Hölder-type approximation for the spatial source term of a backward heat equation

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Abstract

We consider the problem of determining a pair of functions (u, f) satisfying the two-dimensional backward heat equation

$$\begin{aligned} u_t - \Delta u &= \varphi(t)f(x,y), \quad t \in (0,T), (x,y) \in (0,1)\times(0,1), \\ u(x,y,T) &= g(x,y) \end{aligned}$$

with a homogeneous Cauchy boundary condition, where φ and g are given approximately. The problem is severely ill-posed. Using an interpolation method and the truncated Fourier series, we construct a regularized solution for the source term f and provide Hölder-type error estimates in both L^2 and H^1 norms. Numerical experiments are provided.

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1 Introduction

Let $T > 0$ and let $\Omega = (0,1)\times(0,1)$ be a heat conduction body. We consider the problem of determining a pair of functions (u, f) satisfying the system

$$\begin{cases} u_t - \Delta u = \varphi(t)f(x,y), \text{ for } t \in (0,T), (x,y) \in \Omega, \\ u_x(0,y,t) = u_x(1,y,t) = u_y(x,0,t) = u_y(x,1,t) = 0, \\ u(1,y,t) = 0, \\ u(x,y,T) = g(x,y), \end{cases} \tag{1}$$

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where $g \in L^1(\Omega)$ and $\varphi \in L^1(0,T)$ are given data. Note that the over-determination condition $u(1,y,t)$ is necessary to ensure the uniqueness of the problem (see [17], Remark 3, p. 464). Since once the source term f is available one will get a classical backward problem, we therefore only concentrate on finding the function f.

It is a particular problem of finding the source $F(\xi,t)$ satisfying the heat equation

$$u_t - \Delta u = F,$$

where ξ is the spatial variable. The inverse source problem is ill-posed, namely a solution may not exist, and even if the solution exists then it may not depend continuously on the data. Therefore, a regularization is necessary to make the numerical treatment possible. Since the problem is very difficult, ones often restrict the heat source to the separate form

$$F(\xi,t) = \varphi(t) f(\xi)$$

where either φ or f is given. The uniqueness and conditional stability of the heat source of this form were considered by many author [3, 4, 23, 24, 22, 12, 13, 5].

In spite of the uniqueness and stability results, the regularization problem for unstable case is still difficult. To treat the regularization problem, many authors have to assume that the heat source depends only either on time, namely $F(\xi,t) = \varphi(t)$ [20, 14, 6], or on space, namely $F(\xi,t) = f(x)$ [2, 21, 6, 7, 9, 8, 10, 25]. The full separate form $F(\xi,t) = \varphi(t)f(\xi)$, where φ is given, was investigated in [15, 16]. We realize that in the previous works on recovering the spatial source term $f(x)$ [21, 6, 7, 8, 10, 25, 15, 16], ones often have to require both of initial and final temperature. Moreover, error estimates were either not given explicitly, or of logarithm-type only.

A natural and interesting question is to approximate the spatial source term $f(x)$ using either initial or final temperature (but not both). Recently, the regularization using only the initial temperature was considered in [9, 17], and some logarithm-type error estimates were given. In this paper, we shall construct a regularized solution using only the final temperature, and provide Hölder-type estimates. Our work is motivated by the unique determination of the spatial source term in the backward heat equation first established in 1935 by Tikhonov [19]. We shall follow closely the strategy of our previous paper [17] which deals with the heat forward equation. The main difference is that in the backward case we find a refined version of the interpolation inequality (see Lemma 4 below) which allows us to derive the Hölder-type approximation. The one-dimensional setting of our result was already announced in [18].

The remainder of the paper is divided into three sections. In Section 2 we set some notations and state our main results. Section 3 is devoted for the theoretical proof. Some numerical experiments are provided in Section 4 to illuminate the effect of our regularization.

2 Notations and main results

Let $(u,f) \in (C^1([0,T];L^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), L^2(\Omega))$ be a solution to (1). Multiplying the main equation of the system with $W(t,x,y) := e^{(\alpha^2+n^2\pi^2)(t-1)}\cos(\alpha x)\cos(n\pi y)$, then taking the integral over $(t,x) \in (0,T) \times \Omega$ and using the integral by part we obtain

$$\begin{aligned}
&\int_{\Omega} (g(x,y) - e^{-(\alpha^2+n^2\pi^2)T} u(x,y,0)) \cos(\alpha x)\cos(n\pi y) dxdy \\
= &\int_0^T e^{(\alpha^2+n^2\pi^2)(t-1)} \varphi(t) dt. \int_{\Omega} f(x,y) \cos(\alpha x)\cos(n\pi y) dxdy \qquad (2)
\end{aligned}$$

for all $(\alpha, n) \in \mathbb{R} \times \mathbb{Z}$. This formula motivates us to introduce the following notations.

