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Energy Asymptotics for Bose Stars

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Mathematical results of many-body quantum systems

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BOSE STARS

Consider a system of N **bosons** interacting via (Newtonian) gravitational forces. On $L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$,

$$H_N = -\sum_{i=1}^{N} \Delta_i - \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}$$

We are interested in the **ground state energy** $E(N) = \inf \operatorname{spec} H_N$.

For large N, the **mean-field approximation** becomes accurate. In fact, it was shown in **[Lieb, Yau, 1987]** that

$$\lim_{N \to \infty} N^{-3} E(N) = e^{\mathrm{H}}$$

where e^{H} is the minimum of the Hartree functional

$$\mathcal{E}^{\mathrm{H}}(u) = \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx - \frac{1}{2} \iint_{\mathbb{R}^6} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx\, dy$$

We shall investigate the **next order term** as $N \to \infty$.

The Hartree Functional

[Lieb 1977] showed that $\mathcal{E}^H$ has a **unique** minimizer up to symmetries, i.e., every minimizer is of the form $e^{i\theta}u^{\mathrm{H}}(x-y)$ for $y \in \mathbb{R}^3$, $\theta \in [0, 2\pi)$.

Moreover, **[Lenzmann 2009]** showed that the **Hessian** of $\mathcal{E}^{\mathrm{H}}$ at u^{H} is non-degenerate except for the **trivial zero modes** given by iu^{H} and ∇u^{H}.

The **Bogoliubov Hamiltonian** $\mathbb{H}$ on $\mathcal{F}(L^2_{\perp u^{\mathrm{H}}}(\mathbb{R}^3))$ is given by its quantization,

$$\mathbb{H} = \int_{\mathbb{R}^3} \left(\nabla a^\dagger(x)\nabla a(x) - \left(|u^{\mathrm{H}}|^2 * |x|^{-1} + \mu^{\mathrm{H}}\right) a^\dagger(x)a(x)\right) dx$$
$$- \frac{1}{2}\iint_{\mathbb{R}^6} \frac{u^{\mathrm{H}}(x)u^{\mathrm{H}}(y)}{|x-y|} \left(a^\dagger(x)a^\dagger(y) + a(x)a(y) + 2a^\dagger(x)a(y)\right) dx\, dy$$

THEOREM [Brooks, S, 2021]:

$$E(N) = N(N-1)^2 e^{\mathrm{H}} + N^2 \inf \operatorname{spec} \mathbb{H} + o(N^2) \qquad \text{as } N \to \infty$$

Generalizations

Our result applies, more generally, to **translation-invariant** mean-field bosonic systems,

$$H_N = \sum_{i=1}^{N} t(-\Delta_i) - \frac{1}{N-1} \sum_{1\leq i<j\leq N} v(x_i - x_j)$$

for general t and v, **assuming** $e^{\mathrm{H}} < 0$, uniqueness of Hartree minimizers (up to symmetries) and non-degeneracy of the corresponding Hessian.

Example: $t(p^2) = \sqrt{p^2 + m^2} - m$ and $v(x) = \kappa |x|^{-1}$ for κ small enough.

Under natural assumptions on t and v, we show that

$$E(N) = N e^{\mathrm{H}} + \inf \operatorname{spec} \mathbb{H} + o(1) \qquad \text{as } N \to \infty$$

The corresponding question for **confined** systems with unique Hartree minimizer was studied in several earlier works: **[S 2011]**, **[Grech, S, 2013[**, **[Lewin, Nam, Serfaty, Solovej, 2015]**, **[Nam, S, 2015]**

Bose–Einstein Condensation

The **first key step** in the proof is to establish BEC for suitable low-energy states:

THEOREM: There exist states Ψ_N with $\langle \Psi_N | H_N \Psi_N \rangle \leq E(N) + o(1)$ as $N \to \infty$, such that

$$\lim_{N\to\infty} \gamma^{(1)}_{\Psi_N} = |u^{\mathrm{H}}\rangle\langle u^{\mathrm{H}}|$$

