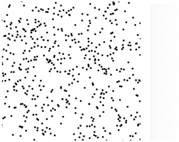
A path-integral approach to interacting Bose gases[1]

June 2022



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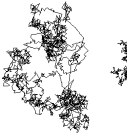

Bosonic atoms in a box

Loop gas

[1] Jürg Fröhlich, ETH Zurich

Some perspective

This talk is about *quantum-mechanical many-body theory*, a subject that includes the theory of interacting quantum gases, quantum liquids (e.g., Landau-Fermi electron liquids), solids, crystalline matter, insulating materials, etc. QM many-body theory is the theoretical basis of *condensed-matter physics*. It is supposed to describe phenomena such as *Bose-Einstein condensation* in three- or higher-dimensional interacting Bose gases, *Kosterlitz-Thouless transitions* in 2D Bose gases and other related systems, various manifestations of *superconductivity*, such as BCS superconductivity or high-T_c supercond., the *quantum Hall effect* in interacting 2D electron liquids, or in rotating Bose gases, etc..

Not a great deal is known about a mathematically rigorous quantum-mechanical description of these and other related phenomena.
A standard strategy is therefore to study *limiting regimes* and *idealized models* that reduce the mathematical complexity and enable one to come up with mathematically precise results. In this talk, this strategy is illustrated on the example of *interacting Bose gases*.

I thank the organizers for inviting me to speak – and, yes, let's get started!

Summary of the talk

I will review recent results, due to *A. Knowles, B. Schlein, V. Sohinger* and myself, on the quantum theory of *interacting Bose gases*[2].

I will make use of: (i) a representation of Bose gases as a kind of scalar field theories resulting from a Hubbard-Stratonovich transformation, (ii) *Ginibre's Brownian loop representation*, (iii) an interpolation in the number N of atom species, and (iv) simplifications appearing in various limiting regimes, such as the mean-field- and the large-N imit.

Our purpose is to study phenomena related to Bose-Einstein condensation ($N \to \infty$) and properties of polymer chains ($N \to 0$).

Some novel support for the conjecture that $\lambda|\varphi|_d^4$-theory in $d \geq 4$ dimensions is non-interacting will be described.

Remark: Related and further results on interacting Bose gases by:

- *T. Balaban, J. Feldman, H. Knörrer* and *E. Trubowitz* (BEC)
- *M. Lewin, P. T. Nam* and *N. Rougerie* (mean-field limit in 2 and 3 D)
- *Manfred Salmhofer* (use of functional integrals); and others.

I thank them for informing me about their important results.

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[2] see also my 1994 Les Houches lectures on general many-body- and transport theory, with applications to the quantum Hall effect, Fermi-liquid theory, BCS theory, etc., and my 2018 lectures at Bad Honnef

Contents and acknowledgements

Contents

Acknowledgements

My collaborators in this endeavor are *A. Knowles*, *B. Schlein*, and *V. Sohinger*. I thank them for very enjoyable collaboration.

I am grateful to *D. Brydges* and *D. Ueltschi* for useful comments, and to *A. Pizzo* for inspiring discussions and collaboration on, among other problems, quantum chains, including lattice Bose gases, and for his friendship. – *And let's take it easy!*

1. What is a non-relativistic Bose gas?

We consider a quantum gas of bosonic atoms, such as atomic *hydrogen*, *helium* (^4He), or *rubidium* (^{85}Ru), confined to a box, Λ, in physical space ($\mathbb{R}^d$) with sides of length L. For simplicity, we assume the atoms to be spinless (e.g., ^4He). The Hamiltonian, H_n, for n atoms on the Hilbert space $\mathcal{H}_n := L^2(\Lambda, d^d x)^{\otimes_s n}$ is:

$$H_n := -\sum_{j=1}^{n} \frac{\Delta_j}{2M} + \frac{\lambda}{2} \sum_{i,j=1}^{n} v(x_i - x_j) \qquad (1)$$

M: mass of atom, ($\hbar = 1$); $\lambda \geq 0$: coupling constant,
v: two-body potential of *positive type* (& pointwise positive), continuous and of rapid decay.[3]
For simplicity, we choose periodic b.c. at $\partial\Lambda$.

Then $H_n > 0$ is *self-adjoint* on $\mathcal{H}_n$, $\forall n$.

We are interested in studying the *statistical mechanics* of such systems in thermal equilibrium at a positive particle density, $\rho := \frac{n}{|\Lambda|}$, and positive temperature, T.

