Nonlinear dynamics of relativistic quantum systems

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The Dirac-Klein-Gordon system

- Relativistic mean field theory of the nuclei
- Nucleon = Dirac particle of mass m, described by spinor $\Psi : \mathbb{R}^3 \to \mathbb{C}^4$
- Nucleons generate and interact via *meson fields*

$\to$ σ-mesons (mass m_σ) described by classical scalar field $S : \mathbb{R}^3 \to \mathbb{R}$;

$\to$ ω-mesons (mass m_ω) described by classical vector field $\omega = (V, \boldsymbol{\omega}) : \mathbb{R}^3 \to \mathbb{R}^4$;

$$\begin{cases} i\partial_t \Psi = \boldsymbol{\alpha} \cdot (-i\nabla - \boldsymbol{\omega})\Psi + (m+S)\beta\Psi + V\Psi, \\ (\partial_t^2 - \Delta + m_\sigma^2)S = -g_\sigma^2 \rho_\sigma(\Psi), \\ (\partial_t^2 - \Delta + m_\sigma^2)\omega = g_\omega^2 J(\Psi), \end{cases}$$

where $g_\sigma, g_\omega \in \mathbb{R}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \beta$ are the 4×4 Dirac matrices, and

$$\rho_\sigma(\Psi) = \langle \beta\Psi, \Psi \rangle_{\mathbb{C}^4}, \quad J(\Psi) = (\rho_v, \boldsymbol{J}), \quad \rho_v(\Psi) = \langle \Psi, \Psi \rangle_{\mathbb{C}^4}, \ \boldsymbol{J}(\Psi) = \langle \Psi, \boldsymbol{\alpha}\Psi \rangle_{\mathbb{C}^4}$$

- Stationary version studied by Rota Nodari, Esteban-Rota Nodari, Le Treust-Rota Nodari, Lewin-Rota Nodari

The strong coupling limit

$$\begin{cases} i\partial_t \Psi = \boldsymbol{\alpha} \cdot (-i\nabla - \boldsymbol{\omega})\Psi + (m+S)\beta\Psi + V\Psi, \\ (\partial_t^2 - \Delta + m_\sigma^2)S = -g_\sigma^2 \rho_\sigma(\Psi), \\ (\partial_t^2 - \Delta + m_\sigma^2)\omega = g_\omega^2 J(\Psi), \end{cases}$$

- Regime m_σ, m_ω, g_σ, g_ω large and of similar magnitude

$\rightarrow$ Expect $S \sim -\gamma_\sigma \rho_\sigma(\Psi)$ and $\omega \sim \gamma_\omega J(\Psi)$ with $\gamma_\sigma = g_\sigma^2/m_\sigma^2$, $\gamma_\omega = g_\omega^2/m_\omega^2$

Leads to the *nonlinear Dirac equation*

$$i\partial_t \Psi = \boldsymbol{\alpha} \cdot (-i\nabla - \gamma_\omega \boldsymbol{J}(\Psi))\Psi + (m - \gamma_\sigma \rho_\sigma(\Psi))\beta\Psi + \gamma_\omega \rho_v(\Psi)\Psi$$

- Implicitly assumes for instance that $\varepsilon(\partial_t^2 - \Delta)S \ll 1$ with $\varepsilon = 1/m_\sigma^2$
- Several related works in the case with $\varepsilon(\partial_t^2 - \Delta)S$ replaced by $(\varepsilon\partial_t^2 - \Delta)S$
- For simplicity, assume in the following $\omega = 0$ and $g_\sigma = m_\sigma = M \gg 1$.

Main result

Theorem

Let $s > 5/2$ and $\Psi_{\rm in} \in H^s(\mathbb{R}^3, \mathbb{C}^4)$, $(S_{\rm in}, \dot{S}_{\rm in}) \in H^s(\mathbb{R}^3, \mathbb{R}) \times H^{s-1}(\mathbb{R}^3, \mathbb{R})$. Let

$$\Psi_{\rm NL} \in C^0((-T^{\rm NL}_{\rm min}, T^{\rm NL}_{\rm max}), H^s(\mathbb{R}^3, \mathbb{C}^4))$$

be the maximal solution to the nonlinear Dirac equation with initial condition $\Psi_{\rm NL}|_{t=0} = \Psi_{\rm in}$. Let $M > 0$ and

$$(\Psi^{(M)}, S^{(M)}) \in C^0((-T^{(M)}_{\rm min}, T^{(M)}_{\rm max}), H^s(\mathbb{R}^3, \mathbb{C}^4) \times H^s(\mathbb{R}^3, \mathbb{R}))$$

