The free energy of the dilute Bose gas

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MBQM 2022
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N bosons in $[-L/2, L/2]^3$, at density $N/L^3 \to \rho$, temperature $T > 0$,

$$H_{N,L} = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \le i < j \le N} V(x_i - x_j)$$

$$\boxed{f(\rho, T) := \lim_{\substack{N \to \infty \\ N/L^3 \to \rho}} -\frac{T}{L^3} \log \operatorname{tr}_{L^2_s(\mathbb{R}^{3N})} e^{-\frac{H_{N,L}}{T}}}$$

When $T = 0$, **dilute limit** $\rho\,\mathfrak{a}^3 \to 0$ [Dyson 57], [Lieb-Yngvason 99], [Yau-Yin 09], [Basti-Cenatiempo-Schlein 21] [Fournais-Solovej 20,21]

$$e(\rho) = f(\rho, 0) = 4\pi\,\mathfrak{a}\,\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho\,\mathfrak{a}^3} + o(\sqrt{\rho\,\mathfrak{a}^3})\right)$$

For $0 < T \lesssim \rho^{2/3} \sim T_c(\rho)$, [Seiringer 08], [Yin 10]

$$f(\rho, T) = f_0(\rho, T) + 4\pi\,\mathfrak{a}\left(2\rho^2 - [\rho - \rho_c(T)]_+^2\right)(1 + o(1))$$

where $f_0(\rho, T)$ is the free energy of the ideal Bose gas.

- Related works in the Gross–Pitaevskii regime: [Deuchert-Seiringer-Yngvason 19], [Deuchert-Seiringer 20]

Second order expansion of the free energy: lower bound

$$f(\rho,T) \simeq f_0(\rho,T) + 4\pi\,\mathfrak{a}\left(2\rho^2 - [\rho-\rho_c(T)]_+^2\right)$$

We want thermal contribution $\lesssim$ Lee-Huang-Yang $4\pi\,\mathfrak{a}\,\rho^2 \times \frac{128}{15\sqrt{\pi}}\sqrt{\rho\,\mathfrak{a}^3}$

$$\begin{cases} \rho_c(T) \sim T^{3/2} \\ f_0(\rho,T) \underset{T \ll T_c(\rho)}{\sim} T^{5/3} \end{cases} \implies T \lesssim \rho\,\mathfrak{a} = \rho^{2/3}\left(\rho\,\mathfrak{a}^3\right)^{1/3}$$

LHY correction at positive temperature (conjecture)

Let $0 \le V \in L^1(\mathbb{R}^3)$, radial, non-increasing, compactly supported, for $T \lesssim \rho\,\mathfrak{a}$ and $\rho\,\mathfrak{a}^3 \to 0$, we have

$$f(\rho,T) \geq 4\pi\,\mathfrak{a}\,\rho^2\left(1 + C_{\mathrm{LHY}}\left(\frac{\rho\,\mathfrak{a}}{T}\right)\sqrt{\rho\,\mathfrak{a}^3} + o\left(\sqrt{\rho\,\mathfrak{a}^3}\right)\right)$$

$$\lim_{\alpha\to 0} C_{\mathrm{LHY}}(\alpha) = \frac{128}{15\sqrt{\pi}}.$$

$$4\pi\,\mathfrak{a}(V) := \inf\left\{\int_{\mathbb{R}^3}|\nabla f|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V|f|^2,\, f \in H^1(\mathbb{R}^3),\, f(x)\to 1 \text{ as } |x|\to\infty\right\}$$

Neumann bracketing

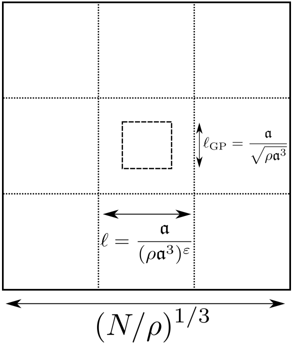
$$e(\rho) = 4\pi\,\mathfrak{a}\,\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\,\mathfrak{a}^3} + o(\sqrt{\rho\,\mathfrak{a}^3})\right)$$

- M^3 boxes of size ℓ with Neumann b.c. ($M \simeq L/\ell$)

$$E(N,L) \geq M^3 \inf_{\substack{\sum c_k = 1 \\ \sum k c_k = \rho\ell^3}} \sum_k c_k E(k,\ell)$$

- after dilation $\mathcal{U}\Psi = \ell^{3n/2}\Psi(\ell\cdot)$, on $L^2_s([-1/2,-1/2]^{3n})$

$$\boxed{\mathcal{U}^* H_{n,\ell}\mathcal{U} = \frac{1}{\ell^2}\left(\sum_{i=1}^n -\Delta_i + \sum_{i<j} \ell^2 V(\ell(x_i - x_j))\right)}$$



