The ground state energy of the dilute Bose gas.

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Based on joint work with Jan Philip Solovej, Copenhagen University, Denmark,
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The dilute Bose gas

Consider N interacting, non-relativistic bosons in a box Λ of sidelength L.
Density $\rho := N/|\Lambda| = N/L^d$, $d = 2, 3$.
The Hamiltonian of the system is, on the symmetric (bosonic) space $\otimes_s^N L^2(\Lambda)$,

$$H_N := \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j),$$

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The energy density in the thermodynamic limit is

$$e(\rho) = \lim_{L\to\infty, N/|\Lambda|=\rho} E_0(N, \Lambda)/L^d.$$

The scattering length I

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$$-\Delta\varphi + \frac{1}{2}v\varphi = 0,$$

where

$$\varphi = \begin{cases} 1 - a/r, & d = 3, \\ \log(r/a), & d = 2, \end{cases} \quad \text{outside } \operatorname{supp} v.$$

Example:

$$v_{\mathrm{hc}}(x) = \begin{cases} +\infty, & |x| < a, \\ 0, & |x| > a. \end{cases}$$

Here $\varphi = 0$ inside hard core and the scattering length equals the radius of the core.

The scattering length II

Let $\varphi_\delta = 2\delta\varphi$, with $\delta = \frac{1}{2}$ $(d=3)$ and $\delta \approx \frac{1}{|\log(\rho a^2)|}$ $(d=2)$ and $\omega = 1 - \varphi_\delta$.
Then, with $g = v(1-\omega) = v\varphi_\delta$, the scattering equation can be reformulated as

$$-\Delta\omega = \frac{1}{2}g, \qquad \text{i.e.} \qquad \widehat{\omega}(k) = \frac{\widehat{g}(k)}{2k^2}, \qquad k \neq 0,$$

and

$$\begin{aligned} 8\pi a &= \int_{\mathbb{R}^3} g < \int_{\mathbb{R}^3} v, \qquad && d=3, \\ 8\pi\delta &= \int_{\mathbb{R}^2} g \ll \int_{\mathbb{R}^2} v, \qquad && d=2. \end{aligned}$$

The 3D Lee-Huang-Yang formula

Theorem

In the dilute limit $\rho a^3 \to 0$,

$$e^{\mathrm{3D}}(\rho) = 4\pi\rho^2 a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\right) + o(\rho^2 a(\rho a^3)^{1/2}).$$

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- Upper bounds giving second order term: Erdős-Schlein-Yau (2008), Yau-Yin (2009), (Aaen 2014), Basti-Cenatiempo-Schlein (2021). Hard core is open but recent progress on hard core in GP-limit (Basti-Cenatiempo-Olgiati-Pasqualetti-Schlein).

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- Lower bound in all cases SF-Solovej 2020-21.

The 2D formula

Theorem (SF, Girardot, Junge, Morin and Olivieri)

In the dilute limit $\rho a^2 \to 0$,

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with

$$\delta_0 := \frac{1}{|\log(\rho a^2|\log(\rho a^2)|^{-1})|} = \frac{1}{|\log(\rho a^2)| + \log(|\log(\rho a^2)|)}$$

where $\Gamma = 0.577\ldots$ *is the Euler-Mascheroni constant.*

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- Leading order by Lieb-Yngvason (2001).
- GP regime Caraci-Cenatiempo-Schlein (2021-22).
- Both upper and lower bound for general potentials including hard core.

2D vs 3D

2D is hard core - even for soft potentials

Using the constant function as trial state gives

$$e(\rho) \leq \frac{1}{2}\rho^2 \int v.$$

This is the wrong **constant** in $3D$ but the wrong **order** in $2D$.

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So

$$L^u \gg a \begin{cases} (\rho a^3)^{-1}, & d=3, \\ (\rho a^2)^{-1/2}(\log(\rho a^2))^{3/2}, & d=2. \end{cases} \qquad N = \rho(L^u)^d \gg \begin{cases} (\rho a^3)^{-2}, & d=3, \\ (\log(\rho a^2))^3, & d=2. \end{cases}$$

Ingredients of lower bound

- Localize to boxes of size $\ell \gg \rho^{-\frac{1}{2}}|\log(\rho a^2)|^{\frac{1}{2}}$. Localization needs to preserve 'Neumann gap'. To get a priori information localize to smaller boxes of size $\ll (\rho)^{-\frac{1}{2}}(\log(\rho a^2))^{-\frac{1}{2}}$. Here Neumann gap can be used to control errors. Rest of analysis carried out on large box. The interaction between localized particles is denoted by $w(x_i, x_j)$.
- Condensation. Let P projection on constant function, Q orthogonal complement.

$$n_0 = \sum P_i, \qquad n_+ = \sum Q_i.$$

A priori bounds control expected values $\langle n_0 \rangle$ and $\langle n_+ \rangle$. Energy error negligible if localizing to subspace where $\cancel{n_+ \leq \mathcal{M}}$ $n_+^L \leq \mathcal{M}$ where $\mathcal{M}$ is between $\langle n_+ \rangle$ and n and where n_+^L counts the number of excitations with low momentum ($\lesssim \ell^{-1}$).

Thank you for your attention.