[3] More natural would be a (regularized) van der Waals potential

2. Thermal equilibrium, limiting regimes

The Gibbs equilibrium state of a Bose gas at density ρ and temperature T is given by a certain *density matrix*, P_n, on $\mathcal{H}_n$:

$$P_n := Z_\Lambda(\beta,\rho)^{-1}\exp\big(-\beta H_n\big), \quad n = \rho \cdot |\Lambda|,\ \beta = \frac{1}{k_B T},$$

where $Z_\Lambda(\beta,\rho) := \mathrm{tr}[\exp\big(-\beta H_n\big)]$ is the *canonical partiton function.*

In the following, we set

$$\frac{\beta}{M} =: \nu, \text{ and either } \lambda = \lambda_0, \text{ or } \lambda = \lambda_0 \nu^2, \tag{2}$$

with $\lambda_0 \geq 0$ fixed. Varying ν and λ_0, we may set $\beta = 1$.

For later purposes, we also consider Bose gases with $N = 1, 2, 3, \ldots$ different species of atoms – all of mass M, and interacting among each other through the two-body potential v.

Range of parameter values and limiting regimes

Parameters (all of them taking arbitrary non-negative values):

$$\boxed{\text{density } \rho, \text{ (or chem. potential } \mu), \nu, \lambda_0, N}$$

Limiting regimes:

(1) $\nu \searrow 0, (M \to \infty), \lambda = \lambda_0$: Classical particle limit.

(2) $\nu \searrow 0$, $\lambda = \lambda_0 \nu^2$: (EFT- or) mean-field limit.

(3) $N \to \infty$: (Spherical-model- or) *Berlin-Kac* limit;
$N \searrow 0$: (SAW- or) *de Gennes* limit.

(4) $\Lambda \nearrow \mathbb{R}^d$: Thermodynamic limit.

Goals:

- Analyzing *limiting regimes* (such as m-f limit) in different orders.
- Understanding *BEC* for large values of N, $\nu \geq 0$, (in $d \geq 4$).
- Understanding the limiting system corresponding to $v(x) \to \delta(x)$, $\nu \geq 0, (\nu = 0 \leftrightarrow \lambda\varphi^4_d$-theory, *triviality* in $d \geq 4$). Etc.

3. Grand-canonical ensemble and 2^{nd} quantization

It will be convenient to use the *grand-canonical ensemble*: Particle number, n, fluctuates, but mean value, $\langle n \rangle_{\beta,\mu} \equiv \rho \cdot |\Lambda|$, tuned by appropriately choosing a *chemical potential*, μ. We set

$$\beta\,\mu =: -\nu\,\kappa\,, \;\; (\text{henceforth} \;\; \beta = 1)$$

2^{nd} quantization: Let

$$\mathcal{F}_\Lambda := \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

be standard *Fock space*, and $\Phi^*_\nu(x), \Phi_\nu(x)$ the usual creation- and annihilation operators acting on $\mathcal{F}_\Lambda$, which satisfy the CCR:

$$[\Phi^\#_\nu(x), \Phi^\#_\nu(y)] = 0, \qquad [\Phi_\nu(x), \Phi^*_\nu(y)] = \nu \cdot \delta(x-y), \qquad (3)$$

$\Phi_\nu(x)\Omega = 0, \forall x,$ where $\Omega = (1, 0, 0, \dots) \in \mathcal{F}_\Lambda$ is the "*vacuum*".

Similarly for N species of bosons: $x \mapsto (x, a), a = 1, 2, ..., N$.

Hamiltonian, partition function, reduced density matrices

Hamiltonian – but not really

$$\mathbb{H}_{\nu,\rho,\Lambda} := \frac{1}{2}\int_{\Lambda} dx \left\{\Phi_{\nu}^{*}(x)\big([-\Delta+1]\Phi_{\nu}\big)(x) + \right.$$

$$\left. +\lambda\nu^{-2}\int_{\Lambda} dy \left[\Phi_{\nu}^{*}(x)\Phi_{\nu}(x)-\rho\right] v(x-y)\left[\Phi_{\nu}^{*}(y)\Phi_{\nu}(y)-\rho\right]\right\} \quad (4)$$

acting on $\mathcal{F}_{\Lambda}$, with $\kappa = 1 - \frac{\lambda}{\nu^2}\rho\int dx\, v(x)$; (henceforth ρ is a parameter that we will vary).