be the maximal solution to the Dirac-Klein-Gordon equation with initial conditions $\Psi^{(M)}_{|t=0} = \Psi_{\rm in}$, $(S^{(M)}, \partial_t S^{(M)})_{|t=0} = (S_{\rm in}, \dot{S}_{\rm in})$. Then, we have

$$\liminf_{M\to+\infty} T^{(M)}_{\rm min/max} \geqslant T^{\rm NL}_{\rm min/max}$$

and, for all $0 < T_1 < T^{\rm NL}_{\rm min}$, $0 < T_2 < T^{\rm NL}_{\rm max}$, and all $0 \leq s' < s$,

$$\lim_{M\to+\infty} \|\Psi^{(M)} - \Psi_{\rm NL}\|_{L^\infty([-T_1, T_2], H^{s'}(\mathbb{R}^3, \mathbb{C}^4))} = 0$$

Main result: comments

• Existence of local-in-time solutions is standard (and only requires $s > 3/2$), with blow-up criterion

$$T^{(M)}_{\mathrm{min/max}} < +\infty \Longrightarrow \limsup_{t \to T^{(M)}_{\mathrm{min/max}}} \|(\Psi(t), S(t))\|_{L^\infty} = +\infty.$$

• $T^{(M)}_{\mathrm{min/max}} = +\infty$ or $T^{\mathrm{NL}}_{\mathrm{min/max}} = +\infty$ only known for *small* initial data (Bejenaru-Herr 2015, 2017)

• Result does *not* require that S is close to $-\rho_\sigma(\Psi)$ at initial time

• Explicit convergence rate $M^{-\min(s-s',1)}$. Can be understood from

$$\left\| \frac{1}{\sqrt{-\Delta + M^2}} u \right\|_{H^{s'}} \leqslant \frac{1}{M^{s-s'}} \|u\|_{H^{s-1}}$$

with $s - 1 \leqslant s' \leqslant s$ and $M \geqslant 1$.

Illustration on a simple ODE

Consider the ODE with unknown $(u, v) : \mathbb{R} \to \mathbb{C} \times \mathbb{R}$

$$\begin{cases} iu' = vu \\ v'' + M^2 v = M^2 |u|^2 \end{cases} \xrightarrow[M \to +\infty]{?} iu' = |u|^2 u$$

Notice that $(|u|^2)' = 2\,\mathrm{Re}\,\overline{u}u' = 2\,\mathrm{Re}\,\overline{u}(-ivu) = 0$, so that $|u(t)|^2 = |u(0)|^2$

$$\Longrightarrow v(t) = \cos(Mt)(v(0) - |u(0)|^2) + \frac{\sin(Mt)}{M} v'(0) + |u(0)|^2$$

$$\begin{aligned} u(t) &= \exp\left(-i \int_0^t v(s)\, ds\right) u(0) \\ &= e^{-it|u(0)|^2} u(0) \exp\left(-i \frac{\sin(Mt)}{M}(v(0) - |u(0)|^2)\right) \exp\left(-i \frac{1 - \cos(Mt)}{M^2} v'(0)\right) \end{aligned}$$

Conclusion: do not need that $v(0)$ is close to $|u(0)|^2$ to obtain $u(t) \to e^{-it|u(0)|^2} u(0)$ as $M \to +\infty$.

Going to PDEs: problems

Assume for simplicity that $S_{\mathrm{in}} = -\rho_\sigma(\Psi_{\mathrm{in}})$ and $\dot{S}_{\mathrm{in}} = 0$.

Duhamel formulation of the equation on S:

$$S(t) = \cos(t\sqrt{-\Delta + M^2})S_{\mathrm{in}} - M^2 \int_0^t \frac{\sin((t-s)\sqrt{-\Delta + M^2})}{\sqrt{-\Delta + M^2}} \rho_\sigma(\Psi(s))\, ds$$

Integrating by parts in s, one obtains

$$\begin{aligned} S(t) + \rho_\sigma(\Psi(t)) = {} & \frac{-\Delta}{-\Delta + M^2}\Big(\rho_\sigma(\Psi(t)) - \cos(t\sqrt{-\Delta + M^2})\rho_\sigma(\Psi_{\mathrm{in}})\Big) \\ & + \frac{M^2}{-\Delta + M^2}\int_0^t \cos((t-s)\sqrt{-\Delta + M^2})\partial_s \rho_\sigma(\Psi(s))\, ds, \end{aligned}$$

Controlling $S(t) + \rho_\sigma(\Psi(t))$ in H^k requires a bound on $\partial_s\rho_\sigma(\Psi(s))$ in H^k, which by the equation on Ψ requires a control on Ψ in H^{k+1}, which itself requires a bound on S in H^{k+1}... Loss of derivatives!

