Our first steps towards bulk-edge correspondence in interacting fermion systems

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Mathematical results of many-body quantum systems

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Based on joint work with **Horia Cornean, Jonas Lampart, Massimo Moscolari**, and **Tom Wessel**

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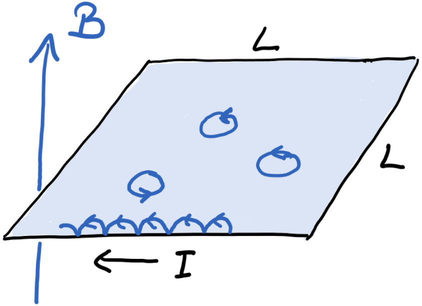
- Magnetic moments of closed orbits: $M_c \approx m_c\, L^2$
- Magnetic moment from boundary current I_{tr} of the skipping orbits: $M_I \approx L^2\, I_{\mathrm{tr}}$



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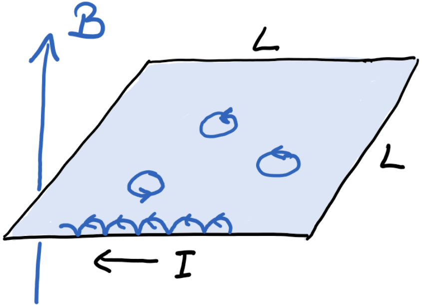

In thermal equilibrium the two contributions cancel exactly:

$$0 = M_{\mathrm{tot}} = (m_c + I_{\mathrm{tr}})L^2\,.$$

(Bohr-van Leeuwen theorem)

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Quantum: Let $H(B)$ denote the **Landau Hamiltonian** on $\mathbb{R}^2$, $H_L(B)$ its restriction to $[0, L]^2$,

$$F_\beta(x) := -\beta^{-1} \ln(1 + \mathrm{e}^{-\beta x}) \,, \quad \text{and} \quad F'_\beta(x) = (1 + \mathrm{e}^{\beta x})^{-1}$$

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The thermodynamic **pressure** is

$$p_L(\beta, \mu, B) := \frac{1}{L^2} \operatorname{tr}\left(F_\beta\big(H_L(B) - \mu\big)\right)$$

resp.

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and the **magnetisation** is

$$m_L(\beta, \mu, B) := \frac{\partial p_L(\beta, \mu, B)}{\partial B} = \tfrac{1}{2L^2} \operatorname{tr}\left((X \wedge J)\, F'_\beta\big(H_L(B) - \mu\big)\right) .$$

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Bulk-edge correspondence for the Landau operator

Macris, Martin, Pulé CMP '88

$p_\infty(\beta, \mu, B) = \lim_{L\to\infty} p_L(\beta, \mu, B)$ and $m_\infty(\beta, \mu, B) = \lim_{L\to\infty} m_L(\beta, \mu, B)$ exist and $m_\infty(\beta, \mu, B) = \frac{\partial p_\infty(\beta,\mu,B)}{\partial B}$.

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$$m_\infty(\beta,\mu,B) \;=\; I_{\mathrm{tot}}(\beta,\mu,B) \;:=\; \lim_{h\to\infty} \mathrm{tr}\left(\chi_h\, J_1\, F'_\beta\big(H_E(B)-\mu\big)\right)\;.$$

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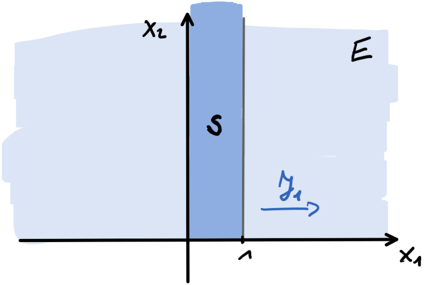

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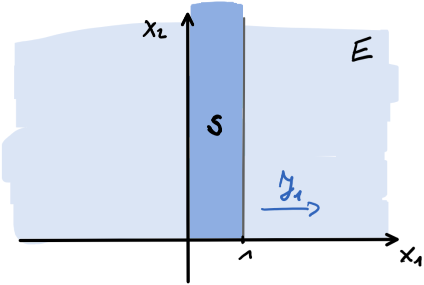

$H_E(B)$ is the Landau operator on the upper halfplane, χ_h the characteristic function of the strip $S_h := [0,1]\times[0,h]$, and $J_1 := \mathrm{i}[H_E(B), X_1]$ the first component of the current operator.