Grand partition function:

$$\boxed{\Xi_{\Lambda}(\nu,\rho) := tr\big(\exp[-\mathbb{H}_{\nu,\rho,\Lambda}]\big)} \quad (5)$$

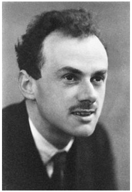
Reduced density matrices:

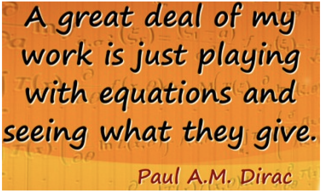
$$\gamma_p(\underline{x};\underline{x}') :=$$

$$= \Xi_{\Lambda}(\nu,\rho)^{-1} tr\Big(\exp[-\mathbb{H}_{\nu,\rho,\Lambda}]\, \Pi_{j=1}^{p}\Phi_{\nu}^{*}(x_j)\, \Pi_{j=1}^{p}\Phi_{\nu}(x_j')\Big) \quad (6)$$

Similar formulae for N species of bosons! (...*Remark on time-dep. correlations*.)

4. Thermal equilibrium in the grand-canonical ensemble – in *functional-integral representation*[4]





Functional-integral representation (for N species of bosons):
We propose to express the grand partition function $\Xi_\Lambda(\nu, \rho)$ (and the reduced p-particle density matrices γ_p) in terms of path integrals. For this purpose, we introduce the formal (Lebesgue) integration measure

$$"\mathcal{D}\overline{\varphi} \wedge \mathcal{D}\varphi := \prod_{x\in\Lambda, \tau\in[0,\nu)} \prod_{a=1}^{N} d\overline{\varphi}_a(\tau, x) \wedge d\varphi_a(\tau, x)"$$

[4] *P.A.M. Dirac:* "The Lagrangian in quantum mechanics"

Functional integrals, ctd.

We then find that (τ: imaginary time)

$$\Xi_\Lambda(\nu,\rho) \propto \qquad\qquad\qquad \swarrow \text{ complex "Gaussian"}$$

$$\propto \int \mathcal{D}\overline{\varphi}\wedge\mathcal{D}\varphi \exp\Big(-\int_0^\nu d\tau \sum_{a=1}^{N}\int_\Lambda dx\Big\{\overline{\varphi}_a(\tau,x)(K(0)\varphi_a)(\tau,x)$$

$$+ \frac{\lambda}{2\nu}\sum_{b=1}^{N}\int_\Lambda dy[|\varphi_a(\tau,x)|^2-\nu^{-1}\rho]\, v(x-y)\, [|\varphi_b(\tau,y)|^2-\nu^{-1}\rho]\Big\}\Big) \tag{7}$$

and we impose periodic b.c. at $\tau = 0, \nu$; define a "one-particle op."

$$K(\sigma) := \frac{\partial}{\partial\tau} - \frac{\Delta}{2} + 1 + i\sigma(\tau,x), \quad K(0) = K(\sigma \equiv 0).$$

Let $d\mu_\lambda(\sigma)$ be the Gaussian probability measure on $\mathcal{S}'([0,\nu)\times\Lambda)$ with mean 0 and covariance, C, given by

$$C(\tau,\tau';x,y) := \delta(\tau-\tau')\cdot\frac{\lambda}{\nu}\, v(x-y)\,. \tag{8}$$

The Hubbard-Stratonovich formula

By (7) and (8),

$$\Xi_\Lambda(\nu,\rho) = \text{const.} \int \mathcal{D}\overline{\varphi} \wedge \mathcal{D}\varphi \int d\mu_\lambda(\sigma)\, e^{iN\theta(\sigma)} \times$$

$$\times \exp\Big[-\int_0^\nu d\tau \int_\Lambda dx \sum_{a=1}^{N} \overline{\varphi}_a(\tau,x)(K(\sigma)\varphi_a)(\tau,x)\Big] \tag{9}$$

$$= \int d\mu_\lambda(\sigma)\, e^{iN\theta(\sigma)}\, [det\, K(\sigma)]^{-N}, \quad \theta(\sigma) := \frac{\rho}{\nu}\int_0^\nu d\tau \int_\Lambda dx\, \sigma(\tau,x)$$

The second equation follows by interchanging integrations over $(\overline{\varphi}_a, \varphi_a)$ and over σ. For an ideal Bose gas ($\lambda = 0$), $\Xi_\Lambda^{(0)}(\nu,\rho) = [det\, K(0)]^{-N}$, hence

$$\frac{\Xi_\Lambda(\nu,\rho)}{\Xi_\Lambda^{(0)}(\nu)} = \int d\mu_\lambda(\sigma)\, e^{iN\theta(\sigma)} \left[\frac{det\, K(\sigma)}{det\, K(0)}\right]^{-N} \tag{10}$$

