On the Huang-Yang energy correction for the dilute Fermi gas

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Mathematical results of many-body quantum systems
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Joint work with

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The Setting

- N interacting spin 1/2 fermions in a box $\Lambda_L := [0,L]^3$, with **periodic** boundary conditions.

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{i<j=1}^{N} V(x_i - x_j), \quad \mathfrak{h}(N_\uparrow, N_\downarrow) := L^2_a(\Lambda_L^{N_\uparrow}) \otimes L^2_a(\Lambda_L^{N_\downarrow})$$

  - $N_\sigma = \#$ particles with spin $\sigma \in \{\uparrow, \downarrow\} \Rightarrow N = N_\uparrow + N_\downarrow$.
  - $L^2_a(\Lambda_L^{N_\sigma})$ is the antisymmetric sector of $L^2(\Lambda_L)^{\otimes N_\sigma}$.
  - $V \geq 0$ is the 'periodization' on Λ_L of a potential V_∞ on $\mathbb{R}^3$, **compactly supported** and **regular** enough:

$$V(x-y) = \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} e^{ip\cdot(x-y)} \hat{V}_\infty(p), \quad \hat{V}_\infty(p) = \int_{\mathbb{R}^3} dx\, e^{-ip\cdot x} V_\infty(x)$$

The Setting

- The **ground state energy** (g.s.e.) of the system is

$$E_L(N_\uparrow, N_\downarrow) := \inf_{\psi \in \mathfrak{h}(N_\uparrow, N_\downarrow)} \frac{\langle \psi, H_N \psi \rangle}{\langle \psi, \psi \rangle}$$

- The **ground state energy density** is

$$e_L(\rho_\uparrow, \rho_\downarrow) := \frac{E_L(N_\uparrow, N_\downarrow)}{L^3},$$

  $\rho_\sigma = N_\sigma / L^3$ denotes the density of particles with spin σ ($\rho = \rho_\uparrow + \rho_\downarrow$ total density).

- We are interested in the **thermodynamic limit**: $N_\sigma, L \to \infty$ with ρ_σ fixed.

- We focus on the **dilute regime**, i.e, $\rho^{1/3} a \ll 1$.

What is expected?

Huang-Yang conjecture (1957)

$$\begin{aligned} e_L(\rho_\uparrow, \rho_\downarrow) &= \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow \\ &+ \frac{4(11 - 2\log 2)}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3} a^2 \rho^{7/3} + o(\rho^{7/3}). \end{aligned}$$

- The first term $\propto \rho^{\frac{5}{3}}$ is a purely kinetic term (Free Fermi Gas)
- The second and the third order corrections depend on the interaction via the *scattering length* a, which describes the effective range of the interaction:

$$8\pi a = \int \mathrm{d}x\, V(x) f(x), \qquad -\Delta f + \frac{1}{2} V f = 0, \qquad \lim_{|x| \to \infty} f(x) = 1$$

Remark: $\hat{V}(0) > 8\pi a \rightsquigarrow$ correlations play a role (see the next slide).

[1] Huang, Yang, *Quantum-Mechanical Many-Body Problem with Hard-Sphere Interaction*, Phys. Rev. (1957).

The Hartree Fock energy

$$H_N = -\sum_{i=1}^{N} \Delta_{x_i} + \sum_{i<j=1}^{N} V(x_i - x_j)$$

- H_N is spin indep. $\rightsquigarrow$ fully antisymmetrized Slater determinant

$$\Phi_{\mathrm{FFG}}(x_1, \cdots x_N) = \frac{1}{\sqrt{N!}} \det\big(f_{k_i}^{\sigma_i})(x_j)\big)_{1 \le i,j \le N}$$

$$f_k^{\sigma_i}(x) = L^{-3/2} e^{ik\cdot x} \text{ with } k \in \mathcal{B}_F^{\sigma_i}$$

$$\sigma_i = \uparrow \text{ for } i \in [1, N_\uparrow],\ \sigma_i = \downarrow \text{ for } i \in [N_\uparrow + 1, N_\downarrow]$$

- **Energy** of Φ_{FFG}: $\langle \Phi_{\mathrm{FFG}}, H_N \Phi_{\mathrm{FFG}} \rangle = E_{\mathrm{HF}}(\omega)$ (Hartree-Fock)

$$\boxed{E_{HF}(\omega) = -\mathrm{tr}\Delta\omega + \frac{1}{2} \sum_{\sigma,\sigma'} \int dxdy\, V(x-y)\big(\omega_{\sigma,\sigma}(x;x)\omega_{\sigma',\sigma'}(y;y) - |\omega_{\sigma,\sigma'}(x;y)|^2\big)}$$

$\det(f_i(x_j))_{1\le i,j\le N} = \sum_{\pi \in S_N} \sigma_\pi f_1(x_{\pi(1)}), \cdots, f_N(x_{\pi(N)})$

The Hartree Fock energy

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- **Energy** of Φ_{FFG}: $\langle \Phi_{\mathrm{FFG}}, H_N \Phi_{\mathrm{FFG}} \rangle = E_{\mathrm{HF}}(\omega)$ (Hartree-Fock)

$$\boxed{\frac{E_L(N_\uparrow, N_\downarrow)}{L^3} \le \frac{E_{HF}(\omega)}{L^3} = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + \underbrace{\widehat{V}(0)}_{\neq 8\pi a} \rho_\uparrow \rho_\downarrow + \mathcal{O}(\rho^{\frac{7}{3}})}$$

$\rightsquigarrow$ we are **missing correlations** between particles

Rigorous Results

Previous Rigorous Results

- It is well-known that [in units s.t. $\hbar = 1$, $m = 1/2$]

$$\boxed{e_L(\rho_\uparrow, \rho_\downarrow) = \underbrace{\frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}})}_{\substack{\textbf{purely kinetic} \\ \text{(Free Fermi Gas)}}} + \underbrace{8\pi a \rho_\downarrow \rho_\uparrow}_{\substack{\textbf{interaction term} \\ \text{a = scattering length of } V}} + o(\rho^2) \qquad (\star)}$$

- The first proof of $(\star)$ was done by Lieb-Seiringer-Solovej in [2].
- Extension to positive temperature Fermi gases by Seiringer in [3].
- Interacting lattice fermions (Hubbard model):

  Giuliani [4] (upper bound) and Seiringer-Yin [5] (lower bound).

[2] Lieb, Seiringer, Solovej, *Ground state energy of the low-density Fermi gas*, Phys. Rev. A (2005).