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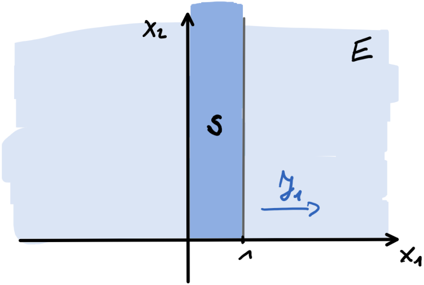

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$$\left|\mathrm{tr}\left(\chi_h\, J_1\, F'_\beta\big(H_E(B)-\mu\big)\right) - \lim_{\tilde h\to\infty} \mathrm{tr}\left(\chi_{\tilde h}\, J_1\, F'_\beta\big(H_E(B)-\mu\big)\right)\right| \le C\mathrm{e}^{-ch}$$

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$$m_\infty \;=\; \sum_{n=1}^{\infty} F_\beta'(e_n) \frac{\mathrm{d}e_n}{\mathrm{d}B} \frac{B}{2\pi} + \frac{p_\infty}{B} \;=:\; m_{\mathrm{c}} + m_{\mathrm{res}}\,.$$

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where the magnetisation current I_mag has density $j_\mathrm{mag} := \mathrm{curl} m_\mathrm{c}$ with $m_\mathrm{c}(x) = m_\mathrm{c}\,\chi_{\{x_2 \geq 0\}}(x)$ and thus

$$I_\mathrm{mag} = \int_{-\infty}^{\infty} (\mathrm{curl}\, m_\mathrm{c}(x))_1 \,\mathrm{d}x_2 = m_\mathrm{c} \int_{-\infty}^{\infty} \delta(x_2)\,\mathrm{d}x_2 = m_\mathrm{c}\,.$$

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Transport coefficients:

The edge conductance is defined by

$$\sigma_E(\beta, \mu, B) := \frac{\partial I_{\mathrm{tr}}(\beta, \mu, B)}{\partial \mu} \, .$$

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$\Rightarrow$ **Bulk-edge correspondence for transport coefficients**
of the Landau Hamiltonian at any temperature, i.e. for all $\beta > 0$, $\mu, B \in \mathbb{R}$

$$\sigma_H(\beta, \mu, B) = \sigma_E(\beta, \mu, B)\,.$$

Bulk-edge corresp. for magnetic Schrödinger operators

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Assumptions: The Hamiltonian describing the bulk system is

$$H_\omega(B) = \frac{1}{2}\left(-\mathrm{i}\nabla - \mathcal{A} - BA\right)^2 + V + V_\omega\,,$$

densely defined on $L^2(\mathbb{R}^2)$.

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Here V and $\mathcal{A}$ are $\mathbb{Z}^2$-periodic potentials, $A(x) = (-x_2, 0)$, and

$$V_\omega(x) = \sum_{\gamma\in\mathbb{Z}^2} \omega_\gamma u(x-\gamma)$$

with u compactly supported and $\{\omega_\gamma\}_{\gamma\in\mathbb{Z}^2}$ a family of i.i.d. random variables with values in $[-1,1]$. The functions $\mathcal{A}, V, u$ are smooth.

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The Hamiltonian describing the edge system is

$$H_{E,\omega}(B) := H_\omega(B)|_{L^2(E)} + W_\omega \quad \text{on} \quad L^2(E)$$

with $E := \{x \in \mathbb{R}^2 \,|\, x_2 \geq 0\}$ and Dirichlet boundary conditions. Here W_ω is a smooth random potential supported in a strip $\mathbb{R} \times [0, d]$ near the edge such that the family $\{H_{E,\omega}(B)\}_\omega$ is still ergodic with respect to integer translations in the x_1-direction.

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Theorem (Cornean, Moscolari, T. '21)

Also for ergodic magnetic Schrödinger operators it holds that for all $\beta > 0$, $\mu, B \in \mathbb{R}$

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If $\mu \notin \sigma(H_B)$, then **(Cornean, Monaco, Moscolari JEMS '21)**

$$\lim_{\beta\to\infty} \partial_\mu m_\infty(\beta,\mu,B) = \lim_{\beta\to\infty} \partial_B \rho_\infty(\beta,\mu,B) = \sigma_H(\infty,\mu,B)$$

and thus

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New proof of bulk-edge correspondence of transport coefficients at zero temperature for such systems!

