# Dilute Bose and Fermi gases in dimensions 1–3

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# Many-body Hamiltonians with 2-body interactions

Consider $N$ identical particles in a box $\Omega_L = [0, L]^d$ (Dirichlet or periodic b.c.). Dimension: $d = 1, 2, 3$.

**The most basic translation invariant many-body Hamiltonian:**

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} v(x_i - x_j)$$

**2-body potential:** $v : \mathbb{R}^d \to [0, \infty]$ spherically symmetric, compact support.

**Hilbert Space Bosons**: $\mathcal{H}_N(\Omega_L) = \bigvee^N L^2(\Omega_L)$ or $L^2(\Omega_L^N)$

**Hilbert Space (spin-$J$) Fermions**: $\mathcal{H}_N(\Omega_L) = \bigwedge^N L^2(\Omega_L; \mathbb{C}^{2J+1})$

# The thermodynamic limit

Our focus will be on the **Ground State Energy:**

$$E_L(N) = \inf \mathsf{Spec}_{\mathcal{H}_N(\Omega_L)}(H_N)$$

We will be interested in the **Thermodynamic Limit:**

$$\boxed{e(\rho) = \lim_{\substack{L\to\infty \\ N/L^d \to \rho}} e_L(N)}$$

of the **energy per volume**

$$e_L(N) = L^{-d} E_L(N).$$

We will write $e_B(\rho)$ or $e_F(\rho)$ for Bosons or Fermions.

# Examples

**Free gas:** $v = 0$

$$e_B^{\text{free}}(\rho) = 0,\ e_F^{\text{free}}(\rho) = C_d \rho^{(d+2)/d}, \qquad d=1: \quad e_F^{\text{free}}(\rho) = \frac{\pi^2}{3}\rho^3$$

The Bosonic state is **pure condensate**:
$\Psi(x_1, \ldots, x_N) = \prod_{i=1}^N \phi(x_i), \quad \phi \approx$ const.
If we use this as a trial state for $v \neq 0$ we get $e_B(\rho) \leq \frac{1}{2}\rho^2 \int v$.
Note for 1D:

$$e_F^{\text{free}}(\rho) \leq \frac{1}{2}\rho^2 \int v, \quad \text{if} \quad \rho \leq \frac{3}{2\pi^2}\int v.$$

**Hard core potential:**

$$v_a(x) = \begin{cases} \infty, & |x| < a \\ 0, & |x| > a \end{cases}$$

# Exactly solvable examples in 1D

For $d = 1$ we have for the **hard core gas**

$$e_B^{\mathrm{HC}}(\rho) = e_F^{\mathrm{HC}}(\rho) = \frac{\pi^2}{3}\rho^3(1-\rho a)^{-2}$$

Like a free Fermi gas $\frac{\pi^2}{3}N\rho^2$ with a smaller volume:
$\rho \to \rho(1-\rho a)^{-1}$.
The **Lieb-Liniger model** for bosons has $v = 2c\delta_0$. It can be solved exactly (Lieb-Liniger 1963). In particular,

$$e_B^{\mathrm{LL}}(\rho) \geq \frac{\pi^2}{3}\rho^3(1-\rho a)^{-2},\ a = -2/c.$$

The lower bound is asymptotically right as $\rho|a| \to 0$.
Main goal here is to explain that $\frac{\pi^2}{3}\rho^3(1-\rho a)^{-2}$ is asymptotically correct for all $v$.

# The scattering length

For $\mathrm{supp}(v) \subseteq B_R$. Let $\phi$ be unique solution to

$$-\Delta\phi + \frac{1}{2}v\phi = 0, \qquad \text{on } B_R, \quad \phi(x) = 1 \text{ for } |x| = R$$

Then $\phi(x) = f(|x|)$ and for $\mathrm{range}(v) < r < R$, we have

$$f(r) = \begin{cases} (r-a)/(R-a) & \text{for } d = 1 \\ \ln(r/a)/\ln(R/a) & \text{for } d = 2 \\ (1-ar^{2-d})/(1-aR^{2-d}) & \text{for } d \geq 3, \end{cases}$$

with some constant $a \in \mathbb{R} \cup \{\infty\}$ called the (s-wave) **scattering length**.