Definition 1. *For $w \in L^1(\Omega)$, $\varphi \in L^1(0,T)$ and $\alpha, \beta \in \mathbb{R}$, define*

$$\begin{aligned}
F(g)(\alpha,\beta) &:= \int_{\Omega} g(x,y)\cos(\alpha x)\cos(\beta y) dx, \\
D(\varphi)(\alpha,\beta) &:= \int_0^T e^{(\alpha^2+\beta^2)(t-T)} \varphi(t) dt, \\
H(\varphi, g)(\alpha,\beta) &:= 1_{\{D(\varphi)\neq 0\}}(\alpha,\beta). \frac{F(g)(\alpha,\beta)}{D(\varphi)(\alpha,\beta)}.
\end{aligned}$$

Observe that if $D(\varphi)(\alpha, n\pi) \neq 0$ then the variational formula (2) may be rewritten as

$$F(f)(\alpha, n\pi) = H(\varphi, g)(\alpha, n\pi) - \frac{e^{-(\alpha^2+n^2\pi^2)T}}{D(\varphi)(\alpha, n\pi)} F(u(.,.,0))(\alpha, n\pi). \qquad (3)$$

On the other hand, since $\{\sqrt{\kappa(m,n)}\cos(m\pi x)\cos(n\pi y)\}_{m,n=0}^{\infty}$ is an orthonormal basis for $L^2(\Omega)$ with $\kappa(m,n) = (2 - 1_{\{m=0\}})(2 - 1_{\{n=0\}})$, the source term $f \in L^2(\Omega)$ may be represented in terms of $F(f)$ by

$$f(x,y) = \sum_{m,n\geq 0} \kappa(m,n) F(f)(n\pi, m\pi) \cos(m\pi x)\cos(n\pi y). \qquad (4)$$

Due to (3), $F(f)(\alpha, n\pi)$ can be approximated by $H(\varphi, g)(\alpha, n\pi)$ when $(\alpha^2 + n^2\pi^2)$ is large enough. This is because the term $e^{-(\alpha^2+n^2\pi^2)T}$ decays very fast and $F(u(.,.,0))$ is bounded uniformly. To ensure that $|D(\varphi)(\alpha, n\pi)|$ is not so small we need a slight condition that

$$\text{either } \liminf_{t\to T^-} \varphi(t) > 0 \text{ or } \limsup_{t\to T^-} \varphi(t) < 0. \qquad (5)$$

Remark 1. *Condition (5) holds for a broad class of functions, for instance when φ is continuous at $t = T$ and $\varphi(T) \neq 0$. This condition should be compared to the condition $\varphi \in C^1[0,T]$ and $\varphi(0) \neq 0$ in [23, 24] and condition (H) in [17] where the heat forward problem was considered.*

We have the uniqueness.

Theorem 1 (Uniqueness)**.** *Let $g \in L^1(\Omega)$ and let $\varphi \in L^1(0,T)$ satisfy (5). Then system* (1) *has at most one solution (u, f) in $(C^1([0,T]; L^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), L^2(\Omega))$.*

In spite of the uniqueness, the problem is still ill-posed, and hence a regularization is necessary. Our strategy is to first approximate $F(f)(\alpha, n\pi)$ by $H(\varphi, g)(\alpha, n\pi)$ for $|\alpha|$ large (which ensures that $\alpha^2 + n^2\pi^2$ is large), and then recover $F(f)(\alpha, n\pi)$ for $|\alpha|$ small. This enables us to approximate the exact solution by a truncated Fourier series. To handle the key point of recovering $F(f)(\alpha, n\pi)$ for $|\alpha|$ small, as in [17, 18] we shall use the Lagrange interpolation polynomial.

Definition 2. *Let $A = \{x_1, x_2, ..., x_m\}$ be a set of m mutually distinct real numbers and let w be a real function. The Lagrange interpolation polynomial $L[A; w]$ is*

$$L[A; w](x) = \sum_{j=1}^{m} \left(\prod_{k \neq j} \frac{x - x_k}{x_j - x_k} \right) w(x_j).$$

Now we are ready to state our main result.

