Matrix Dyson Equation in random matrix theory

László Erdős
IST, Austria

HEINZFEST

Dedicated to the "retirement" of Heinz Siedentop

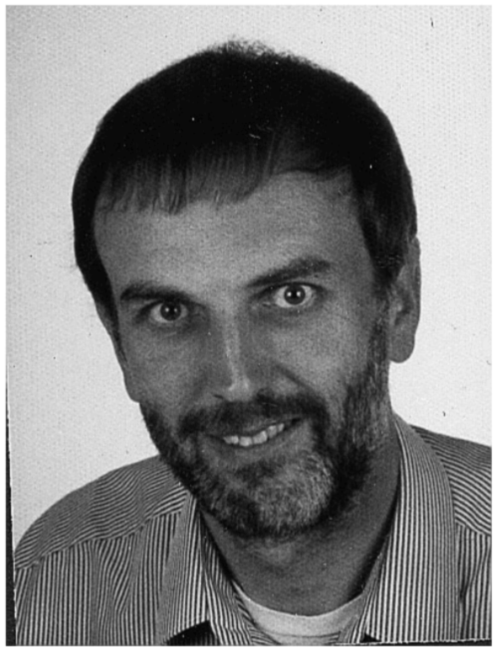
Herrsching, May 9, 2018

Joint work with O. Ajanki, J. Alt, T. Krüger and D. Schröder

Partially supported by ERC Advanced Grant, RANMAT, No. 338804

Neu am Institut:

Prof. Dr. Heinz Siedentop



Seit 1. Juni ist Prof. Siedentop an unserem Institut tätig. Damit konnte das Berufungsverfahren zur Nachfolge von Prof. Wienholtz sehr zügig abgeschlossen werden und der Erste auf der Berufungsliste gewonnen werden. Herr Siedentop vertritt das Arbeitsgebiet Mathematische Physik mit Ausrichtung auf spektraltheoretische Untersuchungen von Mehrteilchen-Hamiltonoperatoren in der Quantenmechanik.

Stellenausschreibung (2002 Jan 16)

An der Fakultät für Mathematik und Informatik ist ab sofort eine Professur (C4) für Angewandte Mathematik (Lehrstuhl)

wieder zu besetzen. Der Lehrstuhl gehört zum Arbeitsbereich

Analysis und Numerik

im Mathematischen Institut. Von den Bewerberinnen und Bewerbern wird erwartet, dass ihre Aktivitäten in der Forschung sich mit den schon vorhandenen in Analysis und mathematischer Physik (Lehrstuhl Prof. Dr. H. Siedentop) sinnvoll ergänzen. Es besteht besonderes Interesse an Bewerbern, die über partielle Differentialgleichungen arbeiten.

In der Lehre wird die Beteiligung an den allgemeinen Lehrverpflichtungen des Mathematischen Instituts erwartet, sowie die Bereitschaft, die Verantwortung für die Numerik-Ausbildung im Mathematik- und Physik-Studium zu übernehmen.

MATRIX DYSON EQUATION

Ingredients:

(i) $z \in \mathbb{C}_+$, "spectral parameter"

(ii) $A \in \mathbb{C}^{N\times N}$ hermitian matrix, "bare Hamiltonian"

(iii) positivity preserving operator $\mathcal{S} : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$, "self-energy"

With these ingredients we consider the matrix Dyson equation (MDE)

$$-\frac{1}{M} = z - A + \mathcal{S}[M], \tag{1}$$

for the unknown matrix-valued function $M = M(z) \in \mathbb{C}^{N\times N}$ with constraint $\operatorname{Im} M \geq 0$. I.e. it is a "renormalized" resolvent:

$$M_{bare} = \frac{1}{A - z} \qquad \Rightarrow \qquad M = \frac{1}{A - \mathcal{S}[M] - z}$$

Motivation: Disordered quantum system

Let H be a random Hamiltonian – for simplicity, $N \times N$ matrix. Write

$$H = \underbrace{A}_{bare} + \underbrace{(H-A)}_{fluct.}, \qquad A := \mathbb{E}H$$

with self energy operator

$$\mathcal{S}[R] := \mathbb{E}\Big[(H-A)R(H-A)\Big], \qquad \text{for any } R \in \mathbb{C}^{N\times N}$$

FACT[1]: the best deterministic approximation of the resolvent is M:

$$G(z) = G := \frac{1}{H-z} = \frac{1}{A+(H-A)-z} \approx M = \frac{1}{A - \mathcal{S}[M] - z}$$

Hence for the density of states we have the self-consistent approx:

$$\varrho_N(E) := \frac{1}{\pi}\mathrm{Im}\,\frac{1}{N}\mathrm{Tr}G(E+i0) \approx \frac{1}{\pi}\mathrm{Im}\,\frac{1}{N}\mathrm{Tr}M(E+i0) =: \varrho(E)$$