Next, we use that, for an arbitrary $m \times m$ - matrix A (with $A + A^* > 0$),

$$det\, A = \exp\big(\mathrm{tr}[\ell n A]\big) = \exp\Big(-\int_0^\infty dt\, \mathrm{tr}\big[(A+t)^{-1} - (1+t)^{-1}\big]\Big).$$

The Green function of $K(\sigma)$

$\rightarrow$ *Functional-integral representation of grand partition function:*

$$\frac{\Xi_\Lambda(\nu,\rho)}{\Xi_\Lambda^{(0)}(\nu)} = \int d\mu_\lambda(\sigma)\, e^{iN\theta(\sigma)}\, e^{N \int_0^\infty dt\, \mathrm{tr}\left[(K(\sigma)+t)^{-1}-(K(0)+t)^{-1}\right]} \quad (11)$$

We note that the trace appearing in the exponent on the R.S. of (11) is finite, and the t-integration converges. Moreover, for

$$\lambda = \mathcal{O}(\frac{1}{N}),$$

one can use the saddle point method to calculate asymptotics of (11), as $N \to \infty$: $\rightarrow$ "$\frac{1}{N}$ - expansion"!

Green function of $K(\sigma)$ with periodic b.c.: For $\tau, \tau' \in [0, \nu)$,

$$[K(\sigma)+t]^{-1}(\tau,\tau') = \sum_{\ell=0}^{\infty} \Theta(\tau-\tau'+\ell\nu)\, \Gamma(\tau,\tau'-\ell\nu; i\sigma+1+t), \quad (12)$$

where $\Theta =$ Heaviside step function, and $\Gamma(\tau,\tau';q)$ is the "heat kernel"

$$\frac{\partial}{\partial\tau}\Gamma(\tau,\tau';q) = \left(\frac{\Delta}{2} - q(\tau,\cdot)\right)\Gamma(\tau,\tau';q), \quad \Gamma(\tau,\tau;q) = \mathbf{1}. \quad (13)$$

Using the Feynman-Kac formula

In (12), the distribution $\sigma(\tau, x) := \sigma([\tau], x)$, $[\tau] = \tau \mod \nu$, is defined to be periodic in τ with period ν; $\Gamma(\tau, \tau'; q)$ can be evaluated by using the *Feynman-Kac formula*:

$$\Gamma(\tau, \tau'; q)_{xy} = \int d\mathbb{W}_{xy}^{\tau-\tau'}(\omega)\, e^{-\int_0^{\tau-\tau'} ds\, q\left(s+\tau', \omega(s)\right)} \tag{14}$$

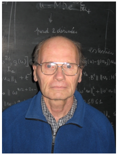
After plugging (14) (with $q := i\sigma + 1 + t$) into (12) and (12) into (11), a straightforward calculation yields the identity:

$$\begin{aligned}
\frac{\Xi_\Lambda(\nu,\rho)}{\Xi_\Lambda^{(0)}(\nu)} &= \\
&= \int d\mu_\lambda(\sigma) e^{iN\theta(\sigma)}\, e^{N\left\{\sum_{\ell=1}^{\infty} \frac{e^{-\ell\nu}}{\ell} \int_\Lambda du \left[\Gamma(\ell\nu,0;i\sigma)_{uu} - \Gamma(\ell\nu,0;0)_{uu}\right]\right\}} \quad (15)\\
&= \int d\mu_\lambda(\sigma) e^{iN\theta(\sigma)}\, e^{N\left\{\sum_{\ell=1}^{\infty} \frac{e^{-\ell\nu}}{\ell} \int_\Lambda du \int d\mathbb{W}_{uu}^{\ell\nu}(d\omega)\left[e^{i\int_0^{\ell\nu} ds\,\sigma([s],\omega(s))} - 1\right]\right\}}
\end{aligned}$$

Note: Ambiguity at $\ell = 0$ cancels when the partition function is normalized by the one of the ideal gas. – Bound: (15) $\Rightarrow$

$$\boxed{\Xi_\Lambda(\nu, \rho)/\Xi_\Lambda^{(0)}(\nu) \le 1}$$

5. The loop ensembles of Ginibre and Symanzik



J. Ginibre

$$\underset{\nu \to 0}{\longrightarrow}$$

K. Symanzik

In order to introduce a gas of interacting Brownian (paths and) loops *equivalent* to the Bose gas, we expand the exponential in the integrand on the R.S. of (15) and then carry out the integration over σ, term by term. To describe the resulting expression, we define a *"2-loop interaction"*

$$\begin{aligned} V(\omega, \omega') &:= \frac{1}{2} \sum_{r=0}^{\ell(\omega)-1} \sum_{s=0}^{\ell(\omega')-1} \int_0^{\nu} dt\, v(\omega(t + r\nu) - \omega'(t + s\nu)), \\ V(\omega, \omega) &:= \sum_{0 \le r < s < \ell(\omega)} \int_0^{\nu} dt\, v(\omega(t + r\nu) - \omega(t + s\nu)) . \end{aligned} \tag{16}$$

In the 1^{st} equation, $\omega \neq \omega'$. And $V(\omega, \omega)$: *self-interaction* of loop ω.

