{D}}S|_{S\mathbb{C}^N} \quad (\Leftarrow \tilde{\mathcal{D}}T = 0,\ T\mathbf{a} = \mathbf{a})$$

Theorem 7. *Suppose that $H_{\alpha,Y}$ is of regular type. Then, wave operators $W_\pm$ are bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$.*

It is known that wave operators for 1-dim SO with point interactions are bounded in $L^p(\mathbb{R}^1)$ for all $1 < p < \infty$ (V. Duchêne, J. L. Marzuola, and M. I. Weinstein 2011) and, in 3-dim dimensions only for $3/2 < p < 3$ (G. Dell'Antonio, A. Michelangeli, R. Scandone and K. Y. 2018) . (For the L^p boundedness of $W_\pm$ for ordinary SO, see the author's paper in Documenta Math. 2016).

The intertwining property $f(P_{ac}H_{\alpha,Y}) = W_\pm f(H_0)W_\pm^*$ reduces the mapping properties of $f(H_{\alpha,Y})$ to those of $f(H_0)$. For example, L^p-L^q and Strichartz estimates for $e^{-itH_{\alpha,Y}}P_{ac}$ follow from those for e^{-itH_0}. p' is the dual exponent of p: $1/p + 1/p' = 1$.

Corollary 8. *Suppose $H_{\alpha,Y}$ is of regular type. For any $2 \le p < \infty$,*

$$\|e^{-itH_{\alpha,Y}} P_{ac}(H_{\alpha,Y}) u\|_p \le C_p |t|^{1/p-1/2} \|u\|_{p'}. \tag{7}$$

We say (p, r) is 2-dimensional Strichartz exponent when

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 < q \le \infty.$$

Corollary 9. *Suppose $H_{\alpha,Y}$ is of regular type. Let (p, q) and (s, r) be 2-dim Strichartz exponents.*

$$\left(\int_{\mathbb{R}} \|e^{-itH_{\alpha,Y}} u\|^q_{L^p(\mathbb{R}^2)} dt \right)^{1/q} \le C\|u\|_2,$$

$$\left\| \int_0^t e^{-i(t-s)H_{\alpha,Y}} P_{ac}\, u(s)\, \mathrm{d}s \right\|_{L^q_t L^p_x} \leqslant\ C\|u\|_{L^{s'}_t L^{r'}_x}.$$

Lemmas Theorems 2, 3, 4 and 5 are proved by studying $\Gamma_{\alpha,Y}(\lambda)^{-1}$ for $\lambda \to 0$. Observe from the property of the Hankel function

$$\begin{aligned}\Gamma(\lambda) &= \left\{\left(\alpha_j + \frac{1}{2\pi}\log\left(\frac{\lambda}{2i}\right) + \frac{\gamma}{2\pi}\right)\delta_{jk} - \mathcal{G}_\lambda(y_j - y_k)\hat{\hat{\delta}}_{jk}\right\}_{jk} \\ &= -Ng\Big(P - N^{-1}g(\lambda)^{-1}\tilde{\mathcal{D}} + \lambda^2\mathcal{G}_1 + \lambda^2 g(\lambda)^{-1}\mathcal{G}_2 + O(\lambda^4)\Big) \\ &=: \; -Ng(\lambda)A(\lambda) \quad \lambda \to 0.\end{aligned}$$

We recall

$$\frac{i}{4}H_0^{(1)}(\lambda|x|) = g(\lambda) + G_0(x) + O(\lambda^2|x|^2 g(\lambda)),$$

$$g(\lambda) = -\frac{1}{2\pi}\log\left(\frac{\lambda}{2i}\right) - \frac{\gamma}{2\pi}, \quad G_0(x) = -\frac{1}{2\pi}\log|x|.$$

$$P = \mathbf{e}\otimes\mathbf{e} = N^{-1}\begin{pmatrix}1 & \dots & 1\\ \vdots & \vdots & \vdots \\ 1 & \dots & 1\end{pmatrix}, \; S = 1 - P.$$

• $\tilde{\mathcal{D}} = \mathcal{D}(\alpha, Y)$ and $\mathcal{G}_1(Y)$ are $N \times N$ real symmetric matrices:

$$\tilde{\mathcal{D}} = \left(\delta_{jk}\alpha_j + \frac{\widehat{\delta}_{jk}}{2\pi} \log|y_j - y_k|\right),$$
$$\mathcal{G}_1 = -\left(\frac{\widehat{\delta}_{jk}}{4N}|y_j - y_k|^2\right).$$

We need invert

$$A(\lambda) = P - N^{-1}g(\lambda)^{-1}\tilde{\mathcal{D}} + \lambda^2\mathcal{G}_1 P + O(\lambda^2 g^{-1})$$

Noticing that $A(\lambda) + S$, $S = 1 - P$ is invertible, we (repeatedly) use the following lemma due to Jensen and Nenciu.