Bulk-edge corresp. for magnetic Schrödinger operators

However, in general for $\beta < \infty$ or $\mu \in \sigma(H(B))$ we expect

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For periodic Schrödinger operators with simple Bloch bands only one can define a splitting $m_\infty = m_{\mathrm{c}} + m_{\mathrm{res}}$ as in the pure Landau case and show that

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For periodic Schrödinger operators with μ in a simple Bloch band or for μ in a mobility gap one can define a natural splitting $m_\infty = m_{\mathrm{c}} + m_{\mathrm{res}}$ locally in energy around μ . Then one finds (CMT '22)

$$|\sigma_H(\beta, \mu, B) - \sigma_E(\beta, \mu, B)| = \mathcal{O}(\mathrm{e}^{-c\beta})\,.$$

Previous rigorous results on $\sigma_H = \sigma_E$ at $T = 0$

The literature on bulk-edge correspondence is vast, and I mention only a few related rigorous results that consider equality of transport coefficients at $T = 0$ defined in microscopic models:

Non-interacting particles:

- Schulz-Baldes, Kellendonk, Richter JPA '00
- Elbau, Graf CMP '02
- Kellendonk, Schulz-Baldes JFA '04
- Elgart, Graf, Schenker CMP '05

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Interacting particles:

- Fröhlich, Studer RMP '93
- Giuliani, Mastropietro, Porta CMP '17
- Antinucci, Mastropietro, Porta CMP '18

Bulk-edge corresp. for magnetic Schrödinger operators

Strategy of the proof:

- Show that

$$\begin{aligned} p_\infty(\beta,\mu,B) &:= \mathbb{E}\,\mathrm{tr}\left(\chi_{[0,1]^2} F_\beta(H_\cdot - \mu)\right) \\ &\overset{\text{a.s.}}{=} \lim_{h\to\infty} h^{-1}\,\mathrm{tr}\left(\chi_{[0,1]\times[0,h]} F_\beta(H_{E,\omega} - \mu)\right) \end{aligned}$$

  and

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  using that "in the bulk"

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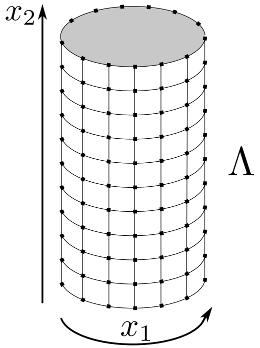
- Show

$$\lim_{h\to\infty} h^{-1}\partial_B \mathbb{E}\,\mathrm{tr}\left(\chi_{[0,1]\times[0,h]} F_\beta(H_{E,\cdot} - \mu)\right) = I_{\mathrm{tot}}(\beta,\mu,B)\,.$$

Bulk-edge corresp. for interacting electrons

We consider systems of interacting fermions on finite cylindrical domains $\Lambda = \{-L, \dots, L\}^2 \subset \mathbb{Z}^2$



The one-particle Hilbert space is

$$\mathfrak{h}_\Lambda := \ell^2(\Lambda, \mathbb{C}^s)\,,$$

the N-particle Hilbert space

$$\mathfrak{h}_{\Lambda,N} := \mathfrak{h}_\Lambda^{\wedge N}$$

and it will be convenient to work on Fock space

$$\mathfrak{F}_\Lambda := \bigoplus_{N=0}^{s|\Lambda|} \mathfrak{h}_{\Lambda,N}\,.$$

Bulk-edge corresp. for interacting electrons

The Hamiltonian describing an interacting gas of fermions on Λ is assumed to be of the form

$$H_B^\Lambda = \sum_{(x,y)\in\Lambda^2} a_x^* T_B(x,y) a_y + \sum_{x\in\Lambda} a_x^* V a_x + \sum_{(x,y)\in\Lambda^2} a_x^* a_x \phi(x-y) a_y^* a_y + \Phi_{\partial\Lambda}$$

where

$$T_B(x,y) := \mathrm{e}^{\mathrm{i}\frac{x_2+y_2}{2}B(x_1-y_1)}\, T(x-y)$$

is a Peierls phase times a translation invariant nearest neighbour hoppig amplitude $T : \mathbb{Z}^d \to \mathcal{L}(\mathbb{C}^s)$, $V \in \mathcal{L}(\mathbb{C}^s)$ an external "periodic" potential, and $\phi : \mathbb{Z}^d \to \mathcal{L}(\mathbb{C}^s)$ a short range interaction potential. Finally, $\Phi_{\partial\Lambda}$ is an arbitrary finite range local interaction supported in a fixed strip at the boundary.