Theorem 2 (Regularization)**.** *Assume that*

$$(u_0, f_0) \in (C^1([0,T]; L^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), L^2(\Omega))$$

is the (unique) solution of system (1) *corresponding to (g_0, φ_0), where φ_0 satisfies* (5)*.*

Let $\varepsilon > 0$ and let $g_\varepsilon \in L^1(\Omega)$, $\varphi_\varepsilon \in L^1(0,T)$ satisfy

$$\|g_\varepsilon - g_0\|_{L^1(\Omega)} \leq \varepsilon, \|\varphi_\varepsilon - \varphi_0\|_{L^1(0,T)} \leq \varepsilon.$$

Let $M_\varepsilon = \varepsilon^{-2/7}$, $N_\varepsilon = T^{-1}\pi^{-2}\ln(\varepsilon^{-1})$, $r_\varepsilon \in [(2/9)\ln(\varepsilon^{-1}), (2/9)\ln(\varepsilon^{-1}) + 1) \cap \mathbb{Z}$, $A_{r_\varepsilon} = \{\pm(r_\varepsilon + j), j = 1, 2, ..., 4r_\varepsilon\}$ and

$$F_{\varepsilon,m,n} = \begin{cases} H(\varphi_\varepsilon, g_\varepsilon)(m\pi, n\pi), & \text{if } N_\varepsilon \leq m^2 + n^2 \leq M_\varepsilon, \\ L[A_\varepsilon; H(\varphi_\varepsilon, g_\varepsilon)(., n\pi)](m\pi), & \text{if } N_\varepsilon > m^2 + n^2. \end{cases}$$

The regularized solution f_ε is constructed from $(g_\varepsilon, \varphi_\varepsilon)$ by

$$f_\varepsilon(x) = \sum_{m,n \geq 0, m^2+n^2 \leq M_\varepsilon} \kappa(m,n) F_{\varepsilon,m,n} \cos(m\pi x)\cos(n\pi y).$$

Then (i) $\lim_{\varepsilon \to 0^+} f_\varepsilon = f_0$ in $L^2(\Omega)$.

(ii) If $f_0 \in H^1(\Omega)$ then $\lim_{\varepsilon \to 0^+} f_\varepsilon = f_0$ in $H^1(\Omega)$ and there is $\varepsilon_0 > 0$ depending only on $(\varphi_0, ||g||_{L^1(\Omega)}, ||f_0||_{L_1(\Omega)}, ||u_0(.,.,0)||_{L_1(\Omega)})$ such that

$$\|f_0 - f_\varepsilon\|_{L^2(\Omega)} \leq \sqrt[10]{\varepsilon} + \frac{1}{\pi}\|f_0\|_{H^1(\Omega)} \sqrt[7]{\varepsilon}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

(iii) If $f_0 \in H^2(\Omega)$ then

$$\|f_0 - f_\varepsilon\|_{H^1(\Omega)} \leq \sqrt[10]{\varepsilon} + 2\sqrt{2}\|f_0\|_{H^2(\Omega)} \sqrt[14]{\varepsilon}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Remark 2. *Since* $\dfrac{\partial f_\varepsilon}{\partial \mathrm{n}} = 0$ *on* $\partial\Omega$*, we do not expect that* $\lim\limits_{\varepsilon\to 0^+} f_\varepsilon = f_0$ *in* $H^2(\Omega)$ *even if* $f_0 \in C^\infty(\overline{\Omega})$ *(unless* $\dfrac{\partial f_0}{\partial \mathrm{n}} = 0$ *on* $\partial\Omega$*, but this condition is not reasonable).*

Remark 3. *To compute the Fourier coefficient* $F_{\varepsilon,m,n}$ *of the regularized solution, we just need to calculate* $H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)$ *for finite points* α*, and then calculate the Lagrange interpolation polynomial of* $H(\varphi_\varepsilon, g_\varepsilon)(., n\pi)$ *at* $m\pi$*. Hence, the computational process is discrete and it can be carried out easily by computer.*

Note also that the uniqueness in Theorem 1 follows from the convergence in Theorem 2 (i). The proof of the main theorem is represented in the next section.

3 Proof of Theorem 2

We first derive some useful properties of $F(w)$ and $D(\varphi)$.

Lemma 1. *Let* $w \in L^1(\Omega)$*. Then for any* $\alpha, \beta \in \mathbb{R}$ *and* $m = 0, 1, 2, ...,$

$$\left| \frac{\partial^m}{\partial \alpha^m} F(w)(\alpha, \beta) \right| \le \|w\|_{L^1(\Omega)} .$$

Proof. It is straightforward to see that

$$\frac{\partial^m}{\partial \alpha^m} F(w)(\alpha,\beta) = \begin{cases} (-1)^{m/2} \displaystyle\int_\Omega w(x,y)x^m \cos(\alpha x)\cos(\beta y)dxdy, & \text{if } m \text{ is even,} \\ (-1)^{(m+1)/2} \displaystyle\int_0^1 w(x,y)x^m \sin(\alpha x)\cos(\beta y)dxdy, & \text{if } m \text{ is odd.} \end{cases}$$