Ginibre's representation of the Bose gas

Grand-canonical partition function as *"loop gas partition function"*:

$$\frac{\Xi_\Lambda(\nu,\rho)}{\Xi_\Lambda^{(0)}(\nu)} = \text{const.} \sum_{n=0}^{\infty} \frac{N^n}{n!} \Bigg\{ \sum_{\ell_1,\dots,\ell_n=1}^{\infty} \int_\Lambda du_1 \cdots \int_\Lambda du_n \times$$

$$\times \left[\prod_{k=1}^{n} \frac{e^{-\ell_k \kappa \nu}}{\ell_k} \int d\mathbb{W}_{u_k u_k}^{\ell_k \nu}(\omega_k) \right] e^{-\sum_{i,j=1}^{n} \frac{\lambda}{\nu} V(\omega_i,\omega_j)} \Bigg\}, \quad (17)$$

with κ as after Eq. (4). This is *Ginibre's representation* of the grand-canonical partition function of the Bose gas with N species of particles as a statistical sum over *Brownian loops* in an interacting "loop gas".
Note: N appears as a parameter that can be given complex values!

It is easy to generalize (17) to a loop gas representation of *reduced density matrices*, $\gamma_p(\underline{x},\underline{a};\underline{x}',\underline{a})$, (see formula (6)): [In addition to the Brownian loops present in (17), there appear *open Brownian paths* connecting a point x_j to a point $x'_{\pi(j)}$, where π is an arbitrary permutation of $\{1,\dots,p\}$ with the property that $a_j = a_{\pi(j)}$, $j = 1,\dots,p$, and in the end one has to sum over all such permutations.]

Passage to Symanzik's representation of scalar Euclidian field theories with quartic self-interaction

We return to expressions (16) (interactions between loops) and (17) (grand partition function). In these formulae, we set

$$\lambda = \frac{\lambda_0}{N+1} \cdot \nu^2 \tag{18}$$

Inspecting (16) and using (18), we see that, as $\nu \to 0$, the sums over $\ell_1, \ldots, \ell_n$ in (17) should be interpreted as *Riemann sum approximations* to the expression

$$\lim_{\nu \to 0} \frac{\Xi_\Lambda(\nu,\rho)}{\Xi_\Lambda^{(0)}(\nu)} = \lim_{\delta \to 0} const_\delta^N \sum_{n=0}^{\infty} \frac{N^n}{n!} \times$$

$$\times \left[\prod_{k=1}^{n} \int_\delta^\infty \frac{dT_k}{T_k} e^{-\kappa_\delta T_k} \int_\Lambda du_k \int d\mathbb{W}_{u_k u_k}^{T_k}(\omega_k) \right] e^{-\sum_{i,j=1}^{n} \frac{\lambda_0}{N+1} V_0(\omega_i,\omega_j)}, \tag{19}$$

where $const_\delta$ is indep. of λ_0, N, κ_δ must be chosen to depend on δ in a way that appropriately implements Wick ordering (see below), and

$$V_0(\omega, \omega') = \frac{1}{2} \int_0^T dt \int_0^{T'} dt'\, v(\omega(t) - \omega'(t')) \, .$$

Reconstructing the functional integral of a Euclidian field theory

Let $d\mu_v(\eta)$ be the Gaussian probability measure on $\mathcal{S}'(\Lambda)$ with mean 0 and covariance $\frac{\lambda_0}{N+1}v$; $(\eta \sim [\sigma]_{\tau\to 0})$. We then find that

$$\text{R.S. of (19)} = \lim_{\delta \searrow 0} const_\delta^N \times$$

$$\times \int d\mu_v(\eta) \exp\Big\{N \int_\delta^\infty \frac{dT}{T} e^{-\kappa_\delta T} \int_\Lambda du \int d\mathbb{W}_{uu}^T(\omega) e^{i\int_0^T \eta(\omega(t))dt}\Big\} \qquad (20)$$