[1]to be proven

Examples

I. Wigner matrices:

$H = H^*$ has centered i.i.d. entries, with $\mathbb{E}|h_{ij}|^2 = \frac{1}{N}$. Then

$$A = \mathbb{E}H = 0, \qquad \mathcal{S}[R] = \mathbb{E}HRH = \Big(\frac{1}{N}\mathrm{Tr}R\Big)\cdot \mathrm{I}$$

Introducing $m = \langle M \rangle := \frac{1}{N}\mathrm{Tr}M$, the Dyson equation simplifies to a scalar equation

$$-\frac{1}{m} = z + m$$

This quadratic equation with $\mathrm{Im}\, m > 0$ has a unique solution:

$$\mathrm{Im}\, m(E + i0) = \frac{1}{2}\sqrt{(4 - E^2)_+}$$

This is the celebrated Wigner semicircle law.

The resolvent $G(z)$ is (close to) the constant $m = m(z)$

II. Wigner-type matrices:

$H = H^*$ has centered independent ~~i.d.~~ entries, with the matrix of variances $S = (s_{ij})$ given by

$$s_{ij} := \mathbb{E}|h_{ij}|^2.$$

Then

$$A = \mathbb{E}H = 0, \qquad \Big(\mathcal{S}[\text{diag}(\mathbf{r})]\Big)_{ik} = \mathbb{E}\Big[H\text{diag}(\mathbf{r})H\Big]_{ik} = \Big(\sum_j s_{ij} r_j\Big)\delta_{ik}$$

i.e. the self-energy operator maps diagonal matrices with $\mathbf{r}$ in the diagonal into diagonal matrices with $S\mathbf{r}$ in the diagonal.

One easily checks that the Dyson equation has a diagonal solution $M = \text{diag}(\mathbf{m})$, where $\mathbf{m}$ satisfies the vector Dyson equation

$$-\frac{1}{\mathbf{m}} = z + S\mathbf{m}, \qquad \mathbf{m} \in \mathbb{C}_+^N$$

(reciprocal of a vector is taken entrywise).
DOS is not the semicircle, but G is (close to) a diagonal matrix

III. Arbitrary self-adjoint random matrices:

$H = H^*$ has ~~centered~~ ~~independent~~ ~~i.d.~~ entries. All covariances of H are encoded in the covariance (self-energy) operator $\mathcal{S}$ and we have the genuine matrix Dyson equation

$$-\frac{1}{M} = z - A + \mathcal{S}[M] \qquad \text{(MDE)}$$

The solution is a genuine matrix, so is the resolvent G

Remarks.

(i) M, hence the (self-consistent) DOS, is solely determined by the first two moments of H (thus it can be computed from Gaussian)

(ii) Can be formulated in infinite dimensions and even on von-Neumann algebras [Alt-E-Kruger, 2018]

Behaviour of the density

We have proved in the vector case [Ajanki-E-Kruger '15] and in the matrix case [Alt-E-Kruger '18] that $\varrho = \langle \mathrm{Im}\, M \rangle$ is

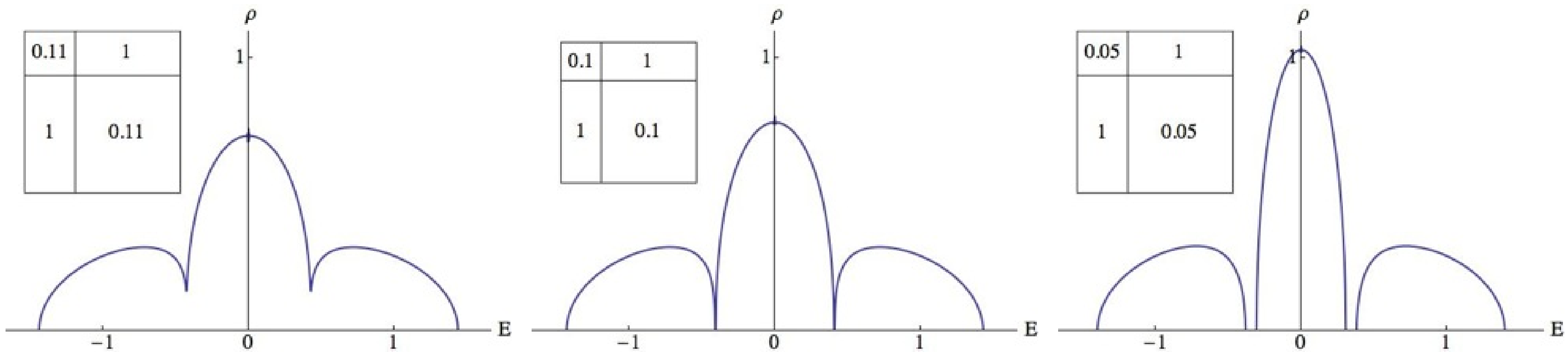
- compactly supported
- real analytic whenever it is positive
- may have only square root and cubic root (cusp) singularity

Remarkable fact: Despite being an algebraic equation of many variables, no higher order singularity can occur.