The desired result follows from the uniform boundedness $|x^m \cos(\alpha x)\cos(\beta y)| \le 1$ and $|x^m \sin(\alpha x)\cos(\beta y)| \le 1$. □

Lemma 2. *Let* $\varphi \in L^1(0,T)$*. Then for all* $\alpha, \beta \in \mathbb{R}$*,*

$$|D(\varphi)(\alpha,\beta)| \le ||\varphi||_{L^1(0,T)}.$$

Moreover, if φ *satisfies (5) then*

$$\liminf_{(\alpha^2+\beta^2)\to\infty} (\alpha^2+\beta^2)\, |D(\varphi)(\alpha,\beta)| > 0.$$

Proof. The first assertion, that $|D(\varphi)(\alpha,\beta)| \le \|\varphi\|_{L^1}$, is obvious. Now assume that φ satisfies the condition (5), for example $\liminf\limits_{t\to T^-} \varphi(t) > 0$. Then there is $T_\varphi \in (0,T)$ and

$C_\varphi > 0$ such that $\varphi(t) \geq C_\varphi$ for all $t \in (T_\varphi, T)$. Thus

$$\begin{aligned} |D(\varphi)(\alpha,\beta)| &\geq -\left|\int_0^{T_\varphi} e^{(\alpha^2+\beta^2)(t-T)}\varphi(t)dt\right| + \left|\int_{T_\varphi}^{T} e^{(\alpha^2+\beta^2)(t-T)}\varphi(t)dt\right| \\ &\geq -\int_0^{T_\varphi} e^{(\alpha^2+\beta^2)(T_\varphi-T)}|\varphi(t)|dt + \int_{T_\varphi}^{T} e^{(\alpha^2+\beta^2)(t-T)}.C_\varphi dt \\ &\geq -e^{(\alpha^2+\beta^2)(T_\varphi-T)}\|\varphi\|_{L^1(0,T)} + C_\varphi.\frac{1-e^{(\alpha^2+\beta^2)(T_\varphi-T)}}{(\alpha^2+\beta^2)}. \end{aligned}$$

It follows that $\liminf_{(\alpha^2+\beta^2)\to\infty} (\alpha^2+\beta^2)\,|D(\varphi)(r)| \geq C_\varphi > 0$, as desired. □

We now validate the observation that $F(f_0)(\alpha, n\pi)$ is approximated by $H(g_\varepsilon, \varphi_\varepsilon)(\alpha, n\pi)$ for $(\alpha^2 + n^2\pi^2)$ large.

Lemma 3. *Let $u_0, f_0, g_0, \varphi_0, g_\varepsilon, \varphi_\varepsilon$ be as in Theorem 2 with $\varepsilon \in (0, 1/2)$. Then there exist $C_0, C_1 > 0$ depending only on $(\varphi_0, ||g_0||_{L^1(\Omega)}, ||u_0(.,.,0)||_{L^1(\Omega)})$ such that if $(\alpha^2+n^2\pi^2) \in [\pi^2 N_\varepsilon, C_1\varepsilon^{-1}]$ then*

$$|F(f_0)(\alpha, n\pi) - H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)| \leq C_0(\alpha^2 + n^2\pi^2)^2\varepsilon.$$

Proof. It follows from Lemma 1 and Lemma 2 that

$$\begin{aligned} |F(g_\varepsilon)(\alpha, n\pi) - F(g_0)(\alpha, n\pi)| &\leq \|g_\varepsilon - g_0\|_{L^1(\Omega)} \leq \varepsilon, \\ |D(\varphi_\varepsilon)(\alpha, n\pi) - D(\varphi_0)(\alpha, n\pi)| &\leq \|\varphi_\varepsilon - \varphi_0\|_{L^1(0,T)} \leq \varepsilon, \end{aligned}$$

and

$$|D(\varphi_0)(\alpha, n\pi)| \geq \frac{2C_1}{\alpha^2 + n^2\pi^2} \quad \text{if} \quad \alpha^2 + n^2\pi^2 \geq R_1$$

for some positive constants C_1 and R_1 depending on φ_0. Thus if $\alpha^2+n^2\pi^2 \in [R_1, C_1\varepsilon^{-1}]$ then

$$\begin{aligned} |D(\varphi_\varepsilon)(\alpha, n\pi)| &\geq |D(\varphi_0)(\alpha, n\pi)| - |D(\varphi_\varepsilon)(\alpha, n\pi) - D(\varphi_0)(\alpha, n\pi)| \\ &\geq \frac{2C_1}{\alpha^2 + n^2\pi^2} - \varepsilon \geq \frac{C_1}{\alpha^2 + n^2\pi^2}. \end{aligned}$$