Bulk-edge corresp. for interacting electrons

For $\beta > 0$ and $\mu, B \in \mathbb{R}$ denote the partition function by

$$Z(\beta, \mu, B) := \mathrm{tr}\left(\mathrm{e}^{-\beta(H_B^\Lambda - \mu\mathcal{N})}\right) ,$$

the Gibbs state by

$$\rho^\Lambda(\beta, \mu, B) := \frac{\mathrm{e}^{-\beta(H_B^\Lambda - \mu\mathcal{N})}}{Z(\beta, \mu, B)} ,$$

the grand canonical pressure by

$$p^\Lambda(\beta, \mu, B) := -L^{-2}\beta^{-1}\ln(Z(\beta, \mu, B)) ,$$

and the magnetisation by

$$m^\Lambda(\beta, \mu, B) := \frac{\partial}{\partial B} p^\Lambda(\beta, \mu, B) .$$

Bulk-edge corresp. for interacting electrons

The first component of the current operator is

$$\begin{aligned} J_1 &= \text{“}\, \mathrm{i}\Big[X_1,\ H_B^\Lambda\Big]\,\text{”} \\ &:= \sum_{x\in\Lambda} \Big(a^*_{x+e_1}\, T_L(b, x+e_1, x)\, a_x - a^*_x\, T_L(b, x, x+e_1)\, a_{x+e_1} \Big) \\ &=: \sum_{x\in\Lambda} j_{1,x} \end{aligned}$$

and we define the total boundary current (at the lower edge) as

$$I^{\Lambda,\ell}_{\mathrm{tot}}(\beta,\mu,B) := \mathrm{tr}\Big(\sum_{x_2=-L}^{-L+\ell} j_{1,(0,x_2)}\, \rho^\Lambda(\beta,\mu,B) \Big)\,.$$

Bulk-edge corresp. for interacting electrons

Locality of the Gibbs state:

We say that $(H_B^\Lambda)_\Lambda$ satisifes locality of the Gibbs state at (β, μ, B), iff there exist a state $\rho_\infty(\beta, \mu, B)$ on the quasi-local algebra $\mathcal{A}$ and constants $c, C > 0$ such that for all $A \in \mathcal{L}(\mathcal{F}_X) \subset \mathcal{A}$ and $\Lambda \supseteq X$

$$\left| \mathrm{tr}(\rho^\Lambda A) - \rho^\infty(A) \right| \leq C \|A\| \mathrm{e}^{-c\, \mathrm{dist}(X, \partial\Lambda)} \, .$$

It follows from results by **Kliesch et al. PRX '14** that there exists $\beta_0 > 0$ such that "locality of the Gibbs state" holds for all $\beta < \beta_0$ uniformly in B and μ.

Bulk-edge corresp. for interacting electrons

Theorem (Lampart, Moscolari, T., Wessel '22)

Assume locality of the Gibbs state for (β, μ, B).
Then there are constants $c, C > 0$ independent of L such that

$$\left| m^{\Lambda}(\beta, \mu, B) \; - \; I_{\mathrm{tot}}^{\Lambda,\ell}(\beta, \mu, B) \right| \leq C \left(L \mathrm{e}^{-c\ell} + \frac{1}{L} \right)$$

and

$$\left| \partial_\mu m^{\Lambda}(\beta, \mu, B) \; - \; \partial_\mu I_{\mathrm{tot}}^{\Lambda,\ell}(\beta, \mu, B) \right| \leq C \left(L \mathrm{e}^{-c\ell} + \frac{1}{L} \right) .$$

Bulk-edge corresp. for interacting electrons

Next steps:

- Relate $\partial_\mu m^\Lambda(\beta, \mu, B)$ and $\partial_\mu I_{\mathrm{tot}}^{\Lambda,\ell}(\beta, \mu, B)$ to transport coefficients.

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  (cf. Henheik, T., Wessel LMP '22; Bachmann, de Roeck, Donvil, Fraas '22)

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  (cf. Bachmann, de Roeck, Fraas CMP '18; T. CMP '20; Henheik, T. SIGMA '22)
  $\Rightarrow$ quantisation then follows from Hastings, Michalakis CMP '15; Bachmann, de Roeck, Fraas CMP '20.

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Thanks for your attention!

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