Along the proof, a canonical cubic equation emerges.

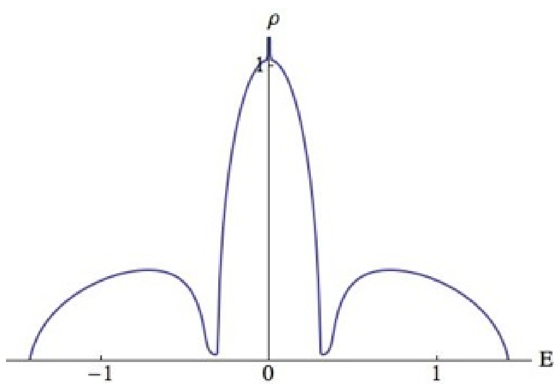
Some features of the DOS

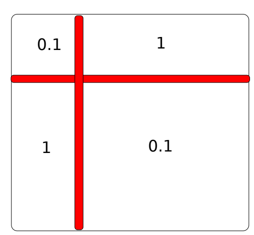
1) Support splits via cusps:



(Vector case: matrices represent the variance matrix S)

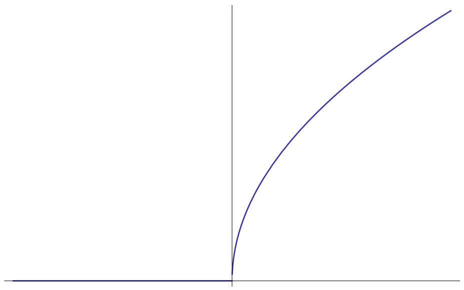
2) Smoothing of the S-profile avoids splitting ($\Rightarrow$ single interval)



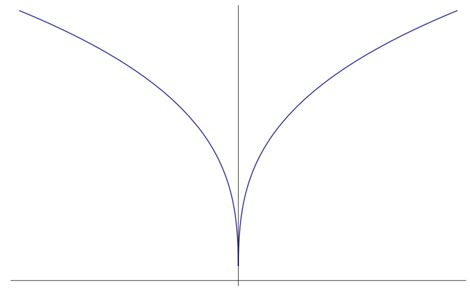


DOS of the same matrix as above but discontinuities in S are regularized

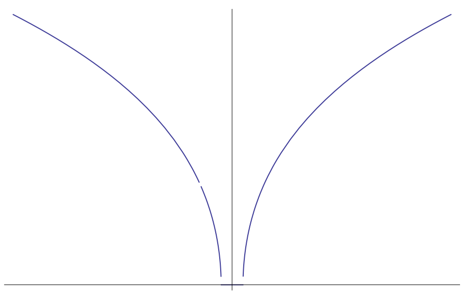
Universality of the DOS singularities



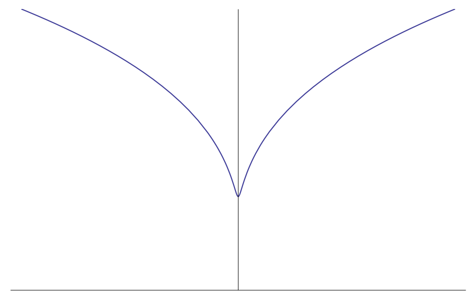
Edge, $\sqrt{E}$ singularity



Cusp, $|E|^{1/3}$ singularity



Small-gap



Smoothed cusp

$$\frac{(2+\tau)\tau}{1+(1+\tau+\sqrt{(2+\tau)\tau})^{2/3}+(1+\tau-\sqrt{(2+\tau)\tau})^{2/3}} \qquad \frac{\sqrt{1+\tau^2}}{(\sqrt{1+\tau^2}+\tau)^{2/3}+(\sqrt{1+\tau^2}-\tau)^{2/3}-1}-1$$

$$\tau := \frac{|E|}{\text{gap}}, \qquad \tau := \frac{|E|}{(\text{minimum})^3}$$

(Heuristic) derivation of MDE (A=0 for simplicity)

$$G(z) := (H-z)^{-1} \qquad I + zG = HG$$

Write it as

$$I + (z + \mathcal{S}[G])G = D \qquad \text{with} \quad D := HG + \mathcal{S}[G]G$$

