On the wave turbulence theory for a stochastic KdV type equation

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OUTLINE OF THE TALK

1. **Brief introduction to wave turbulence**

2. **Sketch of the proof - How to handle the difficulties**

BRIEF INTRODUCTION TO WAVE TURBULENCE

Wave Turbulence: The Physical History

What is Wave Turbulence?

Wave Turbulence: non-equilibrium statistical system of many randomly interacting waves. Kinetic equations of Wave Turbulence describe evolution of the wave energy in Fourier space.

- Wave Equations $\longrightarrow$ Waves
- Kinetic Equations $\longrightarrow$ Particles
- Wave Turbulence (Wave Kinetic Equations)$\longrightarrow$ Using Kinetic Equations to describe (Weak) Nonlinear Waves

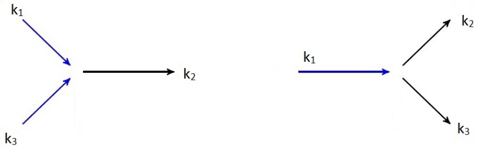
Physical History: Formal Derivations + Applications

- Origin in the works of Peierls (1929) and Hasselmann (1961)
- Modern point of view Benney-Saffman-Newell (1966), Zakharov (1966)
- Recent developments Newell, Zakharov, L'vov, Pomeau, Nazarenko and many others
- Vast range of application:
  - Inertial waves due to rotation
  - Alfvén wave turbulence in the solar wind
  - Waves in plasmas of fusion devices
  - Oceanography and climate science
  - Quantum physics (Pomeau's work)

  and many others

Brief ideas

- Brief ideas: Using kinetic equations to describe weakly nonlinear waves
- Given a wave equation whose nonlinear is quadratic
  $\partial_t\phi(x,t) \;=\; -\Delta\partial_{x_1}\phi(x,t) \;+\; \lambda\partial_{x_1}(\phi^2(x,t)), \qquad x=(x_1,\cdots,x_d)$
  The equation is normally served as the first example for which a 3-wave kinetic equation is derived.[1]
- We obtain a **3**-wave kinetic equation in the Fourier space.

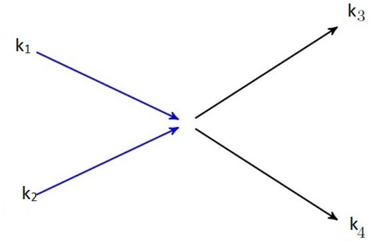


- In the nonlinear wave dynamics, there several type of wave interactions:
  - Any kinds of wave interactions are possible
  - The dominance among those interactions are the above **3**-wave interactions $\longrightarrow$ **3**-wave kinetic equation

[1] Many physical applications concerning drift waves in fusion plasmas and Rossby waves in geophysical fluids, ionic-sonic waves in a magnetized plasma (Nazarenko's book and Zakharov-Kurtnesov 1974). Rigorous derivation by Saut-Lannes 2013.

Brief ideas

- Brief ideas: Using kinetic equations to describe weakly nonlinear waves
- Given a wave equation whose nonlinear is cubic
  (Example: $i\partial_t\psi(x,t) + \Delta\psi(x,t) \ = \ \lambda|\psi(x,t)|^2\psi(x,t)$)
- We obtain a **4**-wave kinetic equation in the Fourier space.



- In the nonlinear wave dynamics, there several type of wave interactions:
  - Any kinds of wave interactions are possible
  - The dominance among those interactions are the above **4**-wave interactions $\longrightarrow$ **4**-wave kinetic equation

Wave Turbulence: The Modern Dispersive PDEs Context

We consider the KdV (KZ) equation in d-dimension

$$\partial_t \phi(x,t) = -\Delta \partial_{x_1} \phi(x,t) + \lambda \partial_{x_1} \left(\phi^2(x,t) \right), \qquad x = (x_1, \cdots, x_d)$$
$$\phi(x,0) = \phi_0(x), x \in \mathbb{T}_L^d : \text{ periodic torus } [0,L]^d.$$

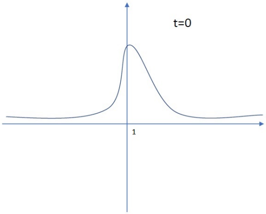
Terence Tao's blog about energy cascade

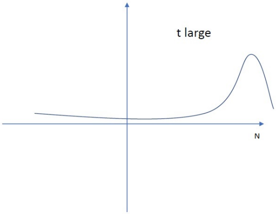
To illustrate how this can happen, let us normalise the torus as $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$. A simple example of a frequency cascade would be a scenario in which solution $\phi(x,t) = \phi(x_1, x_2, t)$ starts off at a low frequency at time zero, e.g. $\phi(x,0) = Ae^{ix_1}$ for some constant amplitude A, and ends up at a high frequency at a later time T, e.g. $\phi(x,T) = Ae^{iNx_1}$ for some large frequency N.