- Gross–Pitaevskii length scale ℓ_{GP}: gap kinetic $\sim$ interaction energy per particle
- 1st order $4\pi\,\mathfrak{a}\,\rho^2$: [Lieb Yngvason 99], [Lieb Seiringer 01] $\ell \ll \ell_{\mathrm{GP}} \implies$ interaction = small perturbation (Temple inequality)
- 2nd order LHY: [Fournais Solovej 20,21], [Fournais 20] $\ell \gg \ell_{\mathrm{GP}}$ (+ sliding technique) = more work

Analysis in small boxes

For $0 \leq \kappa \leq 2/3$, on $L^2_s([-1/2,-1/2]^{3N})$,

$$H_N^\kappa = \sum_{i=1}^N -\Delta_i + \sum_{i<j} N^{2-2\kappa} V(N^{1-\kappa}(x_i - x_j))$$

$\kappa = 0$: Gross–Pitaevskii regime
$\kappa = 2/3$: Thermodynamic limit

- periodic b.c., $\kappa > 0$: [Adhikari Brennecke Schlein 21], [Brennecke Caporaletti Schlein 21] (next order + excitation spectrum)

$$\boxed{\begin{gathered} E(N,\kappa) = 4\pi\,\mathfrak{a}\,N^{1+\kappa} + 4\pi \times \frac{125}{15\sqrt{\pi}}\mathfrak{a}^{5/2}\,N^{5\kappa/2} + o(N^{5\kappa/2}) \\ 1 - \langle \mathbb{1}, \gamma_{\Psi_N}\mathbb{1}\rangle = o(1) \end{gathered}}$$

- Neumann b.c., $\kappa = 0$: optimal rate [Boccato Seiringer 22]

$\longrightarrow$ following [Lieb-Seiringer 01] or [Lieb Yngvason Seiringer Solovej], BEC for small $\kappa > 0$

Analysis of H_N^κ

Denoting $V_N = N^{2-2\kappa} V(N^{1-\kappa}\cdot)$ and $(V_N)_{p,q,r,s} = \langle u_p \otimes u_q, V_N u_r \otimes u_s \rangle$, $u_0 = \mathbb{1}_{[-1/2,1/2]^3}$

$$H_N^\kappa = \sum_p p^2 a_p^\dagger a_p + \frac{1}{2} \sum_{p,q,r,s} (V_N)_{p,q,r,s} a_p^\dagger a_q^\dagger a_r a_s$$

$$\underset{\text{c-numb}}{\simeq} \frac{1}{2} N^{1+\kappa} \widehat{V}(0) + \sum_{p \neq 0} (p^2 + N^\kappa \widehat{V}(0)) a_p^\dagger a_p + \underbrace{\sum_{p,q \neq 0} N (V_N)_{p,q,0,0} a_p^\dagger a_q^\dagger + h.c.}_{=: Q_2} + Q_3 + Q_4$$

c-number substitution $a_0 \simeq a_0^\dagger \simeq \sqrt{a_0^\dagger a_0} \simeq \sqrt{N}$ [Lewin-Nam-Serfaty-Solovej 15]

- Vacuum $|\Omega\rangle$ $(\simeq \mathbb{1}^{\otimes N})$ $\implies$ Hartree energy $\frac{1}{2} N^{1+\kappa} \widehat{V}(0)$
- Quasi-free states $e^{\sum_{p,q} (K_1)_{p,q} a_p^\dagger a_q^\dagger - h.c.} |\Omega\rangle \implies$
  $E(N,\kappa) \leq 4\pi\,\mathfrak{a}\, N^{1+\kappa} + \mathcal{O}(N^{5\kappa/2})$ [Erdös-Schlein-Yau 08] in TL
- $+$ Cubic correlation structure: $T_2 e^{\sum_{p,q,r} (K_c)_{p,q,r} a_p^\dagger a_q^\dagger a_r - h.c.} T_1 |\Omega\rangle \implies$

$$E(N,\kappa) \leq 4\pi\,\mathfrak{a}\, N^{1+\kappa} + \frac{1}{2} \sum_{p \neq 0} [\sqrt{p^4 + 16\pi\,\mathfrak{a}\, p^2 N^\kappa}] - p^2 - 8\pi\,\mathfrak{a}\, N^\kappa + \frac{(8\pi\,\mathfrak{a}\, N^\kappa)^2}{2p^2} + o(N^{5\kappa/2})$$

[Yau Yin 09], [Boccato-Brennecke-Cenatiempo-Schlein 19], [Basti-Cenatiempo-Schlein 21], ...