Note that MDE is equivalent to the same eq. with $D = 0$

$$I + (z + \mathcal{S}[M])M = 0 \qquad (MDE)$$

To conclude $G \approx M$, need to show that

(i) D is small, i.e. G approx. satisfies MDE. (not today)
(ii) The MDE is stable. (today)

In the Gaussian case, a simple integration by parts suffices:

$$\mathbb{E}D = \mathbb{E}\Big[HG + \mathcal{S}[G]G\Big] = \mathbb{E}\Big[- \widetilde{\mathbb{E}}[\widetilde{H}G\widetilde{H}]G + \mathcal{S}[G]G\Big] = 0$$

Then: $\mathbb{E}|D|^p$ are small by tracking cancellations to all orders.

In the general case, one can use cumulant expansion

Basic conditions

- Mean field scaling

$$C_1\langle R\rangle \le \mathcal{S}[R] \le C_2\langle R\rangle, \qquad \forall R \ge 0$$

In the independent case, this means $s_{ij} = \mathbb{E}|h_{ij}|^2 \asymp \frac{1}{N}$

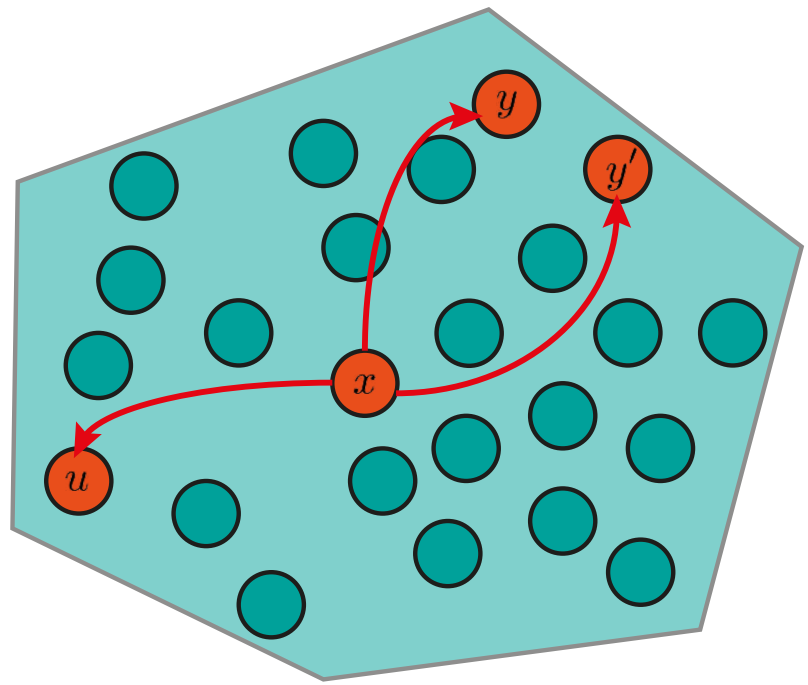
- Decay of correlation in some metric on the configuration space $\Sigma = \{1, 2, \dots, N\}$

(all constants are independent of N).

Mean field quantum Hamiltonian with correlation

Equip the conf. space Σ with a metric to talk about "nearby" states.

It is reasonable to allow that H_{xy} and $H_{xy'}$ are correlated if y and y' are close with a decaying correlation as $\mathrm{dist}(y, y')$ increases.



Non-trivial spatial structure changes the density of states.

There are other natural models leading to correlated structures (structured block matrices).

Main results on correlated random matrices: the bulk

Thm. [Ajanki-E-Krüger '16, E-Krüger-Schröder '17] Consider a general hermitian random matrix with a decaying correlation structure:

$$H = A + \frac{1}{\sqrt{N}} W$$

such that

- $\mathbb{E}W = 0$, $\|A\| \le C$
- $\mathrm{Cov}\Big(\phi(W_A), \psi(W_B)\Big) \le C[1 + \mathrm{dist}(A,B)]^{-12}$ for any $A, B \subset [N]^2$
- $\kappa(\phi_1(W_{A_1}), \phi_2(W_{A_2}), \dots) \lesssim \prod_{\{A_j, A_k\} \in T_{min}} \mathrm{Cov}(\phi_j, \phi_k)$
- $\mathbb{E}|\langle \mathbf{u}, W\mathbf{v}\rangle|^2 \ge c \qquad \forall \mathbf{u}, \mathbf{v} \quad \ell^2-$ normalized vectors