We shall show that the desired estimate follows from the triangle inequality

$$\begin{aligned} &|F(f_0)(\alpha, n\pi) - H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)| \\ \leq\ &|F(f_0)(\alpha, n\pi) - H(\varphi_0, g_0)(\alpha, n\pi)| + |H(\varphi_0, g_0)(\alpha, n\pi) - H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)|\,. \end{aligned}$$

In fact, choosing C_0 such that

$$C_0 \geq \max\left\{\frac{\|u_0(.,.,0)\|_{L^1}}{C_1\pi^2 N_{1/2}}, \frac{\|g_0\|_{L^1} + \|\varphi_0\|_{L^1}}{C_1^2}\right\}$$

where $N_{1/2} = T^{-1}\pi^{-2}\ln(2) > 0$. Using the variational formula (3) we find that

$$|F(f_0)(\alpha, n\pi) - H(\varphi_0, g_0)(\alpha, n\pi)| = \left| \frac{e^{-(\alpha^2+n^2\pi^2)T} F(u_0(.,.,0))(\alpha, n\pi)}{D(\varphi_0)(\alpha, n\pi)} \right|$$

$$\leq \frac{\frac{\alpha^2+n^2\pi^2}{N_{1/2}\pi^2}.e^{-N_\varepsilon \pi^2 T}.\, \|u_0(.,.,0)\|_{L^1(\Omega)}}{\frac{2C_1}{\alpha^2+n^2\pi^2}} \leq \frac{C_0}{2}(\alpha^2+n^2\pi^2)^2\varepsilon$$

where we used $\alpha^2 + n^2\pi^2 \geq \pi^2 N_\varepsilon > \pi^2 N_{1/2}$. It is also straightforward to see that

$$|H(\varphi_0, g_0)(\alpha, n\pi) - H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)| = \left| \frac{F(g_0)(\alpha, n\pi)}{D(\varphi_0)(\alpha, n\pi)} - \frac{F(g_\varepsilon)(\alpha, n\pi)}{D(\varphi_\varepsilon)(\alpha, n\pi)} \right|$$

$$\leq \frac{|F(g_0)| \,.\, |D(\varphi_\varepsilon) - D(\varphi_0)| + |D(\varphi_0)| \,.\, |F(g_\varepsilon) - F(g_0)|}{|D(\varphi_0)(\alpha, n\pi)| \,.\, |D(\varphi_\varepsilon)(\alpha, n\pi)|}$$

$$\leq \frac{\|g_0\|_{L^1}\varepsilon + \|\varphi_0\|_{L^1}\varepsilon}{\frac{2C_1}{\alpha^2+n^2\pi^2}\cdot\frac{C_1}{\alpha^2+n^2\pi^2}} \leq \frac{C_0}{2}(\alpha^2+n^2\pi^2)^2\varepsilon.$$

Thus the desired result follows. □

For each $n = 0, 1, 2, ...$ it has been shown that $F(f_0)(\alpha, n\pi)$ can be approximated by $H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)$ for $|\alpha|$ large. The key point now is that we can recover $F(f_0)(\alpha, n\pi)$ for $|\alpha|$ small from its values for $|\alpha|$ large. The following result is a refined version of Lemma 4 in [17] for real-valued function with bounded derivatives. It was already announced in [18] and for readers' convenience we repeat it again with a proof.

Lemma 4 (Interpolation inequality)**.** *Let $r > 0$ be an integer and $A_r = \{\pm(r+j), j = 1, 2, ..., 4r\}$. Let w, $\widetilde{w}$ be real-valued even function, $w \in C^{8r}(\mathbb{R})$. Then*

$$\sup_{x\in[-r,r]} |w(x) - L[A_r; \widetilde{w}](x)| \leq \sup_{x\in[-5r,5r]} \left|w^{(8r)}(x)\right| e^{-r/2} + re^{4r}. \sup_{x\in A_r} |w(x) - \widetilde{w}(x)|.$$

Proof. Denote $m = 4r$ and $x_j = r + j$ for $1 \leq j \leq m$. For any fixed $x \in [-r, r]$ we have the triangle inequality

$$|w(x) - L[A_r; \widetilde{w}](x)| \leq |w(x) - L[A_r; w](x)| + |L[A_r; (w - \widetilde{w})](x)](x)|. \tag{6}$$