First quadratic transform

goal: renormalize Q_2

$$T_1 = \exp\left(\mathcal{B}_1\right), \qquad \mathcal{B}_1 = \int K_1(x,y) a_x^\dagger a_y^\dagger - h.c.$$

Choose K_1 so that (by the Duhamel formula)

$$\boxed{\begin{gathered}[\mathrm{d}\Gamma(-\Delta) + Q_4, \mathcal{B}_1] + Q_2 \simeq 0 \\ \frac{1}{2}\,\mathrm{ad}^{(2)}_{\mathcal{B}_1}\left(\mathrm{d}\Gamma(-\Delta) + Q_4\right) - \mathrm{ad}_{\mathcal{B}_1}(Q_2) \simeq \left(4\pi\,\mathfrak{a} - \frac{1}{2}\widehat{V}(0)\right) N^{1+\kappa}\end{gathered}}$$

$\longrightarrow$ [Nam T- 21], [Hainzl 21], [Hainzl Schlein T- 22]

- we want to take $K_1(x,y) = -N(1-f)(N^{1-\kappa}(x-y))$

$$-\Delta f + \frac{1}{2} V f = 0, \qquad 4\pi\,\mathfrak{a} = \frac{1}{2}\int Vf.$$

- cut-off at $\ell > 0$, $K_1(x-y) = -N(1-f)(N^{1-\kappa}(x-y))\chi(\lambda^{-1}(x-y))$

$$-\Delta K_1 = \frac{1}{2} N V_N (1 - K_1) + 4\pi\,\mathfrak{a}\, N^\kappa \lambda^{-3} W(\lambda^{-1}\cdot), \qquad \int_{\mathbb{R}^3} W = 1$$

- Neumann b.c. symmetrization by reflexion

$$K_1(x,y) = -\sum_{z\in\mathbb{Z}^3} N(1-f)(N^{1-\kappa}(P_z x - y))\chi(\lambda^{-1}(N^{1-\kappa}(P_z x - y)))$$

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$$\boxed{\begin{gathered}[\mathrm{d}\Gamma(-\Delta) + Q_4, \mathcal{B}_1] + Q_2 \simeq \widetilde{Q}_2 \\ \frac{1}{2}\,\mathrm{ad}^{(2)}_{\mathcal{B}_1}(\mathrm{d}\Gamma(-\Delta) + Q_4) - \mathrm{ad}_{\mathcal{B}_1}(Q_2) \simeq \left(4\pi\,\mathfrak{a} - \frac{1}{2}\widehat{V}(0)\right) N^{1+\kappa} + \underbrace{\widetilde{C}_{N,\lambda}}_{\simeq \lambda^{-1}}\end{gathered}}$$

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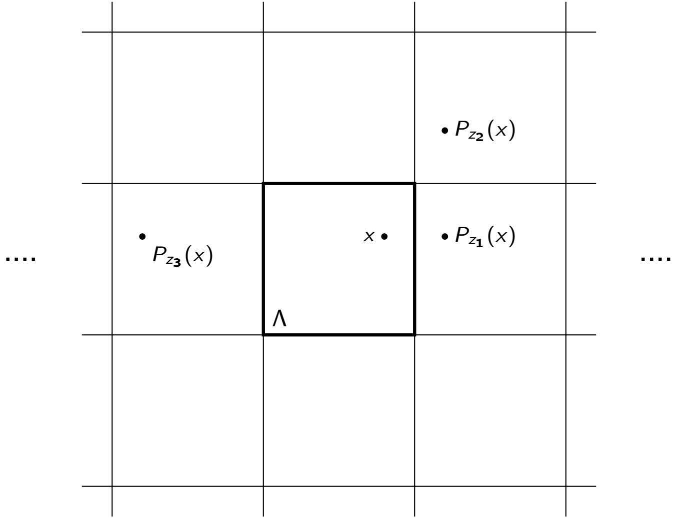

$$K_1(x,y) = -\sum_{z\in\mathbb{Z}^3} N(1-f)(N^{1-\kappa}(P_z x - y))\chi(\lambda^{-1}(N^{1-\kappa}(P_z x - y)))$$

goal: renormalize $Q_3 \lesssim \delta Q_4 + CN^\kappa \mathcal{N}_+$, for (smooth) $\theta \equiv 1$ on $[0,1]$ and $\theta \equiv 0$ on $[2,\infty]$

$$T_c = \exp\left(\mathcal{B}_c\right), \qquad \mathcal{B}_c = N^{-1/2}\theta\left(\frac{\mathcal{N}}{M}\right)\int K_1(x,y)a_x^\dagger a_y^\dagger a_y - h.c. \quad \text{as in [Nam T- 21]}$$