For hard core and Lieb-Liniger $a$ is the scattering length.

For $v = 0$, $\phi = 1$, i.e., $a = 0$ for $d \geq 2$, $a = \pm\infty$ for $d = 1$.

# Dilute limit for bosons

**Theorem (Dilute limit for bosons)**

*If range of $v$ is $R_0$ then as $\rho|a|^d \to 0$*

1D: $e_B(\rho) = \frac{\pi^2}{3}\rho^3 \left[(1-\rho a)^{-2} + O((\rho R)^{6/5})\right]$.
*with $R = \max\{R_0, 2|a|\}$. Error depends on $R \int v_{\text{reg}}$.*

2D: $e_B(\rho) = 4\pi\rho^2 \left(|\ln(\rho a^2)|^{-1} + o(|\ln(\rho a^2)|^{-1})\right)$.
*Error depends on $R_0/a$.*

3D: $e_B(\rho) = 4\pi\rho^2 a \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho a^3})\right)$.
*Upper bound requires $v \in L^1$. Error depends on $R_0/a$.*

The 1D case is joint work with Agerskov and Reuvers.
The 2D case is work of Lieb-Yngvason.
The lower bound in 3D is joint work Fournais.
The upper bound in 3D is Yau-Yin and Basti-Cenatiempo-Schlein.

# Dilute limit for fermions

**Theorem (Spin-$1/2$ fermions in 3D, Lieb-Seiringer-Solovej)**

$$e_F(\rho_\uparrow,\rho_\downarrow) = C_3(\rho_\uparrow^{5/3}+\rho_\downarrow^{5/3}) + 8\pi a\rho_\uparrow\rho_\downarrow + o(a\rho^2)$$

**Theorem (Spin-less fermions in 1D, Agerskov-Reuvers-Solovej)**

$$e_F(\rho) = \frac{\pi^2}{3}\rho^3\left[(1-\rho a_p)^{-2} + O(\rho a_p)^{6/5})\right].$$

*Here $a_p \geq 0$ is the 1D p-wave scattering length.*

# Dilute limit for fermions (Continued)

**Theorem (Spin-$1/2$ fermions in 1D, Agerskov-Reuvers-Solovej 2022)**

*The absolute (total spin zero) ground state energy density satisfies*

$$e_F(\rho) \leq \frac{\pi^2}{3}\rho^3\left[1 - 2e_H\rho a + 2(1+e_H)\rho a_p + o(\rho a_p)\right].$$

*where $e_H$ is the thermodynamic limit of the ground state energy per spin of the Heisenberg antiferromagnet chain with Hamiltonian*

$$\sum_i S_i \cdot S_{i+1}.$$

In fact, $e_H = -\ln 2$ (Bethe-Hulthén). We **conjecture** that this is the correct asymptotics. The result agrees with the hard core case where $a = a_p$ and the delta-function gas ($a_p = 0$) (Yang).

# The 1D upper bound (bosons)

The trial state has to capture free Fermi energy, as well as correction due to scattering process. Hence we consider for $x = (x_1, \ldots, x_N)$

$$\Psi(x) = \begin{cases} \omega(\mathcal{R}(x))\frac{|\Psi_F(x)|}{\mathcal{R}(x)} & \text{if } \mathcal{R}(x) < b \\ |\Psi_F(x)| & \text{if } \mathcal{R}(x) \geq b \end{cases}, \;\; \mathcal{R}(x) = \min_{i<j}(|x_i - x_j|)$$

$\omega$ is the suitably normalized ($\omega(b) = b$) **scattering solution** and $\Psi_F$ is **the free Fermi gas** wave function (Dirichlet b.c.)
We can calculate higher correlation functions for $\Psi_F$ using Wick's Theorem. We use the diamagnetic "inequality" $\int |\nabla|\Psi_F||^2 \leq \int |\nabla \Psi_F|^2$.
**Localization:** We get errors that behave badly in $N$ and we must therefore localize, i.e. write the trial state as products from smaller boxes.