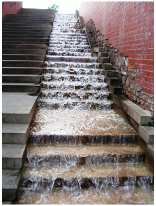
- Energy: $\int |\phi(x,0)|^2 = \int |Ae^{ix_1}|^2 = A^2 = \int |Ae^{iNx_1}|^2 = \int |\phi(x,T)|^2$
- In this example, the energy of ϕ is conserved but the energy goes from frequency 1 to frequency N.
- Energy Cascade Conjecture (Bourgain's 2000): Migration of energy from low to high frequency.

Energy Cascade Conjecture (Bourgain's 2000):

Give a solution $\phi(x, t)$ to a dispersive PDE on a compact manifold M, does a migration of energy occurs from low frequencies to high frequencies?







Two different approaches

Given $\phi(x,t)$: solution of a dispersive equation.
Approach 1: We study

$$\sum_k |\hat{\phi}(k,t)|^2 \langle k \rangle^{2s} = \|\hat{\phi}(t)\|_{H^s}^2, \quad \lim_{t\to\infty} \|\hat{\phi}(t)\|_{H^s}^2$$

- PDE Approach: Bourgain, Staffilani, Colliander-Keel-Staffilani-Takaoka-Tao, Kuksin, Sohinger, Deng-Germain, Carles-Fau, Staffilani-Wilson, Plachon-Tzvetkov-Visciglia ...
- Computational Approach: Pan,...
- Dynamical System Approach: Haise-Procesi, Berti-Maspero,...

Two different approaches

Given $\phi(x,t)$: solution of a dispersive equation.
Approach 2:
Set $a_k = \hat{\phi}(k,t)$ and $n(k,\tau) = |a_k(t)|^2$. At the van Hove limit/ kinetic time

$$t = \tau\lambda^{-2} = \mathcal{O}(\lambda^{-2})$$

derive the wave kinetic equation

$$\partial_t |a_k(t)|^2 \;=\; \lambda^2 Q[|a_k(t)|^2] + \mathcal{O}(\lambda^{2+\delta}) \longrightarrow \partial_\tau n(k,\tau) \;=\; Q[n(k,\tau)] + \mathcal{O}(\lambda^{\delta})$$

From Dispersive Equations to Kinetic Equations

Two different approaches: Second Approach

Dispersive Equation $\phi(x,t) \longrightarrow$ Kinetic equation $(n(k,\tau) \to |\hat{\phi}(k,t)|^2)$.

$$\partial_t \phi(x,t) \; = \; -\Delta\partial_{x_1}\phi(x,t) \; + \; \lambda\partial_{x_1}\left(\phi^2(x,t)\right)$$

Set $|\hat{\phi}(k,t)|^2 \to n(k,\tau)$, $\longrightarrow$ **homogeneous kinetic equation**

$$\partial_\tau n(k,\tau) \; = \; Q[n(k,\tau)],$$

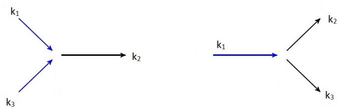
$$Q(n)(k_1) = \int \mathrm{d}k_2 \mathrm{d}k_3 |\mathcal{W}(k_1,k_2,k_3)|^2 \delta(\omega(k_3)+\omega(k_2)-\omega(k_1))$$
$$\times\,\delta(k_2+k_3-k_1)\Big(n_2 n_3 - n_1 n_2 \mathrm{sign}(k_1^1)\mathrm{sign}(k_3^1) - n_1 n_3 \mathrm{sign}(k_1^1)\mathrm{sign}(k_2^1)\Big),$$

where $n_1(\tau) = n(\tau,k_1)$, $n_2(\tau) = n(\tau,k_2)$, $n_3(\tau) = n(\tau,k_3)$.

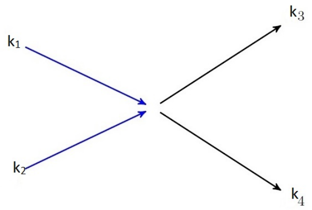
- $\omega = k^1[|k^1|^2 + \cdots + |k^d|^2]$, $k = (k^1,\cdots,k^d)$.
- $\partial_{x_1}\left(\phi^2(x,t)\right)$ quadratic nonlinearity $\longrightarrow Q(n)(k_1)$ quadratic collision operator.

Recalling

Given a wave equation whose nonlinear is **quadratic**, we obtain a 3-wave kinetic equation in the Fourier space.



Given a wave equation whose nonlinear is **cubic**, we obtain a 4-wave kinetic equation in the Fourier space.



Mathematical Literature: Rigorous Derivations

- Erdos-Yau (CPAM 2000), Erdos-Salmhofer-Yau (Acta Math 2008): Random Linear Schrödinger $\rightarrow$ linear Boltzmann (kinetic time) $\rightarrow$ heat equation (diffusion time, $t = \mathcal{O}(\lambda^{-2-\epsilon})$
- Lukkarinen-Spohn (Invent Math 2010): Random Cubic Nonlinear Schrödinger at equilibrium $\rightarrow$ (linearized) wave kinetic equation at (kinetic time).