$$\boxed{\begin{gathered}\left[\mathrm{d}\Gamma\left(-\Delta\right) + Q_4, \mathcal{B}_c\right] + Q_3 \simeq 0 \\ \frac{1}{2}\,\mathrm{ad}^{(2)}_{\mathcal{B}_c}\left(\mathrm{d}\Gamma\left(-\Delta\right) + Q_4\right) - \mathrm{ad}_{\mathcal{B}_c}\left(Q_3\right) \simeq \left(4\pi\,\mathfrak{a} - \widehat{V}(0)\right) N^\kappa \mathcal{N}_+\end{gathered}}$$

$$\begin{aligned} T_c^* T_1^* H_N^\kappa T_1 T_c &\simeq 4\pi\,\mathfrak{a}\,N^{1+\kappa} + \mathrm{d}\Gamma(-\Delta + 8\pi\,\mathfrak{a}\,N^\kappa) + \widetilde{Q}_2 + \widetilde{C}_{N,\lambda} + Q_4 + \mathcal{E}, \\ \pm\mathcal{E} &= o\left(\mathrm{d}\Gamma(-\Delta) + Q_4\right) + CN^{2\kappa}\log N + C\lambda N^{C\kappa}\mathcal{N}_+ \\ &\quad + C\lambda^{-1/2}\frac{M}{N}\mathcal{N}_+ + CN^{C\kappa}\frac{(\mathcal{N}_+ + 1)^2}{N} + N^\kappa\left(1 - \theta\left(\frac{\mathcal{N}}{M}\right)\right)\mathcal{N}_+ + \ldots \end{aligned}$$

- We can use BEC for $\kappa > 0$ small $\langle \mathcal{N}_+ \rangle \ll N^{1-\varepsilon(\kappa)}$ from [Lieb-Seiringer 01] + localization in $\mathcal{N}_+$ from [Lewin-Nam-Serfaty-Solovej]

Diagonalization

The system is essentially quadratic

$$T_2^* T_c^* T_1^* H_N^\kappa T_1 T_c T_2 \simeq 4\pi \mathfrak{a} N^{1+\kappa} + 4\pi \times \frac{128}{15\sqrt{\pi}} \mathfrak{a}^{5/2} N^{5\kappa/2} + \mathrm{d}\Gamma(\sqrt{p^4 + 8\pi \mathfrak{a} N^\kappa p^2})$$

$$\begin{cases} \ell & = N^{1-\kappa} \\ \rho\ell^3 & = N \end{cases} \quad \Longrightarrow \quad N^\kappa = \rho\ell^2$$

Using Neumann bracketing and subadditivity of the entropy ([Hainzl-Lewin-Solovej 09]) ,

$$-\frac{T}{L^3} \log \operatorname{tr} e^{-\frac{H_N}{T}} \gtrsim -\frac{T}{\ell^3} \log \operatorname{tr} e^{-\frac{H_{n,\ell}}{T\ell^2}}$$

$$\simeq 4\pi \mathfrak{a} \rho^2 \left\{ 1 + (\rho \mathfrak{a}^3)^{1/2} \left(\frac{128}{15\sqrt{\pi}} + f_{\mathrm{Bog}}(\rho, T) \right) \right\}$$

$$\boxed{f_{\mathrm{Bog}}(\rho, T) = -\left(\frac{T}{\rho \mathfrak{a}}\right)^{5/2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \log(1 - e^{-(\sqrt{p^4 + \frac{16\pi\rho \mathfrak{a}}{T} p^2})}) \mathrm{d}p}$$

$$= \lim_{\ell\to\infty} \ell^{-3} \sum_{p \in 2\pi\mathbb{Z}^3} \log(1 - e^{-(\sqrt{(p/\ell)^4 + \frac{16\pi\rho \mathfrak{a}}{T} (p/\ell)^2})})$$

Conclusion

- dilute regime $\rho\,\mathfrak{a}^3 \to 0$ and small temperature $T \lesssim \rho\,\mathfrak{a}$

$$f(\rho, T) \geq 4\pi\,\mathfrak{a}\,\rho^2 \left(1 + C_{\mathrm{LHY}}\big(\tfrac{\rho\,\mathfrak{a}}{T}\big)\sqrt{\rho\,\mathfrak{a}^3} + o(\sqrt{\rho\,\mathfrak{a}^3})\right)$$

- using Neumann bracketing we reduce the problem to analysis on boxes much larger than the GP length scale (beyond Gross–Pitaevskii)
- We study H_N^κ with Neumann boundary conditions and $\kappa > 0$ (optimal rate for $\kappa = 0$ [Boccato Seiringer 22])

$$H_N^\kappa = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} N^{2-2\kappa} V(N^{1-\kappa}(x_i - x_j))$$

- new proof for $T = 0$

$$e(\rho) = 4\pi\,\mathfrak{a}\,\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\,\mathfrak{a}^3} + o(\sqrt{\rho\,\mathfrak{a}^3})\right)$$

Thank you for your attention!