We first bound $|w(z) - L(A_r; w)(x)|$. According to the remainder formula of the Lagrange interpolation polynomial (see, e.g., [1] p. 9), there exists $\xi \in [-5r, 5r]$ such that

$$w(x) - L[A_r; w](x) = \frac{w^{(2m)}(\xi)}{(2m)!}.\prod_{j=1}^{m}(x^2 - x_j^2).$$

Using $0 \leq x_j^2 - x^2 \leq x_j^2$ (due to $|x| \leq r < |x_j|$) we deduce that

$$|w(x) - L[A_r; w](x)| \leq \sup_{y\in[-5r,5r]} \left|w^{(8r)}(y)\right| \Psi_1(r) \tag{7}$$

where

$$\Psi_1(r) = \frac{1}{(2m)!}.\prod_{j=1}^{m} x_j^2 = \frac{[(r+1)(r+2)...(5r)]^2}{(8r)!}.$$

It is straightforward to see that $\Psi_1(1) = 4/15 < e^{-1/2}$ and

$$\frac{\Psi_1(r+1)}{\Psi_1(r)} = \frac{25\,[(5r+1)(5r+2)(5r+3)(5r+4)]^2}{(8r+1)(8r+2)...(8r+8)} < \frac{5^{10}}{8^8} < e^{-1/2}$$

for any $r \geq 1$, since

$$\begin{aligned} & 5^8(8r+1)(8r+2)...(8r+8) - 8^8[(5r+1)(5r+2)(5r+3)(5r+4)]^2 \\ = \ & 3276800000000r^7 + 11345920000000r^6 + 16117760000000r^5 + 12084267520000r^4 \\ & +5110135040000r^3 + 1199880928000r^2 + 141123408000r + 6086323584 > 0. \end{aligned}$$

Thus $\Psi_1(r) < e^{-r/2}$ for all $r \geq 1$, and hence (7) reduces to

$$|w(x) - L[A_r; w](x)| \leq \sup_{y \in [-5r,5r]} \left| w^{(8r)}(y) \right| e^{-r/2}. \tag{8}$$

We now bound the second term $|L(A_r; w - \widetilde{w})(z)|$ in the right-hand side of (6). Since w and $\widetilde{w}$ are even, we may write

$$L[A_r; w - \widetilde{w}](x) = \sum_{j=1}^{m} \left(\prod_{k \neq j} \frac{x^2 - x_k^2}{x_j^2 - x_k^2} \right) (w(x_j) - \widetilde{w}(x_j)). \tag{9}$$

For any fixed $1 \leq j \leq m$, using again the estimate $0 \leq x_k^2 - x^2 \leq x_k^2$ we have

$$\begin{aligned} \prod_{k \neq j} \left| \frac{x^2 - x_k^2}{x_j^2 - x_k^2} \right| & \leq \prod_{k \neq j} \frac{x_k^2}{|x_j^2 - x_k^2|} = \left(\prod_{k \neq j} \frac{1}{|x_j - x_k|} \right) \cdot \left(\prod_{k=1}^{m} \frac{x_k^2}{x_j + x_k} \right) . \frac{2}{x_j} \\ & = \frac{[(r+1)(r+2)...(5r)]^2}{(j-1)!(4r-j)!(2r+j+1)(2r+j+2)...(6r+j)} . \frac{2}{r+j} \\ & \leq \frac{[(r+1)(r+2)...(5r)]^2}{(2r-1)!(2r)!(2r+2)(2r+3)...(6r+1)} . \frac{4}{2r+1} \\ & = \frac{4[(r+1)(r+2)...(5r)]^2}{(2r-1)!(6r+1)!} =: \Psi_2(r). \end{aligned}$$

A direct computation shows that $\Psi_2(1) = 80/7 < e^4/4$ and

$$\frac{\Psi_2(r+1)}{\Psi_2(r)} = \frac{25[(5r+1)(5r+2)(5r+3)(5r+4)]^2}{2r(2r+1)(6r+2)(6r+3)...(6r+7)} < \frac{2.5^{10}}{2^3.6^6} < e^4$$

for any $r \geq 1$, since

$$\begin{aligned} & 5^8.2r(2r+1)(6r+2)...(6r+7) - 2^2.6^6.[(5r+1)...(5r+4)]^2 \\ = \ & 72900000000r^7 + 265680000000r^6 + 394065000000r^5 + 302946030000r^4 \\ & +125967060000r^3 + 26004042000r^2 + 1698012000r - 107495424 > 0. \end{aligned}$$

Thus $\Psi_2(r) < e^{4r}/4$ for all $r \geq 1$. It then follows from (9) that

$$|L[A_r; w - \widetilde{w}](x)| \leq m\Psi_2(r) \sup_{y\in A_r} |w(y) - \widetilde{w}(y)| \leq re^{4r} \sup_{y\in A_r} |w(y) - \widetilde{w}(y)|. \qquad (10)$$

Substituting (8) and (10) into (6) we get the desired result. □

The last preparation for the proof of Theorem 2 is the following lemma.