Feynman-Kac shows that

$$\int_\Lambda du \int d\mathbb{W}_{uu}^T(\omega) e^{i\int_0^T \eta(\omega(t))dt} = tr\Big(e^{T(\frac{\Delta}{2}+i\eta)}\Big)$$

Carrying out the T-integration then implies that R.S. of (20) is given by

$$\int d\mu_v(\eta) e^{iN\vartheta(\eta)} \exp\Big\{-N\, tr\Big[\ell n(-\frac{\Delta}{2}+1-i\eta) - \ell n(-\frac{\Delta}{2}+1)\Big]\Big\}$$

$$= \int d\mu_v(\eta) e^{iN\vartheta(\eta)} \left[\frac{det(-\frac{\Delta}{2}+1-i\eta)}{det(-\frac{\Delta}{2}+1)}\right]^{-N}, \qquad (21)$$

where $\vartheta(\eta)$ is chosen such as to cancel the term linear in η in the expo.

The hard-core Bose gas as a regularization of Euclidian $\lambda\varphi^4$-theory

Let $\vec{\phi}(x) = \big(\phi_1(x), ..., \phi_N(x)\big), x \in \Lambda$ be a complex field with N components. Its action functional is defined by

$$
\begin{aligned}
S_\Lambda(\vec{\phi}) \quad &:= \quad \frac{1}{2}\int_\Lambda dx \Big\{ \overline{\vec{\phi}}(x) \cdot \big(-\Delta + 2\big)\vec{\phi}(x) + \\
&+ \quad \frac{\lambda_0}{N+1}\int_\Lambda dy \left[|\vec{\phi}(x)|^2 - \rho\right] v(x-y) \left[|\vec{\phi}(y)|^2 - \rho\right] \Big\}
\end{aligned}
$$

By Hubbard-Stratonovich in reverse,

$$
\text{R.S. of}(21) = \frac{1}{(Z_\Lambda^{(0)})^N} \int \mathcal{D}\overline{\vec{\phi}} \wedge \mathcal{D}\vec{\phi} \exp\big[-S_\Lambda(\vec{\phi})\big] \tag{22}
$$

The $N \to \infty$-limit is the *Berlin-Kac* spherical model, which is known to have a phase transition with spontaneous symmetry breaking; (shown by using saddle-point method!). In the limit, where

$$
v(x-y) \to \delta_{xy},
$$

R.S. of (22) defines Euclidian $\lambda|\vec{\phi}|^4$-theory. Thus: Hard-core Bose gas in d dimensions = *regularization* of Euclidian $\lambda|\vec{\phi}|^4_d$-theory!

6. Summary of results and conjectures

It is time to ask what all this is good for and what it means!

1. *Mean-field limit, $\nu \searrow 0$.* The ideas reviewed in Sect. 5 $\big[$Eqs. (17) through (22)$\big]$ imply that, for a suitable choice of ρ as a function of ν (with $\rho \to \infty$, as $\nu \searrow 0$ in $d \geq 2$), the limit $\nu \searrow 0$ of the Bose gases exists and is given by the Euclidean field theory with action functional $S_\Lambda(\vec{\phi})$ given in (22), above. This is actually a **theorem**! To prove it, we assume that the potential v is continuous, of positive type and of short range. Furthermore, a suitable *choice of ρ ("Wick-ordering" of the quartic term)* is crucial: For $d = 1$, the continuum limit and the mean-field limit are easy to analyze and can be taken in arbitrary order, for arbitrary finite ρ. For $d = 2, 3$, it suffices to *"Wick order"* the factors quadratic in $|\vec{\phi}|$ in the quartic term of the action functional $S_\Lambda(\vec{\phi})$; (subtraction of a divergent ρ). Consider the partition function

$$
\begin{aligned}
\mathcal{Z}_\Lambda(\rho) &= \int d\mu_v(\eta) e^{iN\vartheta(\eta)} \times \\
&\times \quad e^{N \int_0^\infty dt\, \mathrm{tr}\left[(-\frac{\Delta}{2}+1-i\eta+t)^{-1}-(\frac{\Delta}{2}+1+t)^{-1}\right]}
\end{aligned}
\tag{23}
$$

Continuum limt

To make sense of this expression in the continuum limit, in $d = 2, 3$, one must choose $\vartheta(\eta)$ such that the term **linear** in η in the exponent on the R.S. of (23) is **finite**. One then shows that the **remaining term** in the exponent on the R.S. of (23) is well-defined and has a negative real part in dimensions $d = 2, 3$. Thus $\mathcal{Z}_\Lambda(\rho)$ exists and is bounded above by 1 – as direct inspection of the R.S. of (22) also shows.