[3] Seiringer, *The Thermodynamic Pressure of a Dilute Fermi Gas*, CMP (2006).

[4] Giuliani, *Ground state energy of the low density Hubbard model: an upper bound*, J. Math. Phys. (2007).

[5] Seiringer, Yin, *Ground state energy of the low density Hubbard Model*. J. Stat. Phys. (2008).

Main Result

Theorem (M. Falconi, C. Hainzl, E.G., M. Porta, (Ann. Henri Poincaré (2021))

There exist $L_0 > 0$ such that, for $V \in C^\infty(\Lambda_L)$, V compactly supported, $V \geq 0$ and for $L \geq L_0$, the following holds:

$$e_L(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + r_L(\rho_\uparrow, \rho_\downarrow),$$

where a is the scattering length of the potential V, and for some constant C only dependent on V:

$$-C\rho^{2+\frac{1}{9}} \leq r_L(\rho_\uparrow, \rho_\downarrow) \leq C\rho^{2+\frac{2}{9}}$$

- W.r.t. [2], we treat more regular interactions (no hard-core)
- Improved error est. w.r.t. [2]: $-C\rho^{2+\frac{1}{39}} \leq r_L(\rho_\uparrow, \rho_\downarrow) \leq C\rho^{2+\frac{2}{27}}$

[2] Lieb, Seiringer, Solovej, *Ground state energy of the low-density Fermi gas*, Phys. Rev. A (2005).

Towards the Huang-Yang Energy Correction

Optimal rate (E.G., C. Hainzl, P.T. Nam, M. Porta)

Under the same assumptions as before, the following holds:

$$e_L(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + r_L(\rho_\uparrow, \rho_\downarrow),$$

where a is the scattering length of the potential V, and for some constant C only dependent on V:

$$r_L(\rho_\uparrow, \rho_\downarrow) = C\rho^{\frac{7}{3}}$$

- The optimal rate result allows to improve some of the a priori estimates & has to be thought as a toy model for the proof of the Huang-Yang conjecture.

Previous Results: Comparison with Bosons

- Consider now a dilute Bose gas ($3D$):
  - In the **bosonic** case, the interaction contribution to the ground state energy is at the leading order.
  - Bosons minimize the energy occupying the **lowest momentum state**, **forbidden** for fermions (Pauli principle).
- **Lee-Huang-Yang** (LHY) formula ($3D$)

$$e(\rho) = 4\pi a\rho^2\big(1 + (128/15\sqrt{\pi})(\rho a^3)^{\frac{1}{2}} + o(\sqrt{\rho a^3})\big), \qquad \text{as } \rho a^3 \to 0,$$

  - Leading order: upper bound Dyson [6], lower bound Lieb-Yngvason [7]
  - LHY correction: upper bound Yau-Yin [8] & Basti-Cenatiempo-Schlein [9], lower bound Fournais-Solovej [10, 11]

[6] Dyson, *Ground state energy of a Hard-Sphere Gas*, Phys. Rev. (1957).

[7] Lieb, Yngvason, *Ground state energy of the Low Density Bose gas*, Phys. Rev. Lett. (1998).

[8] Yau, Yin, *The Second Order Upper Bound for the Ground Energy of a Bose Gas*, J. Stat. Phys. (2009).

[9] Basti, Cenatiempo, Schlein, *A new second order upper bound for the ground state energy of dilute Bose gases*, Forum Math. Sigma 9, e74 (2021). .

[10] Fournais, Solovej, *The energy of dilute Bose gases*, Ann. Math. 2 (2020).

[11] Fournais, Solovej, *The energy of dilute Bose gases II: the general case*, arXiv:2108.12022 (2021).

Some ideas for the proofs

Particle - Hole Transformation

$$\mathcal{H} = \sum_\sigma \sum_k |k|^2 a^*_{k,\sigma} a_{k,\sigma} + \frac{1}{L^3} \sum_{k,p,q} \hat{V}(k) a^*_{p+k} a^*_{q-k} a_q a_p$$

$$\boxed{\omega_{\sigma,\sigma'} = \delta_{\sigma,\sigma'} \sum_{k \in \mathcal{B}_F^\sigma} |f_k\rangle\langle f_k| \qquad \text{1PDM free Fermi gas}} \qquad f_k = \frac{e^{ik\cdot x}}{L^{\frac{3}{2}}}$$

$u, v : L^2(\Lambda_L; \mathbb{C}^2) \to L^2(\Lambda_L; \mathbb{C}^2)$ s.t. $\overline{v}v = w, \quad u\overline{v} = 0$

- $R : \mathcal{F} \to \mathcal{F}$ (unitary) to compare **many-body g.s.e.** $\leftrightarrow$ **FFG energy**

$$(R\Omega)^{(N)} = \Phi_{\text{FFG}},\ (R\Omega)^{(n)} = 0 \text{ if } n \neq N \Rightarrow \langle R\Omega, \mathcal{H} R\Omega\rangle = E_{\text{HF}}(\omega).$$

$$R^* a^*_{k,\sigma} R = \begin{cases} a^*_{k,\sigma} & k \notin \mathcal{B}_F^\sigma \\ a_{k,\sigma} & k \in \mathcal{B}_F^\sigma \end{cases}$$

(implementing **particle-hole** transformation)

Approximate Ground State

Let $\psi \in \mathcal{F}$ be a normalized state s.t. $\langle \psi, \mathcal{N}_\sigma \psi \rangle = N_\sigma$, $N = N_\uparrow + N_\downarrow$:

$$\boxed{\langle \psi, \mathcal{H}\psi \rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi, \mathbb{H}_0 R^*\psi \rangle + \langle R^*\psi, \mathbb{X} R^*\psi \rangle + \sum_{i=1}^{4} \langle R^*\psi, \mathbb{Q}_i R^*\psi \rangle}$$

$$\mathbb{H}_0 = \sum_{k,\sigma} ||k|^2 - \mu_\sigma| \hat{a}^*_{k,\sigma} \hat{a}_{k,\sigma}, \qquad \mu_\sigma = (k_F^\sigma)^2$$

$$\mathbb{Q}_1 = \frac{1}{2L^3} \sum_{\sigma,\sigma'} \sum_{p, p+k \notin \mathcal{B}_F^\sigma} \sum_{p', p'-k \notin \mathcal{B}_F^{\sigma'}} \hat{V}(k) a^*_{p+k,\sigma} a^*_{p'-k,\sigma'} a_{p',\sigma'} a_{p,\sigma}$$

$$\mathbb{Q}_4 = \frac{1}{2L^3} \sum_{\sigma,\sigma'} \sum_{p \in \mathcal{B}_F^\sigma, p+k \notin \mathcal{B}_F^\sigma} \sum_{p' \in \mathcal{B}_F^{\sigma'}, p'-k \notin \mathcal{B}_F^{\sigma'}} \hat{V}(k) a^*_{p+k,\sigma} a^*_{p,\sigma} a^*_{p'-k,\sigma'} a^*_{p',\sigma'} + \text{h.c.}$$