Recent results - Out of Equilibrium Case

Random initial data

- Buckmaster-Germain-Hani-Shatah (CPAM 2019, Invent Math 2021) $\longrightarrow$ homogeneous wave kinetic equation from NLS: strictly below kinetic time (linear kinetic equation).

$$\partial_\tau n(k,\tau) \;=\; Q[n(k,0)]$$

- Collot-Germain (2019, 2020), Deng-Hani (Forum Pi, 2021) $\longrightarrow$ homogeneous wave kinetic equation from NLS: strictly below kinetic time (linear kinetic equation).
- Ampatzoglou-Collot-Germain 2021, Inhomogeneous kinetic equation, strictly below kinetic time (linear kinetic equation) from NLS :

Stochastic PDEs

- Dymov, Kuksin and collaborators (2019-2021), Faou (CMP 2020) - from KdV.

Random matrix model

- Dubach, Germain, Harrop-Griffiths, 2022 strictly below kinetic time (linear kinetic equation)

Recent results - Out of Equilibrium Case

- (2021) Derivation for the homogeneous nonlinear 4-wave kinetic equation (kinetic time): Deng-Hani - from random NLS (continuum setting)
  $i\partial_t\psi(x,t) + \Delta\psi(x,t) \;=\; \lambda|\psi(x,t)|^2\psi(x,t)$ on the periodic torus $[0,L]^d$, $d \geq 3$, (kinetic time $\approx \lambda^{-2} \approx L^2$), when $L \to \infty$, $\lambda \to 0$
  $\longrightarrow$ propagation of chaos
- (2021) Derivation for the homogeneous nonlinear 3-wave kinetic equation (kinetic time): Staffilani-MBT - from stochastic KdV (lattice setting)
  $\partial_t\phi(x,t) \;=\; -\Delta\partial_{x_1}\phi(x,t) \;+\; \lambda\partial_{x_1}(\phi^2(x,t))$, on the periodic torus $[0,L]^d$, $d \geq 2$, $L \to \infty$, $\lambda \to 0$, λ is independent of L
- (2022) Derivation for the inhomogeneous nonlinear 3-wave kinetic equation (kinetic time) : Hannani-Rosenzweig-Staffilani-MBT - from KdV (lattice setting)

$$\partial_\tau n(r,k,\tau) \;+\; \nabla\omega(k)\cdot\nabla_r n(r,k,\tau) = \; Q[n(r,k,\tau)],$$

All of those works are based on the pioneering works of Spohn, Erdos-Yau, Erdos-Salmhofer-Yau and Lukkarinen-Spohn.

Sketch of the proof - How to handle the difficulties

(I) The setting

(II) The density function - Using the Louville equation to replace the l^1 clustering estimate

(III) Feynman's diagrams

(IV) Crossing estimates (unfortunately do not hold true due to a counter example by Lukkarinen JMPA, 2007) - Establishing new types of crossing estimates in Fourier spaces

(V) The effect of the noise on the Feynman diagrams

(VI) The divergence of the leading diagrams - The resonance broadening technique

(I) The setting

The setting

The KZ (KdV) equation

$$\begin{aligned} \mathrm{d}\psi(x,t) &= -\Delta\partial_{x_1}\psi(x,t)\mathrm{d}t + \lambda\partial_{x_1}\left(\psi^2(x,t)\right)\mathrm{d}t + \sqrt{2c_r}\partial_{x_1}\psi\odot\mathrm{d}W(t), \\ \psi(x,0) &= \psi_0(x), \end{aligned}$$

$c_r = \mathfrak{C}_r\lambda^{\theta_r}$, for some universal constants $\mathfrak{C}_r > 0$ and $1 \geq \theta_r > 0$ is small but non-zero. The lattice system can be rewritten in the Fourier space as a system of ODEs

$$\begin{aligned} \mathrm{d}\hat{\psi}(k,t) &= \mathbf{i}\omega(k)\hat{\psi}(k,t)\mathrm{d}t + \mathbf{i}\bar{\omega}(k)\sqrt{2c_r}\hat{\psi}(k,t)\circ\mathrm{d}W(t) \\ &\quad + \mathbf{i}\lambda\bar{\omega}(k)\frac{1}{|\Lambda_*|^2}\sum_{k=k_1+k_2;k_1,k_2\in\Lambda_*}\hat{\psi}(k_1,t)\hat{\psi}(k_2,t), \\ \hat{\psi}(k,0) &= \hat{\psi}_0(k), \end{aligned}$$

$\mathbf{i}\bar{\omega}(k)\sqrt{2c_r}\hat{\psi}(k,t)\circ\mathrm{d}W(t)$ is the standard Stratonovich product. The mesh and the dispersion relation are

$$\Lambda_* = \Lambda_*(L) = \left\{-\frac{L}{2L+1},\cdots,0,\cdots,\frac{L}{2L+1}\right\}^d,$$

$$\omega_k = \omega(k) = \sin(2\pi k^1)\left[\sin^2(2\pi k^1)+\cdots+\sin^2(2\pi k^d)\right], \quad \bar{\omega}(k) = \sin(2\pi k^1),$$

with $k = (k^1,\cdots,k^d)$.