For $\nu > 0$, we must study

$$\frac{\Xi_\Lambda(\nu, \rho)}{\Xi_\Lambda^{(0)}(\nu)} = \int d\mu_\lambda(\sigma)\, e^{iN\theta(\sigma)}\, e^{N \int_0^\infty dt\, \mathrm{tr}\left[(K(\sigma)+1+t)^{-1} - (K(0)+1+t)^{-1}\right]}, \tag{24}$$

which also has a well-defined integrand whose absolute value is bounded above by 1, uniformly in ν; see (15)!

Thus, it suffices to verify that, in $d = 2, 3$ dim., (24) converges to (23), as $\nu \to 0$, i.e., in the *mean-field limit*. This is a relatively easy problem in CQFT; (thanks to the uniform bounds on the integrands, it suffices to establish L^2-convergence of the exponents ...).

Triviality of scalar field theories; classical particle limit

2. *"Triviality" of $\lambda|\vec{\phi}|_d^4$-theory in dimension $d \geq 4$ – Conjecture.* To construct the continuum limit ($\varepsilon \to 0$) is somewhat challenging for Bose gases with hard-core potentials: $v \to \delta$. It appears that, for Bose gases at finite density and temperature (arbitrary $\nu > 0$), the continuum limit exists in any dimension d! It is known that, for $d \geq 4$, hard-core Bose gases are identical to *ideal* (i.e., non-interacting) Bose gases. This follows from the statement that two Brownian paths in dimension $d \geq 4$ never intersect. In view of the results reviewed in Sect. 5 and in Result 1, this leads to the *conjecture* that $\lambda|\vec{\phi}|_d^4$-theory is trivial, i.e., equivalent to a Gaussian (free) field theory, in dimension $d \geq 4$. (This has been proven only for a single ($N = 1$) complex scalar field in $d > 4$; and for a real scalar field in $d = 4$, $\nearrow$ *Aizenman & Duminil-Copin*)

3. *Classical particle limit.* Consider a Bose gas in the continuum limit at finite temperature ($\beta > 0$), in the limit where the mass, M, of the atoms in the gas tends to ∞, but the interaction strength is kept constant. This corresponds to the limiting theory where

$$\nu \searrow 0\,, \quad \text{with} \quad \lambda = \lambda_0 = \text{const.}$$

Classical particle limit

Recall Ginibre's loop-gas representation (17)

$$\Xi_\Lambda(\nu,\kappa) = \sum_{n=0}^{\infty}\frac{N^n}{n!}\Big\{\Big[\prod_{k=1}^{n}\sum_{\ell_k=1}^{\infty}\int_\Lambda du_k\,\frac{e^{-\ell_k\kappa\nu}}{\ell_k}\int d\mathbb{W}^{\ell_k\nu}_{u_k u_k}(\omega_k)\Big]\times$$
$$\times \quad e^{-\sum_{i,j=1}^{n}\frac{\lambda}{\nu}V(\omega_i,\omega_j)}\Big\}, \qquad (17')$$

for the partition function of the Bose gas, with $\nu=\frac{1}{M},\lambda=\lambda_0,(\beta=1)$. Choose a negative chemical potential, i.e., $\kappa=\kappa(\nu)>0$, in such a way that

$$e^{-\kappa(\nu)\cdot\nu}\nu^{-\frac{d}{2}} =: z = \text{const.},\quad \text{for arbitrary } \nu>0. \qquad (25)$$

Then, as $\nu\searrow 0$, i.e., $M\to\infty$, only the terms with $\ell_k=1,\forall k=1,...,n$, survive, and – in view of definition (16) of the interactions $V(\omega_i,\omega_j)$ – $\Xi_\Lambda(\nu,\kappa(\nu))$ is seen to converge to the *classical partition function*

$$\Xi_\Lambda(z) := \sum_{n=0}^{\infty}\frac{(zN)^n}{n!}\Big[\prod_{k=1}^{n}\int_\Lambda du_k\Big]\exp\{-\frac{\lambda_0}{2}\sum_{i,j=1}^{n}v(u_i-u_j)\} \qquad (26)$$

Thermodynamic limit

4. *Thermodynamic limit.* Focus on Bose gases with $\nu > 0$ and on classical gases. We study convergence of the Gibbs potential

$$\Omega_\Lambda(\nu, \kappa) := \frac{1}{|\Lambda|} \ell n \, \Xi_\Lambda(\nu, \kappa), \quad \kappa > 0,$$

and of the reduced density matrices γ_p, as $\Lambda \nearrow \mathbb{R}^d$ $(\mathbb{Z}^d)$, as well as analyticity properties of the limiting expressions, using a *"cluster expansion"* converging for λ_0 small enough, or for large N, assuming that the two-body potential, v, decays "rapidly" ($\nearrow$ D. Ueltschi).