# The 1D lower bound: Dyson's Lemma

## Lemma (Dyson)

*Let* $R > R_0 = \text{range}(v)$ *and* $\varphi \in H^1(\mathbb{R})$, *then for any interval* $\mathcal{I} \ni 0$

$$\int_{\mathcal{I}} |\partial\varphi|^2 + \frac{1}{2}v|\varphi|^2 \geq \int_{\mathcal{I}} \frac{2}{R-a}\left(\delta_R + \delta_{-R}\right)|\varphi|^2,$$

*where* $a$ *is the s-wave scattering length.*

Hence for a many-body wave function we have, denoting $\mathfrak{r}_i(x) = \min_{j\neq i}(|x_i - x_j|)$

$$\int \sum_i |\partial_i \Psi|^2 + \sum_{i<j} v(x_i - x_j)|\Psi|^2 \geq$$

$$\int \sum_i |\partial_i \Psi|^2 \chi_{\mathfrak{r}_i(x) > R} + \sum_i \frac{2}{R-a}\delta(\mathfrak{r}_i(x) - R)|\Psi|^2.$$

## The 1D lower bound: Reducing to the Lieb-Liniger model

Define $\psi \in L^2([0, \ell - (n-1)R]^n)$ by

$$\psi(x_1, x_2, ..., x_n) = \Psi(x_1, R + x_2, ..., (n-1)R + x_n),$$

for $x_1 \leq x_2 \leq ... \leq x_n$ and symmetrically extended. Then (ignoring some finite volume and lower order corrections)

$$\begin{aligned} \langle \Psi | H_N | \Psi \rangle &\geq E_{LL}(N, \widetilde{L} = L - NR, \widetilde{c} = 2/(R-a)) \langle \psi | \psi \rangle \\ \text{"} \geq \text{"} &N \frac{\pi^2}{3} \rho^2 (1 - \rho R)^{-2} (1 + \rho(R-a))^{-2} \langle \psi | \psi \rangle \\ &\geq N \frac{\pi^2}{3} \rho^2 (1 + 2\rho R - 2\rho(R-a)) \langle \psi | \psi \rangle \\ &= N \frac{\pi^2}{3} \rho^2 (1 + 2\rho a) \langle \psi | \psi \rangle. \end{aligned}$$

We need to control that $\langle \psi | \psi \rangle$ is close to 1.

# A norm estimate

$$1 - \langle\psi|\psi\rangle \leq \text{const. } R^2 \left( \sum_{i<j} \int_{B_{ij}} |\partial_i \Psi|^2 + \sum_{i<j} \int v_{ij} |\Psi|^2 \right).$$

where $B_{ij} = \{x \| \mathfrak{r}_i(x) < R\}$ and we assume $R > \max\{R_0, 2|a|\}$. This allows us to control the missing mass by the energy using a bootstrap argument. Again there are errors that behave badly in $N$ and we must localize by going to a **grand canonical formulation** (choose $\mu$ appropriately):

$$\begin{aligned} E_L(N) &\geq M \inf_n (E_{L/M}^{\text{Neumann}}(n) - \mu n) + \mu N \\ &\geq M \inf_{n \leq (\text{const.})\rho L/M} (E_{L/M}^{\text{Neumann}}(n) - \mu n) + \mu N. \end{aligned}$$

Now the errors will depend only on $\rho L/M$ and this will close the argument.

# Conclusion

I have discussed the asymptotics of the ground state energy of dilute Bose and also Fermi gases in 1–3 dimensions.
Some open problems are:

- General Spin-$1/2$ Fermions in 1–3 D
- Higher order corrections
- Large density limits