The setting

$$\omega_k = \omega(k) = \sin(2\pi k^1)\Big[\sin^2(2\pi k^1) + \cdots + \sin^2(2\pi k^d)\Big], \quad k = (k^1, \cdots, k^d).$$

- When $k^1 = 0$, then $\omega(k) = 0$. We call the degenerate surface for which $k^1 = 0$ the *ghost manifold*. If we take $k_1, \cdots, k_m$ in the ghost manifold, it follows that $\omega(k_1) = \cdots = \omega(k_m) = 0$. Moreover, the sum vector $k_1 + \cdots + k_m$ is also in the ghost manifold, leading to $\omega(k_1 + \cdots + k_m) = \omega(k_1) + \cdots + \omega(k_m)$. On the ghost manifold, not just 3-wave interactions, any m-wave interactions are also allowed, with $m \geq 3$.
- The equation has a resonance broadening effect, thus all quasi-resonance m-wave interactions can also happen in a small neighborhood near the ghost manifold $\longrightarrow$ destroying the structure of 3-wave interactions and there is no 3-wave kinetic equation for the ZK equation without noise (Unlike the NLS case where a result without noise is expected).
- There is no crossing estimates due to a counter example by Lukkarinen (2007).
- The noise does not add energy into the system (L^2 norm), it has the role of only fixing the singularties of the dispersion relation. It vanishes on most pairing diagrams, including all the ladder (leading) and crossing diagrams $\longrightarrow$ The weak noise does not compete with the weak nonlinearity.

(II) The density function

The density function

We now set

$$a_k = \frac{\hat{\psi}(k)}{\sqrt{|\bar{\omega}(k)|}},$$

$$\begin{aligned} \mathrm{d}a_k &= \mathbf{i}\omega(k)a_k\mathrm{d}t + \mathbf{i}\sqrt{2c_r}a_k \circ \mathrm{d}W_k(t) \\ &+ \mathbf{i}\lambda \int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathrm{d}k_2 \mathrm{sign}(k^1)\sqrt{|\bar{\omega}(k)\bar{\omega}(k_1)\bar{\omega}(k_2)|}\delta(k - k_1 - k_2)a_{k_1}a_{k_2}\mathrm{d}t. \end{aligned}$$

$B_{1,k}$ and $B_{2,k}$ such that $a_k = B_{1,k} + \mathbf{i}B_{2,k}$. Since $B_{1,k}$ and $B_{2,k}$ are random variables, we use the variables $b_{1,k}$, $b_{2,k}$ to present their roles in the density function.

$$\begin{aligned} \frac{\partial}{\partial t}\varrho &= c_r \sum_{k\in\Lambda^*}\left(b_{2,k}\frac{\partial}{\partial b_{1,k}} - b_{1,k}\frac{\partial}{\partial b_{2,k}}\right)^2 \varrho + \sum_{k\in\Lambda^*}\omega_k\left(b_{2,k}\frac{\partial}{\partial b_{1,k}} - b_{1,k}\frac{\partial}{\partial b_{2,k}}\right)\varrho \\ &- \lambda\{\mathcal{H}_2, \varrho\}. \end{aligned}$$

The density function

By the change of variables, $b_{1,k} + \mathbf{i} b_{2,k} = \sqrt{2c_{1,k}} e^{\mathbf{i} c_{2,k}}$, with $c_{1,k} \in \mathbb{R}_+$ and $c_{2,k} \in [-\pi, \pi]$, we obtain the optimal transport equation

$$\partial_t \varrho = -\sum_{k \in \Lambda^*} \omega_k \partial_{c_{2,k}} \varrho + c_r \sum_{k \in \Lambda^*} \partial^2_{c_{2,k}} \varrho + \sum_{k \in \Lambda^*} \lambda \mathfrak{H}^a(k) \partial_{c_{1,k}} \varrho + \sum_{k \in \Lambda^*} \lambda \mathfrak{H}^b(k) \partial_{c_{2,k}} \varrho,$$

in which

$$\mathfrak{H}^a(k) = \int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathrm{d}k_2 \mathcal{M}(k, k_1, k_2) \sqrt{2 c_{1,k_1} c_{1,k_2} c_{1,k}} \Big[\delta(k - k_1 - k_2) \sin(c_{2,k_1} + c_{2,k_2} - c_{2,k}) + 2\delta(k + k_1 - k_2) \sin(-c_{2,k_1} + c_{2,k_2} - c_{2,k}) \Big],$$

and

$$\mathfrak{H}^b(k) = -\int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathrm{d}k_2 \mathcal{M}(k, k_1, k_2) \sqrt{c_{1,k_1} c_{1,k_2}} \Big[\delta(k - k_1 - k_2) \sin(c_{2,k_1} + c_{2,k_2}) + 2\delta(k + k_1 - k_2) \sin(-c_{2,k_1} + c_{2,k_2}) \Big] \frac{\sin(c_{2,k})}{\sqrt{2c_{1,k}}}$$
$$- \int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathcal{M}(k, k_1, k_2) \delta(k - k_1 - k_2) \sqrt{c_{1,k_1} c_{1,k_2}} \Big[\delta(k - k_1 - k_2) \cos(c_{2,k_1} + c_{2,k_2}) + 2\delta(k + k_1 - k_2) \cos(-c_{2,k_1} + c_{2,k_2}) \Big] \frac{\cos(c_{2,k})}{\sqrt{2c_{1,k}}}.$$