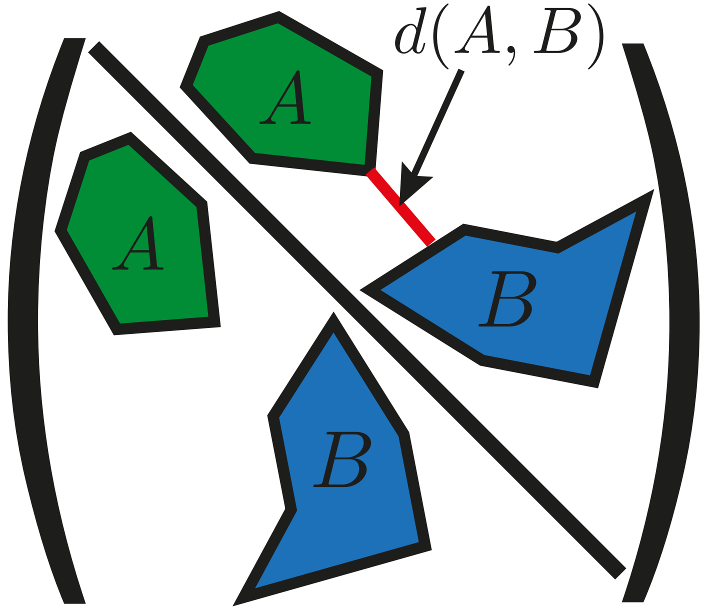
Then optimal local law and its usual corollaries hold in the bulk.

Next pages: more precisely

Distance for index sets in the condition on correlation decay of W

$$\mathrm{Cov}\Big(\phi(W_A), \psi(W_B)\Big) \le \frac{C\|\phi\|_2\|\psi\|_2}{[1+\mathrm{dist}(A,B)]^{12}}$$

for any $A, B \subset \Sigma \times \Sigma$, assuming the usual metric on the set $\Sigma = \{1, 2, \ldots, N\}$ of indices. Here $W_A = \{W_{ij} \ : \ (i,j) \in A\}$.



Clustering of higher order cumulants

The higher order cumulants of (functions of) matrix elements supported in disjoint sets $A_1, A_2, \ldots$ have a decay given by the covariance decay along a minimal spanning tree

$$\kappa(\phi_1(W_{A_1}), \phi_2(W_{A_2}), \ldots) \lesssim \prod_{\{A_j, A_k\} \in T_{min}} \mathrm{Cov}(\phi_j, \phi_k)$$

Standard property in statistical physics (high temperature regime).

Optimal local law: In the bulk spectrum, $\varrho(\Re z) \geq c$, we have

$$\langle \mathbf{v}, [G-M]\mathbf{u}\rangle \lesssim \frac{\|\mathbf{v}\|_2 \|\mathbf{u}\|_2}{\sqrt{N \mathrm{Im}\, z}}, \qquad \frac{1}{N}\,\mathrm{Tr} B(G-M) \lesssim \frac{\|B\|}{N \mathrm{Im}\, z}$$

with very high probability, where M solves the Dyson equation.

The density is given by $\varrho(E) = \frac{1}{\pi N} \mathrm{Tr}\, \mathrm{Im}\, M(E + i0)$.