Lemma 10. *Let A be a closed operator in a Hilbert space $\mathcal{H}$ and S a projection. Suppose $A + S$ has a bounded inverse. Then, A has a bounded inverse if and only if*

$$B = S - S(A + S)^{-1}S \tag{8}$$

has a bounded inverse in $S\mathcal{H}$ and, in this case,

$$A^{-1} = (A+S)^{-1} + (A+S)^{-1}SB^{-1}S(A+S)^{-1}. \tag{9}$$

In this case, the operator corrresponding to B of (8) is

$$\tilde{B} = -N^{-1}g^{-1}\Big(S\tilde{\mathcal{D}}S + SO(g^{-1})S\Big).$$

and, if $S\tilde{\mathcal{D}}S$ is invertible, $\tilde{B}$ is invertible and $A(\lambda)^{-1}$ is obtained by (9). In this way, we have the following lemma which implies Theorem 2. We set

$$F = -N^{-1}g(\lambda)^{-1}\tilde{\mathcal{D}}.$$

Lemma 11. $\Gamma(\lambda)^{-1}$ *can be continuously extended to $\mathbb{R}$ if and only if $S\tilde{\mathcal{D}}S$ is non-singular in $S\mathbb{C}^N$. In this case,*

$$\Gamma(\lambda)^{-1} = -N^{-1}g^{-1}(1+F)^{-1}$$

$$+ (1+F)^{-1} S\Big([S\tilde{\mathcal{D}}S]^{-1} + O(g^{-1})\Big) S(1+F)^{-1}$$
$$+ O(\lambda^2) = [S\tilde{\mathcal{D}}S]^{-1} + O(g^{-1}).$$

and for $0 < |\lambda| < \lambda_0$ *that*

$$\partial_\lambda^\ell [\Gamma^{-1}(\lambda)]_{jk} \leq_{|\,\cdot\,|} C\lambda^{-\ell}, \quad \ell = 0, 1, \ldots. \tag{10}$$

If $S\tilde{\mathcal{D}}S$ is singular, we apply the Jensen-Neciu Lemma to $\tilde{B}$. Note that $S\tilde{\mathcal{D}}S + T$ is invertible in $S\mathbb{C}^N$ and we may repeat the argument.

We do not enter into the proof of Theorems 2,3,4, 5 and Proposition 6 any further. Instead we briefly describe the proof of Theorem 7 which is surprisingly simple.

• **Stationary representation of wave operator** As we are assuming $H_{\alpha,Y}$ is of regular type $\Gamma(\lambda)$ is a non-singular for every $\lambda > 0$ and the function $\Gamma(\lambda)^{-1}$ is analytic in a complex neighbourhood of the positive half line. By the standard argument of scattering theory we have the following stattionary representation formula for the wave operator W_+:

We define for $j,k=1,\dots,N$ by

$$(\Omega_{jk}u)(x)=\frac{1}{\pi\mathrm{i}}\int_0^{+\infty}\lambda\overline{(\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk}}\,\mathcal{G}_{-\lambda}(x)\\ \times\left(\int_{\mathbb{R}^2}\Big(\mathcal{G}_\lambda(y)-\mathcal{G}_{-\lambda}(y)\Big)u(y)\mathrm{d}y\right)\mathrm{d}\lambda\,. \quad (11)$$

Then, with $(T_{x_0}f)(x):=f(x-x_0)$,

$$\langle W^+_{\alpha,Y}u,v\rangle \;=\; \langle u,v\rangle+\sum_{j,k=1}^{N}\langle T_{y_j}\Omega_{jk}T^*_{y_k}u,v\rangle\,.$$

and it suffices to deal with Ω_{jk}. The second line of (11) is the spectral measure of $-\Delta$ and is equal to

$$\frac{i}{2}\int_{\mathbb{S}^1}\widehat{u}(\lambda\omega)\mathrm{d}\omega.$$

Write $\widetilde{\Gamma}_{jk}(\lambda) = [\overline{\Gamma_{\alpha,Y}(\lambda)}^{-1}]_{jk}$. We have

$$\widetilde{\Gamma}_{jk}(\lambda)\hat{u}(\lambda\omega) = \mathcal{F}(\widetilde{\Gamma}_{jk}(|D|u)(\lambda\omega)$$

Define

$$Ku(x) = \frac{1}{\pi \mathrm{i}} \int_0^{+\infty} \mathcal{G}_{-\lambda}(x)\lambda \left(\int_{\mathbb{S}^1} (\mathcal{F}u)(\lambda\omega) d\omega \right) \mathrm{d}\lambda\,.$$

It then follows that

$$\Omega_{jk} = K \circ \widetilde{\Gamma}_{jk}(|D|). \tag{12}$$

Lemma 12. *We have*

$$Ku(x) = \lim_{\varepsilon \downarrow 0} \frac{2i}{\pi^2} \int_{\mathbb{R}^2} \frac{u(y)dy}{x^2 - y^2 - i\varepsilon} \tag{13}$$

and K is bounded from $L^p(\mathbb{R}^2)$ to itself for all $1 < p < \infty$.