- If ψ is an **approximate ground state** (ψ a.g.s.), i.e.,

$$\left| \frac{\langle \psi, \mathcal{H}\psi \rangle}{L^3} - \frac{1}{L^3} \sum_{\sigma=\uparrow\downarrow} \sum_{k \in \mathcal{B}_F^\sigma} |k|^2 \right| \leq C\rho^2$$

$$\Rightarrow \quad \langle \psi, \mathcal{H}\psi \rangle \sim E_{\mathrm{HF}}(\omega) + \langle R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1 + \mathbb{Q}_4) R^*\psi \rangle$$

The Correlation Structure: Pseudo-bosonic Operators

- **Pseudo-bosonic** operators:

$$b^*_{p,\sigma} = \sum_{\substack{k\,:\,k+p\notin\mathcal{B}_F^\sigma \\ k\in\mathcal{B}_F^\sigma}} a^*_{k,\sigma} a^*_{k+p,\sigma}$$

- **Almost** canonical commutation relation (**CCR**)

$$[b_{p,\sigma}, b^*_{q,\sigma'}] = \delta_{p,q}\delta_{\sigma,\sigma'} \underbrace{|\mathcal{B}_F^\sigma|}_{=\mathcal{O}(\rho_\sigma)} + \underbrace{\text{Error},}_{\substack{\psi\ a.g.s. \\ \Rightarrow \text{Error}=o(\rho) \\ [\rho=\rho_\sigma+\rho_{\sigma'}]}} \qquad [b_{p,\sigma}, b_{q,\sigma'}] = [b^*_{p,\sigma}, b^*_{q,\sigma'}] = 0$$

- We rescale the b operators:

$$c_{p,\sigma} = \rho_\sigma^{-1/2} b_{p,\sigma} \Rightarrow [c_{p,\sigma}, c^*_{q,\sigma'}] \sim \delta_{\sigma,\sigma'}\delta_{p,q}$$

$\text{Error} = -\delta_{\sigma,\sigma'} \sum_{\{k,k'\in\mathcal{B}_F^\sigma,\, k+p,k'+q\notin\mathcal{B}_F^\sigma\}} (\delta_{k,k'} \hat{a}^*_{k'+q,\sigma'} \hat{a}_{k+p,\sigma} + \delta_{k+p,k'+q} \hat{a}^*_{k',\sigma'} \hat{a}_{k,\sigma}$

The Correlation Structure: $\mathbb{H}_0$, $\mathbb{Q}_1$, $\mathbb{Q}_4$

- $\mathbb{Q}_4$ is quadratic w.r.t. $c_{p,\sigma}, c^*_{q,\sigma'}$:

$$\boxed{\mathbb{Q}_4 = \frac{1}{2L^3}\sum_{\sigma,\sigma'}\sum_p \rho_\sigma^{\frac{1}{2}}\rho_{\sigma'}^{\frac{1}{2}}\hat{V}(p)c^*_{p,\sigma}c^*_{-p,\sigma'} + \text{h.c.}}$$

- If ψ a.g.s. $\Rightarrow L^{-3}\langle R^*\psi, \mathcal{N}R^*\psi\rangle = o(\rho) \Rightarrow R^*\psi$ state with few particles

$$\rightsquigarrow \langle R^*\psi, \mathbb{H}_0 R^*\psi\rangle \simeq \langle R^*\psi, \mathbb{K}_B R^*\psi\rangle,$$

$$\begin{aligned}\rightsquigarrow \langle R^*\psi, \mathbb{Q}_1 R^*\psi\rangle \simeq \; & \frac{1}{2L^3}\sum_{p,\sigma,\sigma'}\hat{V}(p)\langle R^*\psi, \mathbb{G}_{p,\sigma}\mathbb{G}_{-p,\sigma'}R^*\psi\rangle \\ & -\frac{1}{2L^3}\sum_{p,\sigma}\hat{V}(p)\langle R^*\psi, \mathbb{G}_{0,\sigma}R^*\psi\rangle\end{aligned}$$

$$\boxed{\mathbb{K}_B = \frac{1}{L^3}\sum_\sigma\sum_p |p|^2 c^*_{p,\sigma}c_{p,\sigma}} \qquad \boxed{\mathbb{G}_{p,\sigma} = \frac{1}{L^3}\sum_\sigma\sum_q c^*_{q-p,\sigma}c_{q,\sigma}}$$

Pseudo-bosonic Bogoliubov Transformation

$$\begin{aligned}\langle\psi,\mathcal{H}\psi\rangle \simeq \quad & E_{\mathrm{HF}}(\omega) \quad + \quad \frac{1}{L^3}\sum_{p,\sigma}|p|^2\langle R^*\psi,\hat{c}^*_{p,\sigma}\hat{c}_{p,\sigma}R^*\psi\rangle \\ & + \quad \frac{1}{2L^3}\sum_{p,\sigma,\sigma'}\rho_\sigma^{\frac{1}{2}}\rho_{\sigma'}^{\frac{1}{2}}\hat{V}(p)\langle R^*\psi,\hat{c}^*_{-p,\sigma'}\hat{c}^*_{p,\sigma}R^*\psi\rangle+\mathrm{h.c.} \\ & + \quad \frac{1}{2L^9}\sum_{\substack{p,q,q'\\ \sigma,\sigma'}}\hat{V}(p)\langle R^*\psi,\hat{c}^*_{q-p,\sigma}\hat{c}_{q,\sigma}\hat{c}^*_{q'+p,\sigma'}\hat{c}_{q',\sigma'}R^*\psi\rangle \\ & - \quad \frac{1}{2L^6}\sum_{p,q,\sigma}\hat{V}(p)\langle R^*\psi,\hat{c}^*_{q,\sigma}\hat{c}_{q,\sigma}R^*\psi\rangle\end{aligned}$$

■ $\simeq \mathbb{H}_0$, ■ $\simeq \mathbb{Q}_4$, ■ $\simeq \mathbb{Q}_1$

- **Pseudo-bosonic** Bogoliubov transformation

$$T(\equiv T(\varphi)) = \exp\left\{\frac{1}{L^3}\sum_p \hat{\varphi}(p)\rho_\uparrow^{\frac{1}{2}}\rho_\downarrow^{\frac{1}{2}}\hat{c}_{p,\uparrow}\hat{c}_{-p,\downarrow}-\mathrm{h.c.}\right\}$$

[12] Benedikter, Nam, Porta, Schlein, Seiringer *Optimal upper bound for the correlations energy of a Fermi gas in the mean-fiel regime*, CMP (2020).