These bounds are optimal and effective down to $\mathrm{Im}\, z \gg \frac{1}{N}$

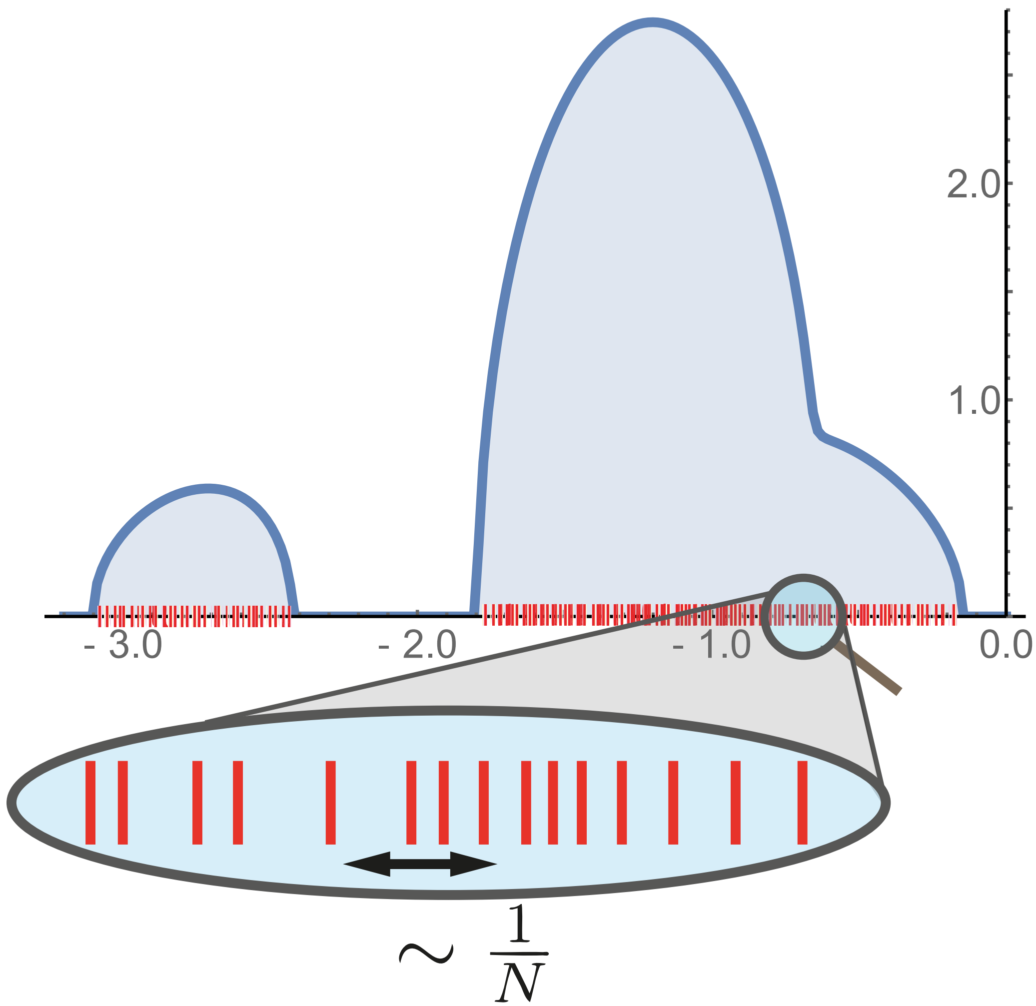
Corollaries

(i) Eigenvector delocalization: $\|\mathbf{v}\|_\infty \lesssim N^{-1/2}\|\mathbf{v}\|_2$

(ii) Eigenvalue rigidity: $|\lambda_i - \gamma_i| \lesssim \frac{1}{N}$ in bulk (γ_i – quantiles of ϱ)

(iii) Universality of local correlation functions and gaps.

Bulk universality: Dyson sine-kernel statistics holds on the level of individual eigenvalues:



Main results on correlated random matrices: the edges

Theorem [Alt-E-Krüger-Schröder, 2018]
Under the same conditions, optimal local law holds for any energy $\Re z$, including spectral edges, but away from the (possible) cusps of ϱ and blowups of M (we have conditions to exclude the latter).

Every edge (internal ones allowed) eigenvalue follows the Tracy-Widom distribution ("edge universality")

Band rigidity: if E is away from suppϱ, then

$$\Big|\text{Spec}\, H \cap (-\infty, E)\Big| = N \int_{-\infty}^{E} \varrho(x)\mathrm{d}x$$

with very high probability. The RHS is an integer — quantization.

Note: Band rigidity is much stronger than usual rigidity and it does not hold for invariant ensembles.

Long history (incomplete)

[Wigner 1954] – semicircle

[Dyson-Gaudin-Mehta 1960's] – sine kernel bulk universality for Gaussian

[Tracy-Widom 1994] – edge universality for Gaussian

[Soshnikov 1995] – edge universality for Wigner (moment method)

[Johansson 2001] – bulk universality with large Gaussian component

[E-Schlein-Yau-Yin, 2008-10] – local laws, bulk universality for Wigner, DBM

[Tao-Vu, 2009] – universality for hermitian via 4 moment comparison

[E-Knowles-Yau-Yin 2011-13] – rigidity, gen. Wigner matrices, sparse matrices

[E-Yau, 2011] – gap universality

[Bourgade-E-Yau-Yin '13, Landon-Sosoe-Yau '16] – fixed energy universality

[Bourgade-E-Yau '13] – TW for gen. Wigner, [Lee-Yin '14] optimal moments.

[Lee-Schnelli 15-16] [Knowles-Yin '15] – TW for deformed Wigner

[Landon-Yau 15-17, E-Schnelli 15] – "Black box" theorems for universality

[Huang-Landon-Yau, '17] – TW for very sparse case

[Che 16-17] – bulk and edge univ for special corr structure

Analysis of the MDE

Recall: $A = A^*$ and $\mathcal{S}$ pos. preserving map on $\mathbb{C}^{N\times N}$ are given,

$$-\frac{1}{M} = z - A + \mathcal{S}[M], \qquad \text{Im}\, z > 0, \qquad \text{Im}\, M > 0$$

Old fact: [Girko, Pastur, Wegner, Helton-Far-Speicher] The MDE has a unique solution with $\text{Im}\, M \geq 0$. It is the Stieltjes transform of a matrix-valued measure

$$M(z) = \frac{1}{\pi} \int \frac{V(\omega)\mathrm{d}\omega}{\omega - z}, \qquad z \in \mathbb{C}_+$$

Clearly, the (self-consistent) density of states is given by

$$\varrho(\omega) := \frac{1}{\pi N} \mathrm{Tr} V(\omega), \qquad \omega \in \mathbb{R}$$

Stability operator of the MDE

$$\mathcal{L} := I - M\mathcal{S}[\cdot]M$$

KEY ISSUE: invertibility of $\mathcal{L}$.