Proof. We rewite $Ku(x)$ as follows (modulo constants)

$$\int_0^{+\infty}\int_{\mathbb{R}^9}\frac{e^{ix\xi-iy\eta}}{\xi^2-\lambda^2-i0}\delta\Big(\eta^2-\lambda^2\Big)u(y)dyd\eta d\xi\lambda d\lambda$$

$$=\int_{\mathbb{R}^6}\frac{e^{ix\xi-iy\eta}}{\xi^2-\eta^2-i0}u(y)dyd\eta d\xi$$

$$=\int_{\mathbb{R}^6}\frac{e^{ix(\xi+\eta)-iy\eta}}{\xi^2+2\xi\eta-i0}u(y)dyd\eta d\xi$$

$$=\int_{\mathbb{R}^6}\int_0^\infty e^{-it(\xi^2+2\xi\eta-i0)}e^{ix\xi+i(x-y)\eta}u(y)dyd\eta d\xi dt$$

$$=\int_{\mathbb{R}^3}\int_0^\infty e^{-it(\xi^2-i0)+ix\xi}u(x-2\xi t)d\xi dt$$

Then, we make change of variables $x-2\xi t=y$ or $\xi=(x-y)/2t$. This makes

$$-t\xi^2+x\xi=-\frac{(x-y)^2}{4t}+\frac{x(x-y)}{2t}=\frac{x^2-y^2+i0}{4t}$$

Thus, after changing variables t to t^{-1},

$$\begin{aligned} Ku(x) &= \int_{\mathbb{R}^3} \int_0^\infty e^{i(x^2-y^2+i0)/4t} f(y) \frac{d\xi}{4t^2} dt \\ &= \int_{\mathbb{R}^2} \int_0^\infty e^{it(x^2-y^2+i0)} u(y) dy dt \\ &= \int_{\mathbb{R}^2} \frac{u(y)}{i(x^2-y^2+i0)} dy \end{aligned}$$

This proves (13). To prove the bound we continue to rewrite $Ku(x)$ by using the spherical mean

$$Mu(r) = \frac{1}{2\pi} \int_{\mathbb{S}^1} u(r\omega) d\omega.$$

Then

$$Ku(x) = \int_0^\infty \frac{r(Mf)(r)}{i(x^2-r^2+i0)} dr$$

$$= \int_0^\infty \frac{(Mu)(\sqrt{r})}{i(x^2 - r + i0)} dr$$

and

$$\int_{\mathbb{R}^2} |Ku(x)|^p dx = \int_{\mathbb{S}^1} \left(\int_0^\infty |Ku(\sqrt{\mu})|^p d\mu \right) d\omega$$

But

$$K(\sqrt{r}) = \int_0^\infty \frac{(Mu)(\sqrt{r})}{i(\mu - r + i0)} dr$$

and L^p-boundedness of Cauchy integral implies

$$\int_0^\infty |K(\sqrt{\mu})|^p d\mu \leq \int_0^\infty |(Mu)(\sqrt{r})|^p dr \leq \|u\|_{L^p}.$$

$\square$

We still need study $\tilde{\Gamma}(|D|)$. For small λ, $\tilde{\Gamma}(\lambda)$ satisfies Mikhlin's condition for the L^p multiplier and $\chi(|D|)\tilde{\Gamma}(|D|)$ with $\chi C_0^\infty(\mathbb{R})$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. For large λ it has the form

$$(1 - \chi(\lambda))\widetilde{\Gamma}(\lambda) = \sum_{\text{finite sum}} e^{ia_l\lambda}(1 - \chi(\lambda))b_l(\lambda) + \text{Mikhlin multiplier}$$

where b_l are symbols of order $1/2$.

Finally we apply the following lemma by J. C. Peral (1980) to the homogenenous Fourier integral operator produced by the large part of λ, which completes the proof.

Lemma 13 (Peral). *The translation invariant Fourier integral operator*

$$(Tf)(x) = \int_{\mathbb{R}^n} e^{ix\xi + i|\xi|} \frac{\psi(\xi)}{|\xi|^b} \widehat{f}(\xi) d\xi, \tag{14}$$

where $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\psi(\xi) = 0$ in a neighbourhood of $\xi = 0$ and $\psi(\xi) = 1$ for $|\xi| \geq 2$, is bounded in $L^p(\mathbb{R}^n)$ if and *only if*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{b}{n-1}\,. \tag{15}$$

For our application, $n = 2$ and $b = 1/2$, which produces $1 < p < \infty$.