[13] Benedikter, Nam, Porta, Schlein, Seiringer, *Correlation energy of a weakly interacting Fermi gas*, Inventiones mathematicae (2021).

[14] Christiansen, Hainzl, Nam, *The Random Phase Approximation for Interacting Fermi Gases in the Mean-Field Regime*, arXiv:2106.11161 (2021)

Extracting $8\pi a\rho_\uparrow\rho_\downarrow$

$$\langle\psi,\mathcal{H}\psi\rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi,(\mathbb{H}_0+\mathbb{Q}_1+\mathbb{Q}_4)R^*\psi\rangle + o(L^3\rho^2)$$

$$T_\lambda := \exp\left\{\frac{\lambda}{L^3}\sum_p \hat{\varphi}(p)\rho_\uparrow^{\frac{1}{2}}\rho_\downarrow^{\frac{1}{2}}\hat{c}_{p,\uparrow}\hat{c}_{-p,\downarrow} - \mathrm{h.c.}\right\} \equiv \exp\{\lambda B\}, \qquad T\equiv T_1$$

$$\langle R^*\psi,(\mathbb{H}_0+\mathbb{Q}_1+\mathbb{Q}_4)R^*\psi\rangle = \quad \langle T^*R^*\psi,(\mathbb{H}_0+\mathbb{Q}_1)T^*R^*\psi\rangle - \int_0^1 d\lambda\,\langle T_\lambda^*R^*\psi,[\mathbb{H}_0+\mathbb{Q}_1,B]T_\lambda^*R^*\psi\rangle$$

Extracting $8\pi a \rho_\uparrow \rho_\downarrow$

$$\langle \psi, \mathcal{H}\psi\rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1 + \mathbb{Q}_4)R^*\psi\rangle + o(L^3\rho^2)$$

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$$\begin{aligned}\langle R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1 + \mathbb{Q}_4)R^*\psi\rangle = \quad & \langle T^*R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1)T^*R^*\psi\rangle - \int_0^1 d\lambda\, \langle T_\lambda^* R^*\psi, [\mathbb{H}_0 + \mathbb{Q}_1, B]T_\lambda^* R^*\psi\rangle \\ & + \langle T^*R^*\psi, \mathbb{Q}_4 T^*R^*\psi\rangle - \int_0^1 d\lambda\, \langle T_\lambda^* R^*\psi, [\mathbb{Q}_4, B]T_\lambda^* R^*\psi\rangle\end{aligned}$$

Extracting $8\pi a \rho_\uparrow \rho_\downarrow$

$$\langle \psi, \mathcal{H}\psi \rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1 + \mathbb{Q}_4) R^*\psi \rangle + o(L^3\rho^2)$$

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$$\begin{aligned}
\langle R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1 + \mathbb{Q}_4) R^*\psi \rangle &= \langle T^* R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1) T^* R^*\psi \rangle - \int_0^1 d\lambda \, \langle T_\lambda^* R^*\psi, [\mathbb{H}_0 + \mathbb{Q}_1, B] T_\lambda^* R^*\psi \rangle \\
&\quad + \langle T^* R^*\psi, \mathbb{Q}_4 T^* R^*\psi \rangle - \int_0^1 d\lambda \, \langle T_\lambda^* R^*\psi, [\mathbb{Q}_4, B] T_\lambda^* R^*\psi \rangle \\
&= \langle T^* R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1) T^* R^*\psi \rangle \quad (\geq 0 \rightsquigarrow \text{useful to est. the errors}) \\
([\mathbb{H}_0 + \mathbb{Q}_1, B] \simeq -\mathbb{Q}_4) \qquad &\quad - \langle T^* R^*\psi, [\mathbb{H}_0 + \mathbb{Q}_1, B] T^* R^*\psi \rangle + \langle T^* R^*\psi, \mathbb{Q}_4 T^* R^*\psi \rangle
\end{aligned}$$

[ψ = appr. g.s. (lower bound) / $\psi = RT\Omega$ (upper bound), i.e. $\psi = \widetilde{T} R \Omega$ = N-part. state]

Extracting $8\pi a \rho_\uparrow \rho_\downarrow$

$$\langle \psi, \mathcal{H}\psi \rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi, (\mathbb{H}_0 + \mathbb{Q}_1 + \mathbb{Q}_4) R^*\psi \rangle + o(L^3\rho^2)$$

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[ψ = appr. g.s. (lower bound) / $\psi = RT\Omega$ (upper bound), i.e. $\psi = \widetilde{T}R\Omega = N$-part. state]

Extracting $8\pi a\rho_\uparrow\rho_\downarrow$

$$\langle\psi,\mathcal{H}\psi\rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi, (\mathbb{H}_0+\mathbb{Q}_1+\mathbb{Q}_4)R^*\psi\rangle + o(L^3\rho^2)$$

$$T_\lambda := \exp\left\{\frac{\lambda}{L^3}\sum_p \hat{\varphi}(p)\rho_\uparrow^{\frac12}\rho_\downarrow^{\frac12}\hat{c}_{p,\uparrow}\hat{c}_{-p,\downarrow} - \mathrm{h.c.}\right\} \equiv \exp\{\lambda B\}, \qquad T\equiv T_1$$

$$\begin{aligned}
\langle R^*\psi,(\mathbb{H}_0+\mathbb{Q}_1+\mathbb{Q}_4)R^*\psi\rangle &= \langle T^*R^*\psi,(\mathbb{H}_0+\mathbb{Q}_1)T^*R^*\psi\rangle - \int_0^1 d\lambda\,\langle T_\lambda^*R^*\psi,[\mathbb{H}_0+\mathbb{Q}_1,B]T_\lambda^*R^*\psi\rangle \\
&\quad + \langle T^*R^*\psi,\mathbb{Q}_4T^*R^*\psi\rangle - \int_0^1 d\lambda\,\langle T_\lambda^*R^*\psi,[\mathbb{Q}_4,B]T_\lambda^*R^*\psi\rangle \\
&= \langle T^*R^*\psi,(\mathbb{H}_0+\mathbb{Q}_1)T^*R^*\psi\rangle \quad (\geq 0 \rightsquigarrow \text{useful to est. the errors})
\end{aligned}$$