Lemma 5. *For each $w \in L^2(\Omega)$ and $M > 0$ define*

$$\Gamma_M(w)(x,y) = \sum_{m,n\geq 0, m^2+n^2\leq M} \kappa(m,n) F(w)(m\pi, n\pi) \cos(m\pi x)\cos(n\pi y)$$

Then (i) $\lim_{M\to+\infty} \|\Gamma_M(w) - w\|_{L^2(\Omega)} = 0$.
(ii) If $w \in H^1(\Omega)$ then $\lim_{M\to+\infty} \|\Gamma_M(w) - w\|_{H^1(\Omega)} = 0$ *and*

$$\|\Gamma_M(w) - w\|_{L^2(\Omega)} \leq \frac{1}{\pi\sqrt{M}} \|w\|_{H^1(\Omega)}.$$

(iii) If $w \in H^2(\Omega)$ then

$$\|\Gamma_M(w) - w\|_{H^1(\Omega)} \leq \frac{2\sqrt{2}}{\sqrt[4]{M}} \|w\|_{H^2(\Omega)}.$$

Note that $\{\cos(m\pi x)\cos(n\pi y)\}_{m,n=0}^{\infty}$ is an orthogonal basis for both of $L^2(\Omega)$ and $H^1(\Omega)$.

Proof. (i) The convergence follows from the Parseval identity

$$\|w\|^2_{L^2(\Omega)} = \sum_{m,n\geq 0} \kappa(m,n) |F(w)(m\pi, n\pi)|^2 < \infty$$

and

$$\|\Gamma_M(w) - w\|^2_{L^2(\Omega)} = \sum_{m,n\geq 0, m^2+n^2>M} \kappa(m,n) |F(w)(m\pi, n\pi)|^2. \qquad (11)$$

(ii) Now assume that $w \in H^1(\Omega)$. In this case we have

$$\|w\|^2_{H^1(\Omega)} = \sum_{m,n\geq 0} \left(1 + \pi^2(m^2+n^2)\right) \kappa(m,n) |F(w)(m\pi, n\pi)|^2 < \infty$$

and

$$\|\Gamma_M(w) - w\|^2_{H^1(\Omega)} = \sum_{m,n\geq 0, m^2+n^2>M} \left(1 + \pi^2(m^2+n^2)\right) \kappa(m,n) |F(w)(m\pi, n\pi)|^2. \qquad (12)$$

Thus

$$\lim_{M\to+\infty} \|\Gamma_M(w) - w\|_{H^1(\Omega)} = 0$$

and (11) reduces to

$$
\begin{aligned}
\|\Gamma_M(w)-w\|^2_{L^2(\Omega)} &= \sum_{m,n\geq 0, m^2+n^2>M} \kappa(m,n)\,|F(w)(m\pi,n\pi)|^2 \\
&\leq \frac{1}{1+\pi^2 M} \sum_{m,n\geq 0, m^2+n^2>M} (1+\pi^2(m^2+n^2))\kappa(m,n)\,|F(w)(m\pi,n\pi)|^2 \\
&\leq \frac{1}{1+\pi^2 M} \|w\|^2_{H^1(\Omega)}.
\end{aligned}
$$

(iii) Now assume that $w \in H^2(\Omega)$. If $M \leq 64$ then the desired inequality is trivial since $\|\Gamma_M(w)-w\|_{H^1(\Omega)} \leq \|w\|_{H^1(\Omega)} \leq \|w\|_{H^2(\Omega)}$. Therefore it suffices to assume that $M \geq 64$. Using the integral by part we get

$$
\begin{aligned}
\pi^2(m^2+n^2)F(w)(m\pi,n\pi) &= -\int_\Omega \Delta w(x,y)\cos(m\pi x)\cos(n\pi y)dxdy \\
&\quad + \int_0^1 ((-1)^m w_x(1,y)-w_x(0,y))\cos(n\pi y)dy \\
&\quad + \int_0^1 ((-1)^n w_y(x,1)-w_y(x,0))\cos(m\pi x)dx.
\end{aligned}
$$