The density function and the noise

Definition

For any observable $F : \mathbb{R}^{2|\Lambda^*|} \to \mathbb{C}$, we define the average

$$\langle F \rangle \; = \; \langle F \rangle_t \; = \; \int_{\mathbb{R}^{2|\Lambda^*|}} \mathrm{d}b_1 \mathrm{d}b_2 F(b_1, b_2) \varrho(t, b_1, b_2).$$

Define

$$\mathfrak{P} \; = \; \exp\left(c_{\mathfrak{P}} \int_{\Lambda^*} \mathrm{d}k c_{1,k} \right),$$

for any $c_{\mathfrak{P}} \in \mathbb{R}$.

Proposition (Moment bounds)

Let m be an arbitrary positive natural number. Let $\{k_{i_1}, \cdots, k_{i_m}\}$ be a subset of Λ^*, σ_{i_j} with $j \in \{1, \cdots, m\}$ be either 1 or -1. We then have

$$\int_{(\Lambda^*)^m} \prod_{j=1}^{m} \mathrm{d}k_{i_j} \left| \left\langle a_{k_{i_1}, \sigma_{i_1}} \cdots a_{k_{i_m}, \sigma_{i_m}} \right\rangle_t \right|^2 \; \lesssim \; |2/c_{\mathfrak{P}}|^m \left| \int_{(\mathbb{R}_+ \times [-\pi, \pi])^{|\Lambda^*|}} \mathrm{d}c_1 \mathrm{d}c_2 \mathfrak{P} \varrho(0) \right|^2,$$

where the constant in the inequality is universal,$a(k, 1, t) \; = \; a_k^*(t)$, $a(k, -1, t) \; = \; a_k(t)$. The inequality holds true for both cases $c_r = 0$ and $c_r > 0$.

The density function

Some comments

- In the pioneering work of Lukkarien and Spohn, the control comes from imposing the assumption on the l^1-clustering estimate at equilibrium. Indeed, one of the main technical obstacles that enforces Lukkarinen and Spohn to put the system at equilibrium is the difficulty in having the l^1-clustering estimate out of equilibrium.
- To obtain the l^2 moment estimate, we focus on the analysis of the Liouville equation of the density function.

(III) Feynman's diagrams

Feynman's diagrams

We recall

$$
\begin{aligned}
\mathrm{d}a_k \; &= \mathbf{i}\omega(k)a_k\mathrm{d}t \; + \; \mathbf{i}\sqrt{2c_r}a_k \circ \mathrm{d}W_k(t) \\
&\quad + \; \mathbf{i}\lambda \int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathrm{d}k_2 \mathrm{sign}(k^1)\sqrt{|\bar{\omega}(k)\bar{\omega}(k_1)\bar{\omega}(k_2)|}\delta(k-k_1-k_2)a_{k_1}a_{k_2}\mathrm{d}t.
\end{aligned}
$$

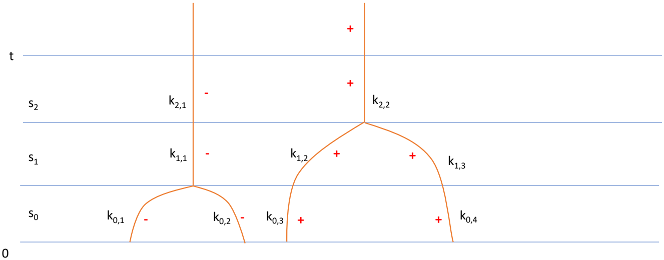
In order to absorb the quantity $\mathbf{i}\omega(k)a_k\mathrm{d}t$, we set

$$
\begin{aligned}
\alpha(k,1,t) \; &= \; a^*(k,t)e^{\mathbf{i}\omega(k)t}, \alpha(k,-1,t) \; = \; a(k,t)e^{-\mathbf{i}\omega(k)t}, \\
\mathrm{d}\alpha_t(k,\sigma) \; &= -\,\mathbf{i}\sigma\lambda \int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathrm{d}k_2 \delta(k-k_1-k_2) \times \\
&\quad \times \mathcal{M}(k,k_1,k_2)\alpha_t(k_1,\sigma)\alpha_t(k_2,\sigma)e^{\mathbf{i}t\sigma(-\omega(k_1)-\omega(k_2)+\omega(k))}\mathrm{d}t \\
&\quad - \; \mathbf{i}\sqrt{2c_r}\alpha_t(k,\sigma) \circ \mathrm{d}W_k(t),
\end{aligned}
$$