$(-\Delta(1-\varphi)+V(1-\varphi)=0)$

$$- \langle T^*R^*\psi,[\mathbb{H}_0+\mathbb{Q}_1,B]T^*R^*\psi\rangle + \langle T^*R^*\psi,\mathbb{Q}_4T^*R^*\psi\rangle$$

$$+\int_0^1\int_1^\lambda d\lambda d\lambda'\,\langle T_{\lambda'}^*R^*\psi,[\mathbb{H}_0+\mathbb{Q}_1,B]T_{\lambda'}^*R^*\psi\rangle$$

$(8\pi a\rho_\uparrow\rho_\downarrow + \text{error} \;\leftsquigarrow)$

$$-\int_0^1 d\lambda\,\langle T_\lambda^*R^*\psi,[\mathbb{Q}_4,B]T_\lambda^*R^*\psi\rangle$$

[ψ = appr. g.s. (lower bound) / $\psi = RT\Omega$ (upper bound), i.e. $\psi = \widetilde{T}R\Omega = N$-part. state]

Comments

$$T = \exp\left\{ \frac{\lambda}{L^3} \sum_p \hat{\varphi}(p) \rho_\uparrow^{\frac{1}{2}} \rho_\downarrow^{\frac{1}{2}} \hat{c}_{p,\uparrow} \hat{c}_{-p,\downarrow} - \text{h.c.} \right\}, \ \hat{c}_{p,\uparrow} = \sum_k \hat{u}_\uparrow(k+p) \hat{v}_\uparrow(k) \hat{a}_{k+p,\uparrow} \hat{a}_{k,\uparrow}$$

- All the bosonic properties of T are only almost true:
  - ⋆ we work in the thermodynamic limit $\rightsquigarrow$ we cannot approximate T as
  $$T^*(\varphi) a^*_{p,\sigma} T(\varphi) = \cosh(\varphi) a_{p,\sigma} + \sinh(\varphi) a^*_{-p,-\sigma} + \text{Err}$$
  $\rightsquigarrow$ several error terms not trivial to control.
- We need uniform bounds w.r.t. the volume $\rightsquigarrow$ cut-off of the momenta of the quantities involved in T, i.e., $\hat{\varphi}, \hat{u}, \hat{v}$ (*we do not need to use any localization, our result is self-contained*)
- In the constant term we extract using T we have to remove all the cut-off in the momenta $\Rightarrow$ the interaction V has to be regular

The Role of the Cut-off

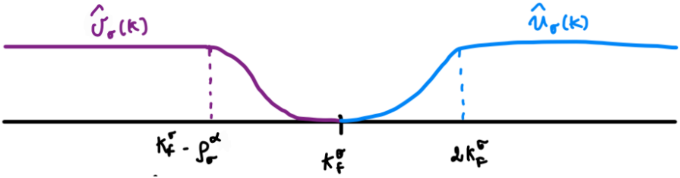
$$T = \exp\left\{\frac{\lambda}{L^3}\sum_p \hat{\varphi}(p)\rho_\uparrow^{\frac{1}{2}}\rho_\downarrow^{\frac{1}{2}}\hat{c}_{p,\uparrow}\hat{c}_{-p,\downarrow} - \text{h.c.}\right\}, \quad \hat{c}_{p,\uparrow} = \sum_k \hat{u}_\uparrow(k+p)\hat{v}_\uparrow(k)\hat{a}_{k+p,\uparrow}\hat{a}_{k,\uparrow}$$

- To induce a cut-off over $\hat{\varphi}(p)$ ($|p| \geq \rho^\gamma$) we take φ as the solution of the scattering equation in a ball centered at zero and with radius $\rho^{-\gamma}$ ($0 \leq \gamma \leq 1/3$), satisfying Neumann boundary conditions:

$$8\pi a_\gamma \rightsquigarrow 8\pi a + \mathcal{O}(\rho^\gamma)$$

- The cut-off over $\hat{u}$, $\hat{v} \rightsquigarrow \hat{u}^r$, $\hat{v}^r$:

$$\hat{u}^r_\sigma(k) = \begin{cases} 0 & \text{for } |k| \leq k_F^\sigma, \\ 1 & \text{for } 2k_F^\sigma \leq |k| \leq \rho^{-\beta}, \\ 0 & \text{for } |k| \geq 2\rho^{-\beta}, \end{cases} \quad \hat{v}^r_\sigma(k) = \begin{cases} 1 & \text{for } |k| \leq k_F^\sigma - \rho^\alpha, \\ 0 & \text{for } |k| \geq k_F^\sigma, \end{cases} \quad (\alpha \geq 1/3).$$



Towards the Huang-Yang Energy Correction

Huang-Yang Conjecture

$$e_L(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + \frac{4(11 - 2\log 2)}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3} a^2 \rho^{7/3} + o(\rho^{7/3}).$$

- A first step towards the proof of the Huang-Yang correction it is to prove (*optimal rate*)

$$e_L(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + C\rho^{7/3}.$$

(upper & lower bound)

[8] Huang, Yang, *Quantum-Mechanical Many-Body Problem with Hard-Sphere Interaction*, Phys. Rev. (1957).

Upper Bound (Optimal Rate)

Upper bound (E.G., C. Hainzl, P.T. Nam, M. Porta)

There exist $L_0 > 0$ such that, for $V \in C^\infty(\Lambda_L)$, V compactly supported, $V \geq 0$ and for $L \geq L_0$, the following holds:

$$e_L(\rho_\uparrow, \rho_\downarrow) \leq \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + C\rho^{\frac{7}{3}},$$

- We use again an almost bosonic Bogoliubov transformation of the form

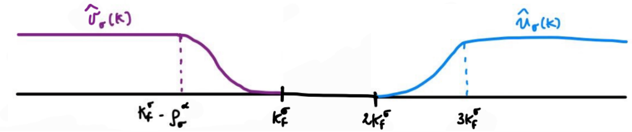
$$T(\equiv T(\varphi)) = \exp\left\{\frac{\lambda}{L^3}\sum_p \hat{\varphi}(p)\rho_\uparrow^{\frac{1}{2}}\rho_\downarrow^{\frac{1}{2}}\hat{c}_{p,\uparrow}\hat{c}_{-p,\downarrow} - \text{h.c.}\right\},\ \hat{c}_{p,\uparrow} = \sum_k \hat{u}_\uparrow(k+p)\hat{v}_\uparrow(k)\hat{a}_{k+p,\uparrow}\hat{a}_{k,\uparrow}$$