Basic idea: Set

$$\exp\{-\frac{\lambda_0}{\nu} V(\omega_i, \omega_j)\} =: 1 + G_{\lambda_0}(\omega_i, \omega_j) \,. \tag{27}$$

Then

$$|G_{\lambda_0}(\omega_i, \omega_j)| = \mathcal{O}(\lambda_0) \,, \forall i, j \,, \tag{28}$$

uniformly in the loops ω_i, ω_j, with *rapid decay* in $\mathrm{dist}(\omega_i, \omega_j)$. The cluster expansion is an expansion in powers of the small quantities $G_{\lambda_0}(\omega_i, \omega_j)$, with $i \neq j$. Using the *"linked cluster theorem"* and standard combinatorics of the *Mayer expansion*, convergence is seen to hold for small λ_0...! (Similarly for classical gases – but simpler.)

$N \to \infty$ and Bose-Einstein condensation

5. *$N \to \infty$, and BEC – Conjecture.* Choosing $\lambda = \frac{\lambda_0}{N+1}$ in expression (17') for $\Xi_\Lambda(\nu, \kappa)$, and using (27) and

$$|G_{\lambda_0}(\omega_i, \omega_j)| = \mathcal{O}(\frac{\lambda_0}{N+1}), \forall i, j,$$

one shows by inspection that, in the limit $N \to \infty$, only *"cactus diagrams"* survive in the cluster expansion. These diagrams can be resummed, with the result that the system approaches an ideal Bose gas of one species, but with a *renormalized chemical potential*. This limiting system is known to exhibit *BEC*. – What about large, but finite values of N? To study this question, one would attempt to use renormalization group methods in the Hubbard-Stratonovich rep., see (10), (11). (Perhaps, a variant of the *"lace expansion"*, due to *D. Brydges* and *T. Spencer*, in the Ginibre rep. might also lead to results.)

I conjecture that one can prove *BEC*, for large enough N, in $d \underset{(=)}{>} 4$.

But this remains to be proven.

$N \to 0$ and regularised SAW

6. *$N \to 0$, and SAW – Conjecture.* In the limit $N \to 0$ all diagrams with loops disappear. Thus the partition function Ξ_Λ tends to 1, as $N \to 0$.
   More interesting is the study of the reduced density matrix $\gamma_2(x, y)$! Studying the selfinteractions, $V(\omega, \omega)$, of Brownian paths contributing to $\gamma_2(x, y)$, we see that this is a regularised version of the *Edwards-Anderson model* of *SAW*. I expect that, in dimension $d \geq 4$, the critical properties of this theory are identical to those of ordinary Brownian motion. But a proof remains to be developed.

That's it for today!

7. [*Remark on definition of $\mathcal{Z}_\Lambda(\rho)$.* Integrand in exponent on the R.S. of Eq. (23), **after** subtraction of term **linear** in η, is given by

$$-\int_0^\infty dt\,\mathrm{tr}\Big[\big(-\frac{\Delta}{2}+\kappa+t\big)^{-1}\eta\,\big(-\frac{\Delta}{2}+\kappa+i\eta+t\big)^{-1}\eta\,\big(-\frac{\Delta}{2}+\kappa+t\big)^{-1}\Big],$$

which has a **negative-definite real part** and is square-integrable in η with respect to $d\mu_v(\eta)$.

7. Concluding remarks

The first statement follows by noticing that

$$\text{N. R.}(A(t)) = \{z \,|\, \Re z \geq \kappa + t\} \Rightarrow \text{N. R.}(A(t)^{-1}) = \{z \,|\, \Re z \geq 0\},$$

with $A(t) := -\frac{\Delta}{2} + \kappa + i\eta + t$, & N. R. := "num. range" ($\nearrow$ Graf).]

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And this is really more important: "Survivre et Vivre" – 50 years later:

"... depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *Survivre*, fondé en juillet à Montréal. Son but est la lutte pour la survie de l'espèce humaine, et même de la vie tout court menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des mécanismes sociaux suicidaires, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d'armements. ..." *(Alexandre Grothendieck)*

Réveillez-vous, indignez-vous! *(Stéphane Hessel)*