$\sigma = \pm 1$

The effect of the noise on the Feynman diagrams

$$
\begin{aligned}
\mathrm{d}\alpha_t(k,1) \;=&\; -\,\mathbf{i}\lambda\int_{\Lambda^*}\mathrm{d}k_1\int_{\Lambda^*}\mathrm{d}k_2\delta(k-k_1-k_2)\mathcal{M}(k,k_1,k_2)\times\\
&\times\,\alpha_t(k_1,1)\alpha_t(k_2,1)e^{\mathbf{i}t(-\omega(k_1)-\omega(k_2)+\omega(k))}\mathrm{d}t-\;\mathbf{i}\sqrt{2c_r}\alpha_t(k,1)\circ\mathrm{d}W_k(t),\\
\mathrm{d}\alpha_t(k,-1) \;=&\; \mathbf{i}\sigma\lambda\int_{\Lambda^*}\mathrm{d}k_1\int_{\Lambda^*}\mathrm{d}k_2\delta(k-k_1-k_2)\mathcal{M}(k,k_1,k_2)\times\\
&\times\,\alpha_t(k_1,-1)\alpha_t(k_2,-1)e^{-\mathbf{i}t(-\omega(k_1)-\omega(k_2)+\omega(k))}\mathrm{d}t\;-\;\mathbf{i}\sqrt{2c_r}\alpha_t(k,-1)\circ\mathrm{d}W_k(t),
\end{aligned}
$$



Feynman's diagrams

The leading diagrams (ladder diagrams) are obtained by iteratively applied the following graphs

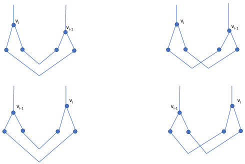


Figure: Each uses two cluster vertices.

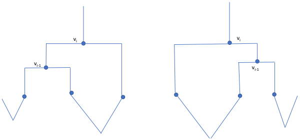
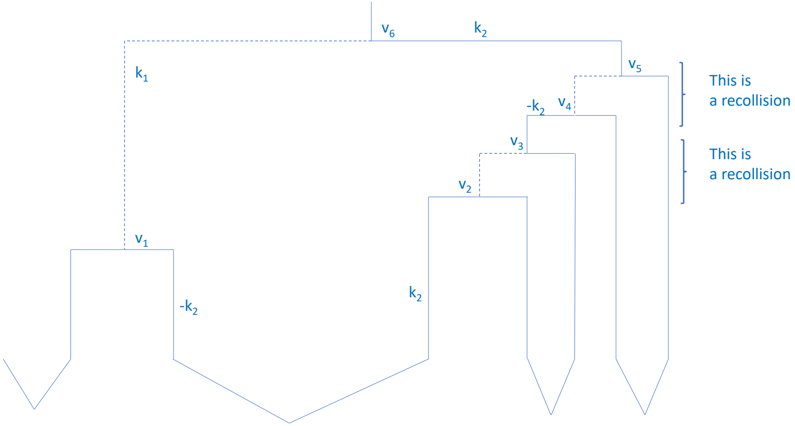


Figure: Each uses only one cluster vertex.

Feynman's diagrams

The leading diagrams (ladder diagrams)



Feynman's diagrams

- Following Erdos-Yau, we have to eliminate the other graphs (crossing and nested). We use crossing estimates, that unfortunately do not hold true due to a counter example by Lukkarinen (JMPA, 2007)
- The leading diagrams diverge by the same reason.
- $\longrightarrow$ The problem is to restore the convergence of the leading diagrams and to reestablish the crossing estimates.

(IV) Crossing estimates

Crossing estimates for linear Schrödinger equation with a random potential

For the linear Schrödinger equation with a random potential, the estimate takes the form

$$\sup_{(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3}\int_{(\mathbb{T}^d)^2}\frac{\mathrm{d}k_1\mathrm{d}k_2}{|\alpha_1-\omega(k_1)+\mathbf{i}\lambda^2||\alpha_2-\omega(k_2)+\mathbf{i}\lambda^2||\alpha_3-\omega(k_1-k_2+k_0)+\mathbf{i}\lambda^2|}$$
$$\lesssim \langle \ln\lambda\rangle^{\gamma_a}\lambda^{\gamma_b},$$

The validity of the corresponding estimate in the earlier continuum Schrödinger setting, with

$$\omega(k)=|k|^2,\quad k\in\mathbb{R}^d \tag{1}$$

is fairly straightforward to prove, but the lattice case turns out to be much more involved since

$$\omega(k)=\sin^2(2\pi k^1)+\cdots+\sin^2(2\pi k^d),\qquad \text{for } k=(k^1,\cdots,k^d). \tag{2}$$

Crossing estimates for linear Schrödinger equation with a random potential

The bound has been proved

- $\gamma_b = -4/5$ and $\gamma_a = 2$ by Chen (JSP 2005)
- $\gamma_b = -3/4$ and $\gamma_a = 6$ by Erdos-Salmhofer-Yau (Acta Math 2008)

Lukkarinen's theorem (JMPA 2007)

An analytic dispersion relation suppresses crossings if and only if it is not a constant on any affine hyperplane.