Bounds for the reduced equation

- Introduce the *reduced unknown* $\overline{S} = S + \rho_\sigma(\Psi)$, satisfying

$$(\partial_t^2 - \Delta + M^2)\overline{S} = (\partial_t^2 - \Delta)\rho_\sigma(\Psi)$$

- Ψ satisfies a Dirac equation $\Rightarrow (\partial_t^2 - \Delta)\rho_\sigma(\Psi) = P(\Psi, \nabla\Psi, \nabla S)$

(*Gain of one derivative*)

Lemma

Let $s > 5/2$ and $s' \in [s-1, s]$. Then, for any $R > 0$ there exists $C(R) > 0$ independent of M such that for all $M \geqslant 1$ we have

- $\|(\Psi, \overline{S})\|_{L_t^\infty([0,T],W^{1,\infty})} \leqslant R$
$$\Longrightarrow \forall t \in [0, T],\ \|(\Psi, \overline{S})(t)\|_{H^s} \leqslant \|(\Psi, \overline{S})(0)\|_{H^s} e^{C(R)t}$$
- $\|(\Psi, \overline{S})\|_{L_t^\infty([0,T],H^s)} \leqslant R$
$$\Longrightarrow \|\overline{S}(t) - \cos(t\sqrt{-\Delta + M^2})\overline{S}(0)\|_{L_t^\infty([0,T],H^{s'})} \leqslant C(R)M^{s'-s}$$
- $\|(\Psi, \overline{S})\|_{L_t^\infty([0,T],H^s)} \leqslant R \Longrightarrow \|\Psi - \Psi_{\mathrm{NL}}\|_{L^\infty([0,T],H^{s'})} \leqslant C(R)M^{s'-s}$.

The many-body problem

- Replace Ψ by γ non-negative operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ (*one-body reduced density matrix*)

- "Many-body" Dirac-Klein-Gordon system

$$\begin{cases} i\partial_t \gamma = [\boldsymbol{\alpha} \cdot (-i\nabla) + (m+S)\beta, \gamma] \\ (\partial_t^2 - \Delta + M^2)S = -M^2 \rho_{\beta\gamma} \end{cases}$$

where $\rho_{\beta\gamma}(x) := \mathrm{Tr}_{\mathbb{C}^4}(\beta\gamma(x,x))$.

- Analogue of Sobolev spaces for density matrices:

$$\|\gamma\|_{\mathfrak{H}^s} := \|(1-\Delta)^{s/2}\gamma(1-\Delta)^{s/2}\|_{\mathfrak{S}^2}.$$

- Similar statement for initial data $(\gamma_{\mathrm{in}}, S_{\mathrm{in}}, \dot{S}_{\mathrm{in}}) \in \mathfrak{H}^s \times H^s \times H^{s-1}$ ($s > 5/2$):

$$\lim_{M \to +\infty} \|\gamma - \gamma_{NL}\|_{L^\infty([-T_1, T_2], \mathfrak{H}^{s'})} = 0$$

Analogue of blow-up criterion in L^∞ for density matrices?

Blow-up criterion for density matrices

For wavefunctions, use the *Kato-Ponce inequality*

$$\|uv\|_{H^s} \leqslant C(\|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty})$$

to infer $\|\langle \beta\Psi, \Psi\rangle \beta\Psi\|_{H^s} \lesssim \|\Psi\|_{L^\infty}^2 \|\Psi\|_{H^s}$ and deduce a blow-up criterion in L^∞.

Replacement for density matrices:

Lemma

Let $3/2 < s' < s$. Then, there exists $C > 0$ such that for all $\gamma \in \mathfrak{H}^s$ with $\gamma \geqslant 0$ and for all $f \in H^s$ we have

$$\|f\gamma\|_{\mathfrak{H}^s} + \|\gamma f\|_{\mathfrak{H}^s} \leqslant C(\|f\|_{H^s}\|\gamma\|_{\mathfrak{H}^{s'}}^{1/2}\|\gamma\|_{\mathfrak{H}^s}^{1/2} + \|f\|_{L^\infty}\|\gamma\|_{\mathfrak{H}^s})$$

$$\|\rho_\gamma\|_{H^s} \leqslant C\|\gamma\|_{\mathfrak{H}^{s'}}^{1/2}\|\gamma\|_{\mathfrak{H}^s}^{1/2}$$