Mechanism for stability I. Vector case

$$-\frac{1}{\mathbf{m}} = z + S\mathbf{m}, \qquad S = (s_{ij}), \qquad \mathbf{m} = (m_i)$$

Why is $L = 1 - \mathbf{m}^2 S$ invertible at all? [here $(\mathbf{m}^2 S)_{ij} := m_i^2 s_{ij}$]

Take Im-part and symmetrize

$$\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|} = \eta|\mathbf{m}| + |\mathbf{m}|S|\mathbf{m}|\,\frac{\operatorname{Im}\mathbf{m}}{|\mathbf{m}|}$$

Since $\operatorname{Im}\mathbf{m} \ge 0$, by Perron-Frobenius, $F := |\mathbf{m}|S|\mathbf{m}| \le 1 - c\eta$

Lemma. If F is self-adjoint with $Ff = \|F\|f$ and a gap, then

$$\left\|\frac{1}{U - F}\right\| \le \frac{C}{\mathrm{Gap}(F)\,\Big|1 - \|F\|\langle f, Uf\rangle\Big|}, \qquad \text{for any } U \text{ unitary}$$

Thus, we have stability with $\mathbf{m} = e^{i\varphi}|\mathbf{m}|$

$$\left\|\frac{1}{1 - \mathbf{m}^2 S}\right\| = \left\|\frac{1}{e^{-2i\varphi} - F}\right\| \le \frac{C}{\min(\sin\varphi_j)^2}$$

Mechanism for stability II. Matrix case

Lemma: $M = M(z)$ be the solution to MDE, then

$$\Big\|\frac{1}{1-M\mathcal{S}[\cdot]M}\Big\| = \Big\|\frac{1}{1-\mathcal{C}_M\mathcal{S}}\Big\| \le \frac{C}{[\varrho(z)+\mathrm{dist}(z,\mathrm{supp}\varrho)]^2}$$

with C depending on M in a controlled way. Here $\mathcal{C}_M[T] := MTM$.

Key: find the "right" symmetrization $\mathcal{F}$ despite the noncommutative matrix structure.

Need the analogue of

$$\mathbf{m} = e^{i\boldsymbol{\varphi}}|\mathbf{m}|, \qquad F = |\mathbf{m}|S|\mathbf{m}|, \qquad |1-\mathbf{m}^2S| = |e^{-2i\boldsymbol{\varphi}} - F|$$

We needed that F is symmetric (for spectral analysis of $U-F$), positivity preserving (for Krein-Rutman), and

$$\frac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|} = \eta|\mathbf{m}| + |\mathbf{m}|S\Big(|\mathbf{m}|\frac{\mathrm{Im}\,\mathbf{m}}{|\mathbf{m}|}\Big)$$

Bringing to matrix Perron-Frobenius form

Try to write the equation for $\operatorname{Im} M$

$$\operatorname{Im} M = \eta M^* M + M^* \mathcal{S}(\operatorname{Im} M) M$$

with some Q as ($\eta = 0$)

$$X = Y^* Q^* \mathcal{S}[QXQ^*] QY, \quad \text{with} \quad X := \frac{1}{Q} (\operatorname{Im} M) \frac{1}{Q^*}, \quad Y := \frac{1}{Q} M \frac{1}{Q^*}$$

i.e.

$$X = Y^* \mathcal{F}[X] Y, \qquad \text{with} \quad \mathcal{F}[\cdot] := Q^* \mathcal{S}[Q \cdot Q^*] Q$$

Notice that $X = \operatorname{Im} Y$ and if Y is unitary, then X and Y commute,

$$X = \mathcal{F}[X]$$

so Perron-Frobenius applies and $\mathcal{F} = \mathcal{F}^*$ is bounded. We get

$$I - \mathcal{C}_M \mathcal{S} \text{ is stable} \iff I - \mathcal{C}_Y \mathcal{F} \text{ is stable}$$

All we need is a "balanced polar decomposition" of $M = QYQ^*$.