- The proof is a refinement of the one of the previous Theorem:
  1. Better way to choose the cut-off
  2. Technical improvements in estimating the error terms

Some Remarks - Upper Bound (Optimal Rate)

- The new cut-off over $\hat{u}(k)$, $\hat{v}(k)$ are

$$\hat{u}^r_\sigma(k) = \begin{cases} 0 & \text{for } |k| \leq 2k_F^\sigma, \\ 1 & \text{for } 3k_F^\sigma \leq |k| \leq \rho^{-\beta}, \\ 0 & \text{for } |k| \geq 2\rho^{-\beta}, \end{cases} \quad \hat{v}^r_\sigma(k) = \begin{cases} 1 & \text{for } |k| \leq k_F^\sigma - \rho^\alpha, \\ 0 & \text{for } |k| \geq k_F^\sigma, \end{cases} \quad (\alpha \geq 1/3).$$



$\Rightarrow$ we can take φ as the solution of the zero energy scattering equation $-\Delta(1-\varphi) + V(1-\varphi) = 0 \rightsquigarrow$ effectively this is supported for $|k| \geq \rho^{\frac{1}{3}}$

$\Rightarrow$ we then can improve several error estimates using that

$$\sum_{|k|\geq 2k_F} \hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma} \leq C\rho^{-\frac{2}{3}} \sum_k ||k|^2 - k_F^2| \hat{a}^*_{k,\sigma}\hat{a}_{k,\sigma} \equiv \rho^{-\frac{2}{3}} \mathbb{H}_0$$

Lower Bound (Optimal Rate)

Lower bound (E.G., C. Hainzl, P.T. Nam, M. Porta)

There exist $L_0 > 0$ such that, for $V \in C^\infty(\Lambda_L)$, V compactly supported, $V \geq 0$ and for $L \geq L_0$, the following holds:

$$e_L(\rho_\uparrow, \rho_\downarrow) \geq \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + C\rho^{\frac{7}{3}},$$

1. The first step is to proceed as for the upper bound, using the Bogoliubov transformation quadratic w.r.t. the almost bosonic operators.

   $$\Rightarrow \langle \psi, \mathcal{H}\psi \rangle \geq E_{\mathrm{HF}} + \langle T^* R^* \psi, (\mathbb{H}_0 + \mathbb{Q}_1) T^* R^* \psi \rangle + 8\pi a \rho_\uparrow \rho_\downarrow L^3 + \widetilde{\mathbb{Q}}_4 + \mathcal{E},$$

$$\widetilde{\mathbb{Q}}_4 = \underbrace{\left(\frac{1}{2L^3} \sum_{\sigma,\sigma'} \sum_p \rho_\sigma^{\frac{1}{2}} \rho_{\sigma'}^{\frac{1}{2}} \big(\hat{V}(p) - \hat{V} * \hat{\varphi}\big) c^*_{p,\sigma} c^*_{-p,\sigma'} + \mathrm{h.c.} \right)}_{\text{same structure of } \mathbb{Q}_4 \ (\text{quadratic in the } \hat{c}, \hat{c}^*)} \Bigg|_{ck_F \leq |p| \leq Ck_F}$$

$$|\mathcal{E}| \leq CL^3 \rho^{\frac{7}{3}}$$

Lower Bound (Optimal Rate)

Lower bound (E.G., C. Hainzl, P.T. Nam, M. Porta)

There exist $L_0 > 0$ *such that, for* $V \in C^\infty(\Lambda_L)$, V *compactly supported,* $V \geq 0$ *and for* $L \geq L_0$, *the following holds:*

$$e_L(\rho_\uparrow, \rho_\downarrow) \geq \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + C\rho^{\frac{7}{3}},$$

2. To show that $\widetilde{\mathbb{Q}}_4$ is $\mathcal{O}(\rho^{\frac{7}{3}} L^3)$ is not trivial in the lower bound (*not good a priori estimates as for the upper bound*)

   $\rightsquigarrow$ one has to cancel it using a modified scattering equation

   $\rightsquigarrow$ this induces several technical problems.

3. One has to deal more carefully with the non-bosonizable term of the form "$\hat{c}^* \hat{a}^* \hat{a}$"

Work in Progress
Rigorous Proof of the Huang-Yang Formula

$$\boxed{e_L(\rho_\uparrow, \rho_\downarrow) = \frac{3}{5}(6\pi^2)^{\frac{2}{3}}(\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + \frac{4(11 - 2\log 2)}{35\pi^2}\left(\frac{3}{4\pi}\right)^{4/3} a^2 \rho^{7/3} + o(\rho^{7/3})}$$

- An adaptation of the proof of the optimal rate should give the right upper bound
- The lower bound is more involved (already true at the level of the optimal rate)
- A long term project is to go beyond $\rho^{\frac{7}{3}}$.

Thank you for your attention!

How to calculate higher-order corrections?

Work in progress with C. Hainzl, P.T. Nam, M. Porta

- One has to use a different transformation:

$$T = \exp\left\{ \frac{1}{L^3} \sum_k \sum_{\substack{r \in \mathcal{B}_F^\uparrow \\ r+k \notin \mathcal{B}_\uparrow^F}} \sum_{\substack{r' \in \mathcal{B}_F^\downarrow \\ r'-k \notin \mathcal{B}_F^\downarrow}} \hat{\varphi}_{r,r'}(k) a_{k+r,\uparrow} a_{k,\uparrow} a_{k-r',\downarrow} a_{r',\downarrow} - \text{h.c.} \right\}$$

where $\hat{\varphi}_{r,r'}(k)$ satisfies

$$(|k|^2 + 2k \cdot (r - r'))\hat{\varphi}_{r,r'}(k) + \hat{V}(k) - \frac{1}{L^3} \sum_q \hat{V}(p-q)\hat{\varphi}_{k,k'}(q) = 0,$$

known as *Bethe-Goldstone* equation $\rightsquigarrow$ the leading term is now

$$-\frac{1}{L^6} \sum_k \sum_{\substack{r \in \mathcal{B}_F^\uparrow \\ r+k \notin \mathcal{B}_\uparrow^F}} \sum_{\substack{r' \in \mathcal{B}_F^\downarrow \\ r'-k \notin \mathcal{B}_F^\downarrow}} \hat{V}(k)\hat{\varphi}_{r,r'}(k)$$

How to calculate higher-order corrections?