Lukkarinen's counterexample

A counterexample, in which the crossing estimate fails to hold true, has also been introduced, which unfortunately covers the lattice ZK dispersion relation

$$\omega_k \;=\; \omega(k) \;=\; \sin(2\pi k^1)\Big[\sin^2(2\pi k^1) + \cdots + \sin^2(2\pi k^d)\Big], \quad \text{for } k = (k^1, \cdots, k^d).$$

Crossing estimates for the nonlinear Schrödinger equation in the lattice setting

Crossing estimates in the work of Lukkarinen-Spohn

In the context of the lattice nonlinear Schrödinger equation, the crossing estimate takes the form

$$\sup_{(\alpha_1,\alpha_2)\in\mathbb{R}^2}\int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\mathrm{d}k_2 \frac{1}{\left|\alpha_1 \pm \omega(k_1) \pm \omega(k_2) \pm \omega(k_3) + \mathbf{i}\lambda^2\right|}$$
$$\times \frac{1}{\left|\alpha_2 \pm \omega(k_1) \pm \omega(k_2) \pm \omega(k_3) \pm \omega(k_1+k_4) \pm \omega(k_1+k_5) + \mathbf{i}\lambda^2\right|} \lesssim \langle \ln\lambda\rangle^{\gamma_a}\lambda^{\gamma_b},$$

- Those crossing estimates have only two denominators instead of three as in the linear case.
- Even though the strategy is still to show the dominance of the leading (ladder) diagrams, the classification of other types of graphs (crossing and nested diagrams) is much more complicated and involved.

Crossing estimates for the ZK equation in the lattice setting

Crossing estimates in our work

In the context of the lattice nonlinear ZK equation, the crossing estimate takes the form

$$\sup_{(\alpha_1,\alpha_2)\in\mathbb{R}^2}\int_{\mathbb{T}^d}\mathrm{d}k_1\,\frac{1}{\left|\alpha_1\pm\omega(k_1)\pm\omega(k_1+k_2)+\mathbf{i}\lambda^2\right|}$$

$$\times\frac{1}{\left|\alpha_2\pm\omega(k_1)\pm\omega(k_1+k_2)\pm\omega(k_1+k_3)+\mathbf{i}\lambda^2\right|}\lesssim\langle\ln\lambda\rangle^{\gamma_a}\lambda^{\gamma_b}.$$

The lattice ZK crossing estimate contains two denominators instead of three denominators and one integration in k_1 instead of two integrations in k_1, k_2. The loss of one integration is due to the nature of the quadratic nonlinearity.

Crossing estimates for the ZK equation in the lattice setting

- An "easy-to-see" technical difficulty is that, from (36), one could use an L^3 estimate

$$\Big|\int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\mathrm{d}k_2 e^{\mathbf{i}t(\pm\omega(k_1)\pm\omega(k_2)\pm\omega(k_3))+\mathbf{i}s(\omega(k_1+k_4)\pm\omega(k_1+k_5))}\Big|$$
$$\lesssim \|p_t\|_3^2\|K(\pm t,\pm s,\pm s,k_4,k_5)\|_3,$$

thanks to the presence of the double integral $\int_{(\mathbb{T}^d)^2}\mathrm{d}k_1\mathrm{d}k_2$, where

$$K(x,t_0,t_1,t_2,k,k_*) := e^{-\mathbf{i}d(t_0+t_1+t_2)}$$
$$\times\prod_{i=1}^d\int_0^{2\pi}\frac{\mathrm{d}p}{2\pi}e^{\mathbf{i}px^i+\mathbf{i}(t_0\cos(2\pi p)+t_1\cos(2\pi p+k^i)+t_2\cos(2\pi p+k_*^i))},$$

with $k=(k^1,\cdots,k^d)$ and $k_*=(k_*^1,\cdots,k_*^d)$, and $p_t=K(x,t,0,0,0,0)$.

- On the other hand, the ZK crossing estimate only involves one integration, and as thus, a straightforward bound would be an L^2 estimate, *which leads to the divergence of the sum of all the leading, crossing and nested diagrams. As the noise has no influence on pairing graphs, and therefore we are only left with the use of delicate arguments to prove very fine estimates needed to restore the convergence of the (leading and non-leading) diagrams.*

A main portion of our work is dedicated to establishing several different and new types of crossing estimates, which are more flexible than those used in the previous works, for the singular lattice ZK dispersion relation under the low dimensional assumption $d \geq 2$. These novel types of crossing estimates allow us to go around the situation encountered previously in Lukkarinen's counterexample and are embedded into new (and sophisticated) types of graph estimates.