From **[Lewin, Nam, Rougerie, 2014]** we know that necessarily

$$\lim_{N\to\infty} \gamma^{(1)}_{\Psi_N} = \int_{\mathbb{R}^3} |u^{\mathrm{H}}_y\rangle\langle u^{\mathrm{H}}_y| \, d\mu(y)$$

for some measure with $\mu(\mathbb{R}^3) \leq 1$, where $u^{\mathrm{H}}_y(x) = u^{\mathrm{H}}(x-y)$. In order to ensure that $\mu = \delta_0$, we apply an IMS localization technique to (a regularized version of) the **median**: For $x_1 \leq x_2 \leq \cdots \leq x_N$,

$$M_k(x_1, \ldots, x_N) = \frac{1}{2k+1} \sum_{j=N/2-k}^{N/2+k} x_j$$

Dealing with Zero Modes, Part I

Since $\mathcal{E}^{\mathrm{H}}$ is defined on the sphere $\|u\| = 1$, it is convenient to introduce the **embedding**

$$\iota : \{z \in \{u^{\mathrm{H}}\}^{\perp},\ \|z\| \leq 1\} \to \{u \in L^2(\mathbb{R}^3),\ \|u\| = 1, \langle u^{\mathrm{H}}|u\rangle \geq 0\}$$
$$\iota(z) := \sqrt{1 - \|z\|^2} u^{\mathrm{H}} + z$$

eliminating the $U(1)$ symmetry.

Its **quantum counterpart** is the map $U_N : \mathcal{F}_{\leq N}(L^2_{\perp u^{\mathrm{H}}}(\mathbb{R}^3)) \to L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$ introduced in **[Lewin, Nam, Serfaty, Solovej]**

$$U_N u_1^{\otimes i_1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} u_m^{\otimes i_m} := (u^{\mathrm{H}})^{\otimes N - \sum_j i_j} \otimes_{\mathrm{s}} u_1^{\otimes i_1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} u_m^{\otimes i_m}$$

for $u_j \perp u^{\mathrm{H}}$ and $\sum_j i_j \leq N$, effectively eliminating the particle number constraint.

The **main problem** concerns the fact that not even for very small z

$$\mathcal{E}^{\mathrm{H}}(\iota(z)) \geq e^{\mathrm{H}} + (1-\varepsilon)\mathrm{Hess}|_{u^{\mathrm{H}}}\mathcal{E}^{\mathrm{H}}(z) \qquad \text{for } 0 < \varepsilon < 1$$

Dealing with Zero Modes, Part II

To deal with this problem, we introduce a second, **canonical** transformation F that "straightens" the manifold of minimizers. It satisfies, for small enough $a \in \mathbb{R}^3$,

$$\iota \circ F(a \cdot \nabla u^{\mathrm{H}}) = u^{\mathrm{H}}_{y(a)}$$

and, for small enough z,

$$\mathcal{E}^{\mathrm{H}}(\iota \circ F(z)) \geq e^{\mathrm{H}} + (1-\varepsilon)\mathrm{Hess}|_{u^{\mathrm{H}}}\mathcal{E}^{\mathrm{H}}(z) \qquad \text{for any } 0 < \varepsilon < 1$$

Its quantum counterpart is a **unitary** map $\mathcal{W}_N$ on $\mathcal{F}(L^2_{\perp u^{\mathrm{H}}}(\mathbb{R}^3))$ (reminiscent of the Gross-Transformation) which acts as a Weyl transform for modes orthogonal to ∇u^{H}, parametrized by $a(\nabla u^{\mathrm{H}}) + a^*(\nabla u^{\mathrm{H}})$.

The **key estimate** we prove is the lower bound

$$\mathcal{W}_N^{-1} U_N^{-1} H_N U_N \mathcal{W}_N \gtrsim N e^{\mathrm{H}} + (1-\varepsilon)\mathbb{H}$$

up to errors $o(1)$, for states that satisfy **strong BEC** in the sense that $U_N^{-1}\Psi \in \mathcal{F}_{\leq M}(L^2_{\perp u^{\mathrm{H}}}(\mathbb{R}^3))$ with $M \ll N$.

Conclusions

- We show the validity of the **Bogoliubov approximation** for translation-invariant Bose gases in the mean-field limit.
- As an example, our results apply to gravitating bosons ("**Bose stars**")
- To establish **Bose–Einstein condensation** for low-energy states, we introduce a localization procedure with respect to the median.
- To deal with the problem of zero modes, we introduce a **novel unitary transformation**, which can be interpreted as the quantum analogue of the canonical transformation straightening (locally) the manifold of Hartree minimizers.