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where $\hat{\varphi}_{r,r'}(k)$ satisfies

$$(|k|^2 + 2k \cdot (r - r'))\hat{\varphi}_{r,r'}(k) + \hat{V}(k) - \frac{1}{L^3} \sum_q \hat{V}(p-q)\hat{\varphi}_{k,k'}(q) = 0,$$

known as *Bethe-Goldstone* equation $\rightsquigarrow$ the leading term is now

$$-\frac{1}{L^6} \sum_k \sum_{\substack{r \in \mathcal{B}_F^\uparrow \\ r+k \notin \mathcal{B}_\uparrow^F}} \sum_{\substack{r' \in \mathcal{B}_F^\downarrow \\ r'-k \notin \mathcal{B}_F^\downarrow}} \hat{V}(k)\hat{\varphi}_{r,r'}(k)$$

Thank you for the attention!

Some Comments on the scattering equation

- $\mathbb{H}_0, \mathbb{Q}_1$ can be written in terms of c, c^* provided they act on state with few particles, with momenta $|q| \gg \rho^{1/3}$

- $R^*\psi \simeq T\Omega$ which is a superposition of states with momenta supported in the support of $\hat{\varphi}(p)$

  $\rightsquigarrow$ we need to regularise $\hat{\varphi}$ so that it is supported on a ball of radius $1 \ll R \ll \rho^{-1/3}$

  $\rightsquigarrow$ φ_γ sol. of the scattering equation in the ball $B_{\rho^{-\gamma}}(0)$, $0 \leq \gamma \leq 1/3$

$$-\Delta(1-\varphi_\gamma)+\frac{1}{2}V_\infty(1-\varphi_\gamma) = \lambda_\gamma(1-\varphi_\gamma), \quad \varphi_\gamma = \nabla\varphi_\gamma = 0 \quad \text{on } \partial B_{\rho^{-\gamma}}(0)$$

- The φ we take is the periodization of φ_γ, i.e.,

$$\varphi(x) = \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} e^{ip\cdot x} \hat{\varphi}_\gamma(p)$$

Scattering equation in a ball

⚠ Remember that we are looking to states with $|q| \gg \rho^{1/3}$

$$\boxed{T = \exp\left\{ \frac{1}{L^3} \sum_p \hat{\varphi}(p) \rho_\uparrow^{\frac{1}{2}} \rho_\downarrow^{\frac{1}{2}} \hat{c}_{p,\uparrow} \hat{c}_{-p,\downarrow} - \text{h.c.} \right\}, \quad \hat{\varphi} \text{ s.t. } 2|p|^2 \hat{\varphi}(p) - \hat{V}(p) + (V * \hat{\varphi})(p) = 0}$$

$$\boxed{\frac{\langle R^* \psi, \mathcal{H} R^* \psi \rangle}{L^3} = \frac{3}{5}(6\pi^2)^{\frac{2}{3}} (\rho_\uparrow^{\frac{5}{3}} + \rho_\downarrow^{\frac{5}{3}}) + 8\pi a \rho_\uparrow \rho_\downarrow + o(\rho^2)}$$

- $R^*\psi \simeq T\Omega \rightsquigarrow$ momenta of $R^*\psi$ in the support of $\hat{\varphi}(p) \Rightarrow$ we would like $\hat{\varphi}(p)$ to be supported for $|p| \gg \rho^{1/3}$

$$\Downarrow$$

we take into account φ_∞: sol. of the scat. eq. in the ball $B_{\rho^{-\gamma}}(0)$ $(0 \leq \gamma \leq 1/3$, $\varphi_\infty = \nabla \varphi_\infty = 0$ on $\partial B_{\rho^{-\gamma}}(0))$. In p-space the eq. is:

$$2|p|^2 \hat{\varphi}(p) - \hat{V}_\infty(p) + (\hat{V}_\infty * \hat{\varphi}_\infty)(p) = \lambda_\gamma (1 - \hat{\varphi}_\infty),$$

$$\varphi(x) := \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} e^{ip\cdot x} \hat{\varphi}_\infty(p), \quad \int dx\, V_\infty(x)(1 - \varphi_\infty(x)) = 8\pi a_\gamma, \quad |a - a_\gamma| \leq C\rho^\gamma$$

$V(x-y) = \frac{1}{L^3} \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} e^{ip\cdot(x-y)} \hat{V}_\infty(p)$

Ideas for the rigorous proof

$$\langle\psi,\mathcal{H}\psi\rangle = E_{\mathrm{HF}}(\omega) + \langle R^*\psi, (\mathbb{H}_0+\mathbb{Q}_1+\mathbb{Q}_4)R^*\psi\rangle + o(L^3\rho^2)$$

$$T_\lambda := T = \exp\left\{\frac{\lambda}{L^3}\sum_p \hat{\varphi}(p)\rho_\uparrow^{\frac12}\rho_\downarrow^{\frac12}\hat{c}_{p,\uparrow}\hat{c}_{-p,\downarrow} - \mathrm{h.c.}\right\} \equiv \exp\{\lambda B\}, \qquad T\equiv T_1$$

- We can discard the interaction between particles with the same spin
- We need to regularized (cut-off in the momenta) the quantities involved in T, i.e., φ, u, ω

$$\langle R^*\psi, (\mathbb{H}_0+\mathbb{Q}_1+\mathbb{Q}_4)R^*\psi\rangle = \langle T^*R^*\psi, (\mathbb{H}_0+\mathbb{Q}_1)T^*R^*\psi\rangle - \int_0^1 \langle T_\lambda^* R^*\psi, [\mathbb{H}_0+\mathbb{Q}_1, B]T_\lambda^*R^*\psi\rangle$$
$$+\langle R^*\psi, \mathbb{Q}_4 R^*\psi\rangle$$

$$= \langle T^*R^*\psi, (\mathbb{H}_0+\mathbb{Q}_1)T^*R^*\psi\rangle \quad (\geq 0 \rightsquigarrow \text{useful to est. the errors})$$

$$(\simeq 0 \text{ due to the scat. eq.}) \qquad - \int_0^1 d\lambda\, \langle T_\lambda^*R^*\psi, ([\mathbb{H}_0+\mathbb{Q}_1, B] - \mathbb{Q}_4)T_\lambda^*R^*\psi\rangle$$

$$(8\pi a\rho_\uparrow\rho_\downarrow + \text{error} \ \leftsquigarrow) \qquad + \langle R^*\psi, \mathbb{Q}_4R^*\psi\rangle - \int_0^1 d\lambda\, \langle T_\lambda^*R^*\psi, \mathbb{Q}_4 T_\lambda^*R^*\psi\rangle$$

