Kakeya sets and applications

(Reading seminar Winter 2021-2022)

Introduction: A Kakeya set, or a Besicovitch set, is a subset in Euclidean space which contains a unit line segment in every direction. In 1919, Besicovitch showed that there are Kakeya sets which are very small, namely they are of measure zero. On the other hand, the Kakeya conjecture states that Kakeya sets are not too small, namely every Kakeya set in n-dimensional space must be n-Hausdorff dimensional. In the seminar, we will discuss basic properties of Kakeya sets as well as some interesting applications in real and harmonic analysis.

Some possible topics:

Besicovitch-Perron's construction of Besicovitch's sets. Ref. [3,4].

Baire category theorem and Korner's construction of Besicovitch's sets. Ref. [5,6].

Fourier transform and the disc conjecture. Ref. [7]

Hausdorff dimension, Minkowski dimension, and Kakeya set conjecture. Ref. [1]

Kakeya maximal function conjecture. Ref. [8,9]

The Bochner-Riesz conjecture. Ref. [2]

Schrödinger equation and the local smoothing conjecture. Ref. [2]

Kakeya sets in vector spaces over finite fields. Ref. [10,11]

References:

[1] https://en.wikipedia.org/wiki/Kakeya_set

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[4] O. Perron."Über einen Satz von Besicovitch". Mathematische Zeitschrift. 28 (1928), 383–386.

[5] T. W. Körner. "Besicovitch via Baire". Studia Mathematica 158 (2003), 65-78. https://eudml.org/doc/284499.

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