Balanced Polar Decomposition of M

Goal: $M = QYQ^*$, Y unitary, $|Q| \sim 1$

Explicitly: use that $M = A + iB$ and $B > 0$ to write

$$M = \sqrt{B}\Big(\frac{1}{\sqrt{B}}A\frac{1}{\sqrt{B}} + i\Big)\sqrt{B}$$

and make the middle factor unitary by dividing its absolute value:

$$M = \sqrt{B}WYW\sqrt{B} =: QYQ^*$$

$$W := \left[1 + \Big(\frac{1}{\sqrt{B}}A\frac{1}{\sqrt{B}}\Big)^2\right]^{\frac{1}{4}}, \quad Y := \frac{\frac{1}{\sqrt{B}}A\frac{1}{\sqrt{B}} + i}{W^2}$$

In the regime, where $c \leq B \leq C$ and $\|A\| \leq C$, we have

$$Q = \sqrt{B}W \sim 1$$

Finally: Noncommutative generalization of the Stability Lemma

$$\left\| \frac{1}{1 - \mathcal{C}_M \mathcal{S}} \right\|_{sp} \lesssim \left\| \frac{1}{\mathcal{U} - \mathcal{F}} \right\|_{sp} \lesssim \frac{1}{\text{Gap}(\mathcal{F}) \ \left| 1 - \|\mathcal{F}\| \langle F, \mathcal{U}(F) \rangle \right|}$$

with $\mathcal{U} = \mathcal{C}_U$, then we prove that (noncommutative!)

$$|1 - \|\mathcal{F}\| \langle F, UFU \rangle| \geq c\rho^2, \qquad \text{Gap}(\mathcal{F}) \geq c$$

if $\rho = \langle \text{Im}\, M \rangle > 0$ (i.e. in the bulk). □

Remark: This proof also gives Hölder-$\frac{1}{3}$ regularity for ρ:

$$\frac{d}{dz}(MDE) \quad \Rightarrow \quad \mathcal{L}(\partial_z M) = -M^2$$

thus

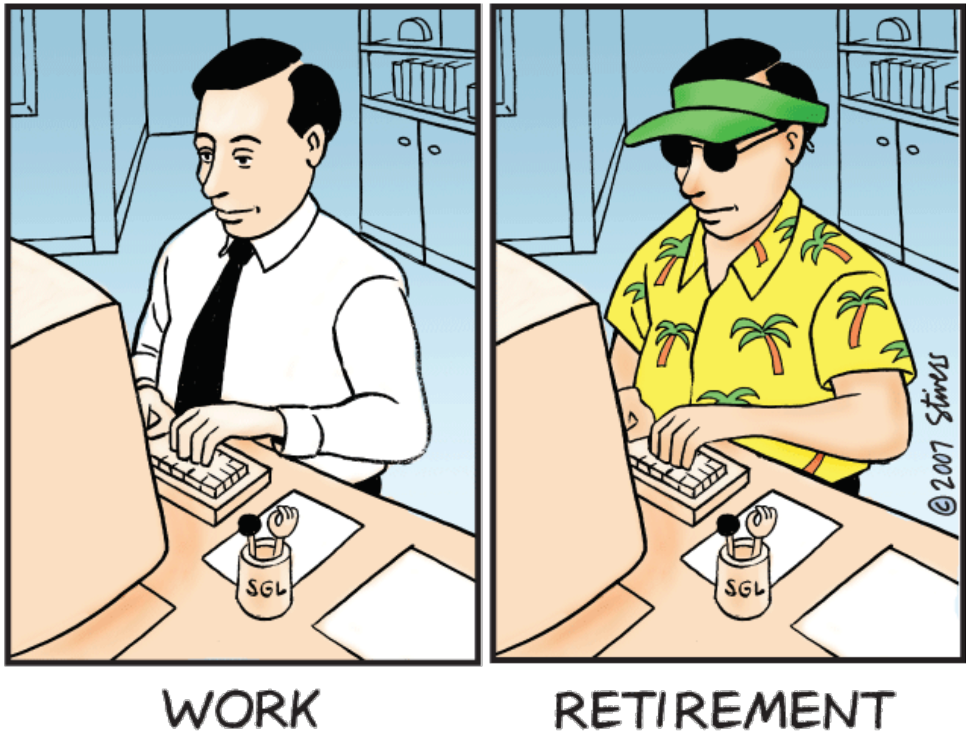
$$\partial_z \rho = -\langle \text{Im}\, \mathcal{L}^{-1}(M^2) \rangle \lesssim \rho^{-2}$$

SUMMARY

- Quantitative analysis of the solution of the Matrix Dyson Equation and its stability. Universality of singularities.

- For correlated random matrices with slow correlation decay in both symmetry classes we proved
  - Optimal local law (rigidity, delocalization)
  - Band rigidity
  - Wigner-Dyson-Mehta bulk universality
  - Tracy-Widom edge universality

Happy Emeritierung, Heinz



Vielen Dank für alles!