[ψ = appr. g.s. (lower bound) / $\psi = RT\Omega$ (upper bound), i.e. $\psi = \widetilde{T}R\Omega = N$-part. state]

The correlation structure: $\mathbb{H}_0$

- It is convenient to rescale the b operators:

$$\hat{c}_{p,\sigma} = \rho_\sigma^{-1/2}\hat{b}_{p,\sigma} \Rightarrow [\hat{c}_{p,\sigma}, \hat{c}^*_{q,\sigma'}] = \delta_{\sigma,\sigma'}\delta_{p,q}$$

- $\mathbb{H}_0, \mathbb{Q}_1$ do not have the same structure as $\mathbb{Q}_4$ **but** they behave as **almost bosonic operators** on a **suitable class of states**.

Recall that $\mathbb{H}_0 = \sum_{k,\sigma} ||k|^2 - (k_F^\sigma)^2|a^*_{k,\sigma}a_{k,\sigma}$. Moreover,

$$[\mathbb{H}_0, \hat{c}^*_{q,\sigma}] = \rho_\sigma^{-\frac{1}{2}} \sum_{\substack{k: k+q\notin\mathcal{B}_F^\sigma \\ k\in\mathcal{B}_F^\sigma}} (|k+q|^2 - |k|^2)\hat{a}^*_{k+q,\sigma}\hat{a}^*_{k,\sigma} \underbrace{\simeq}_{\substack{|k|\leq C\rho_\sigma^{1/3} \\ |q|\gg\rho_\sigma^{1/3}}} |q|^2\hat{c}^*_{q,\sigma}$$

$$\mathbb{K}_B = \frac{1}{L^3}\sum_{p,\sigma}|p|^2\hat{c}^*_{p,\sigma}\hat{c}_{p,\sigma}$$

$$\mathbb{H}_0\hat{c}^*_{q,\sigma}\Omega = [\mathbb{H}_0, \hat{c}^*_{q,\sigma}]\Omega \simeq [\mathbb{K}_B, \hat{c}^*_{q,\sigma}]\Omega = \mathbb{K}_b\hat{c}^*_{q,\sigma}\Omega$$

$\rightsquigarrow$ if $R^*\psi$ is a state with few particles (if the c operators are true bosonic operators $\Rightarrow \hat{c}^*_{q,\sigma}\Omega$ is a state with one boson)

$$\langle R^*\psi, \mathbb{H}_0 R^*\psi\rangle \simeq \langle R^*\psi, \mathbb{K}_B R^*\psi\rangle$$

Extracting $\mathcal{O}(\rho^2)$

$$\boxed{T = \exp\left\{ \frac{1}{L^3} \sum_p \hat{\varphi}(p) \rho_\uparrow^{\frac{1}{2}} \rho_\downarrow^{\frac{1}{2}} \hat{c}_{p,\uparrow} \hat{c}_{-p,\downarrow} - \text{h.c.} \right\}}$$

- If ψ a.g.s. $\Rightarrow L^{-3}\langle R^*\psi, \mathcal{N} R^*\psi\rangle = o(\rho) \rightsquigarrow R^*\psi \overbrace{\simeq}^{L^{-3}\langle T\Omega, \mathcal{N} T\Omega\rangle = o(\rho)} T\Omega$

$$\langle R^*\psi, \mathcal{H} R^*\psi\rangle \simeq \langle T\Omega, \mathcal{H} T\Omega\rangle = E_{\mathrm{HF}}(\omega) + 2L^3 \rho_\uparrow \rho_\downarrow e(\varphi)$$

$$\boxed{e(\varphi) = \frac{1}{L^3} \sum_p \left(|p|^2 \hat{\varphi}(p)^2 - \hat{V}(p)\hat{\varphi}(p) + \frac{1}{2}(\hat{V} * \hat{\varphi})(p)\hat{\varphi}(p) \right)}$$

- If φ is the sol. of: $\underline{2|p|^2\hat{\varphi}(p) - \hat{V}(p) + (V * \hat{\varphi})(p) = 0}$ (scatt. eq.)

$$\Rightarrow e(\varphi) = -\frac{1}{2L^3} \sum_p \hat{V}(p)\hat{\varphi}(p)$$

Extracting $\mathcal{O}(\rho^2)$

$$\boxed{T = \exp\left\{\frac{1}{L^3}\sum_p \hat\varphi(p)\rho_\uparrow^{\frac12}\rho_\downarrow^{\frac12}\hat c_{p,\uparrow}\hat c_{-p,\downarrow} - \text{h.c.}\right\}}$$

- If ψ a.g.s. $\Rightarrow L^{-3}\langle R^*\psi, \mathcal{N}R^*\psi\rangle = o(\rho) \rightsquigarrow R^*\psi \overbrace{\simeq}^{L^{-3}\langle T\Omega,\mathcal{N}T\Omega\rangle = o(\rho)} T\Omega$

$$\langle R^*\psi, \mathcal{H}R^*\psi\rangle \simeq \langle T\Omega, \mathcal{H}T\Omega\rangle = E_{\text{HF}}(\omega) + 2L^3\rho_\uparrow\rho_\downarrow e(\varphi)$$

$$\boxed{e(\varphi) = \frac{1}{L^3}\sum_p \left(|p|^2\hat\varphi(p)^2 - \hat V(p)\hat\varphi(p) + \frac12(\hat V * \hat\varphi)(p)\hat\varphi(p)\right)}$$

- If φ is the sol. of: $\underline{2|p|^2\hat\varphi(p) - \hat V(p) + (V * \hat\varphi)(p) = 0}$ (scatt. eq.)

$$\frac{\langle R^*\psi, \mathcal{H}R^*\psi\rangle}{L^3} = \frac35(6\pi^2)^{\frac23}(\rho_\uparrow^{\frac53}+\rho_\downarrow^{\frac53}) + \rho_\uparrow\rho_\downarrow\Bigg(\underbrace{\hat V(0) - \frac{1}{L^3}\sum_p \hat V(p)\hat\varphi(p)}_{=\int dx\, V_\infty(x)(1-\varphi(x)) = 8\pi a,\ L\to\infty}\Bigg) + o(\rho^2)$$