(V) The effect of the noise on the Feynman diagrams

The effect of the noise on the Feynman diagrams

$$\frac{\partial}{\partial t}\langle \alpha_t(k,1),\alpha_t(k,-1)\rangle \ = \ \mathbf{i}\lambda\int_{(\Lambda^*)^2}\mathrm{d}k_1\mathrm{d}k_2\delta(-k+k_1+k_2)$$
$$\times\,\mathcal{M}(k,k_1,k_2)e^{\mathbf{i}t(\omega(k_1)+\omega(k_2)-\omega(k))-\tau_{k,k_1k_2}c_rt}\langle\alpha_t(k,-1)\alpha_t(k_1,1)\alpha_t(k_2,1)\rangle,$$

- The quantity τ_{k,k_1k_2} denotes the square $|1_k - 1_{k_1} - 1_{k_2}|^2$ of the length of the vector $1_k - 1_{k_1} - 1_{k_2}$.
- In general, we set

$$\tau_{k_{i_1}\cdots k_{i_m},k_{j_1}\cdots k_{j_{m'}}} = \left|\sum_{l=1}^{m} 1_{i_l} - \sum_{l'=1}^{m'} 1_{j_{l'}}\right|, \quad \forall k_{i_1}\cdots k_{i_m}, k_{j_1}\cdots k_{j_{m'}} \in (\Lambda^*)^{m+m'},$$

- The quantity vanishes for pairing graphs $\longrightarrow$ it has the role of only fixing the singularties of the dispersion relation and vanishes on the ladder and crossing diagrams

(VI) Resonance broadening

Resonance broadening

- Unlike the Schrödinger dispersion relation, the lattice ZK dispersion relation not only creates major obstacles in obtaining the crossing estimates but also has another serious problem: it prevents the convergence of the leading diagrams.
- It is common practice to assume that

$$\begin{aligned}&\int_{\mathbb{T}^{2d}} \mathrm{d}k_2\mathrm{d}k_3\delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))F(k_2+k_3,k_2,k_3)\\ =&\int_{\mathbb{R}}\mathrm{d}s\int_{\mathbb{T}^{2d}}\mathrm{d}k_2\mathrm{d}k_3 e^{-\mathbf{i}s(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))}F(k_2+k_3,k_2,k_3),\end{aligned}$$

  for any test function $F(k_2+k_3,k_2,k_3)\in C^\infty(\mathbb{T}^{2d})$ and for any dispersion relation ω.
- For most dispersion relations, the quantity $\delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ cannot be defined as a positive measure.

Resonance broadening

- Let us define
  and
  $$\tilde{\delta}_\ell(\omega(k_3)+\omega(k_2)-\omega(k_1)) \ := \ \frac{\ell}{\ell^2+(\omega(k_3)+\omega(k_2)-\omega(k_1))^2}, \quad \ell>0.$$
  One writes
  $$\int_{\mathbb{T}^{2d}} \mathrm{d}k_2\mathrm{d}k_3\tilde{\delta}_\ell(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))F(k_2+k_3,k_2,k_3)$$
  $$= \int_{\mathbb{R}} \mathrm{d}s e^{-|s|\ell} \int_{\mathbb{T}^{2d}} \mathrm{d}k_2\mathrm{d}k_3 e^{-\mathbf{i}s(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))}F(k_2+k_3,k_2,k_3),$$
- When ω is sufficiently good, the oscillatory integral $\int_{\mathbb{T}^{2d}}\mathrm{d}k_2\mathrm{d}k_3$ $e^{-\mathbf{i}s(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))}F(k_2+k_3,k_2,k_3)$ produces sufficient decay in s, yielding the convergence of $\tilde{\delta}_\ell(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ and $\tilde{\delta}_\ell(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ to the positive measure $\delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ in the limit $\ell\to 0$.
- When ω is the lattice ZK dispersion relation, the delta function $\delta(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))$ cannot be defined as a positive measure, yielding the divergence of the leading graphs, that contains oscillatory integrals of the form (with $\ell=0$) $\int_{\mathbb{T}^{2d}}\mathrm{d}k_2\mathrm{d}k_3\ e^{-\mathbf{i}s(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))}\ F(k_2+k_3,k_2,k_3)$.

Resonance broadening

There are a few common resonance broadening strategies

$$\delta_\ell(\omega(k_3)+\omega(k_2)-\omega(k_1)) \;:=\; \frac{1}{2\ell}\int_{-\ell}^{\ell}\mathrm{d}\xi\int_{\mathbb{R}}\mathrm{d}s e^{-\mathbf{i}s(\omega(k_3)+\omega(k_2)-\omega(k_2+k_3))-\mathbf{i}2\pi s\xi}, \quad \ell>0.$$

We say that a function f_ℓ^∞ solves the "resonance broadening" 3-wave equation if and only if

$$\frac{\partial}{\partial\tau}f_\ell^\infty(k,\tau)=\mathcal{C}_\ell\big(f_\ell^\infty\big)(k,\tau),$$

with the collision operator

$$\mathcal{C}_\ell(f^\infty)(k_1)=\int_{(\mathbb{T}^d)^2}\mathrm{d}k_2\mathrm{d}k_3|\mathcal{M}(k_1,k_2,k_3)|^2\frac{1}{\pi}\delta_\ell(\omega(k_3)+\omega(k_2)-\omega(k_1))$$
$$\times\,\delta(k_2+k_3-k_1)\Big(f_2^\infty f_3^\infty-f_1^\infty f_2^\infty\mathrm{sign}(k_1^1)\mathrm{sign}(k_3^1)-f_1^\infty f_3^\infty\mathrm{sign}(k_1^1)\mathrm{sign}(k_2^1)\Big).$$

THANK YOU SO MUCH FOR YOUR ATTENTION!