It then follows from the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for $a,b,c \in \mathbb{R}$ that

$$
\begin{aligned}
&\pi^4 \sum_{m,n\geq 0, m^2+n^2>M} (m^2+n^2)\kappa(m,n)\,|F(w)(m\pi,n\pi)|^2 \\
\leq & \sum_{m,n\geq 0, m^2+n^2>M} \frac{3\kappa(m,n)}{m^2+n^2}\left|\int_0^1 ((-1)^m w_x(1,y)-w_x(0,y))\cos(n\pi y)dy\right|^2 \\
&+ \sum_{m,n\geq 0, m^2+n^2>M} \frac{3\kappa(m,n)}{m^2+n^2}\left|\int_0^1 ((-1)^n w_y(x,1)-w_y(x,0))\cos(m\pi x)dx\right|^2 \\
&+ \sum_{m,n\geq 0, m^2+n^2>M} \frac{3\kappa(m,n)}{m^2+n^2}\left|\int_\Omega \Delta w(x,y)\cos(m\pi x)\cos(n\pi y)dxdy\right|^2. \qquad (13)
\end{aligned}
$$

We shall bound three terms of the right-hand side of (13). We first have

$$
\begin{aligned}
&\sum_{m,n\geq 0, m^2+n^2>M} \frac{3\kappa(m,n)}{m^2+n^2}\left|\int_\Omega \Delta w(x,y)\cos(m\pi x)\cos(n\pi y)dxdy\right|^2 \\
\leq & \frac{3}{M}\sum_{m,n\geq 0, m^2+n^2>M} \kappa(m,n)\left|\int_\Omega \Delta w(x,y)\cos(m\pi x)\cos(n\pi y)dxdy\right|^2 \\
\leq & \frac{3}{M}\|\Delta w\|^2_{L^2(\Omega)}.
\end{aligned}
$$

To bound the second term, we use the Parseval identity in $L^2(0,1)$ to get

$$\begin{aligned}
&\sum_{n\geq 0}\sqrt{\kappa(n,n)}\left|\int_0^1 ((-1)^m w_x(1,y)-w_x(0,y))\cos(n\pi y)dy\right|^2\\
&=\|(-1)^m w_x(1,.)-w_x(0,.)\|^2_{L^2(0,1)}\\
&=\int_0^1\left|\int_0^1 ((-1)^m-1)\,w_x(x,y)dx+\int_0^1 ((-1)^m x+1-x)\,w_{xx}(x,y)dx\right|^2 dy\\
&\leq \int_0^1\left|2\int_0^1 |w_x(x,y)|dx+\int_0^1 |w_{xx}(x,y)|dx\right|^2 dy\\
&\leq 5\left(\|w_x\|^2_{L^2(\Omega)}+\|w_{xx}\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

where the last inequality is due to $(2a+b)^2\leq 5(a^2+b^2)$ for $a,b\in\mathbb{R}$. Employing the fact that

$$\kappa(m,n)\leq 2\sqrt{K(n,n)},\qquad \sum_{m\geq\sqrt{M}+1}\frac{1}{m^2}\leq\sum_{m\geq\sqrt{M}+1}\frac{1}{m(m-1)}\leq\frac{1}{\sqrt{M}},$$

we have

$$\begin{aligned}
&\sum_{m,n\geq 0,m^2+n^2>M}\frac{3\kappa(m,n)}{m^2+n^2}\left|\int_0^1 ((-1)^m w_x(1,y)-w_x(0,y))\cos(n\pi y)dy\right|^2\\
&\leq \sum_{\sqrt{M}+1>m\geq 0}\left(\frac{6}{M}\sum_{n\geq 0}\sqrt{\kappa(n,n)}\left|\int_0^1 ((-1)^m w_x(1,y)-w_x(0,y))\cos(n\pi y)dy\right|^2\right)\\
&+\sum_{m\geq\sqrt{M}+1}\left(\frac{6}{m^2}\sum_{n\geq 0}\sqrt{\kappa(n,n)}\left|\int_0^1 ((-1)^m w_x(1,y)-w_x(0,y))\cos(n\pi y)dy\right|^2\right)\\
&\leq \frac{30(\sqrt{M}+2)}{M}\left(\|w_x\|^2_{L^2(\Omega)}+\|w_{xx}\|^2_{L^2(\Omega)}\right)+\frac{30}{\sqrt{M}}\left(\|w_x\|^2_{L^2(\Omega)}+\|w_{xx}\|^2_{L^2(\Omega)}\right)\\
&=\frac{60(\sqrt{M}+1)}{M}\left(\|w_x\|^2_{L^2(\Omega)}+\|w_{xx}\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

The third term can be bound by the same way. Thus (13) reduces to

$$\begin{aligned}
&\pi^4\sum_{m,n\geq 0,m^2+n^2>M}(m^2+n^2)\kappa(m,n)\,|F(w)(m\pi,n\pi)|^2\\
\leq\ &\frac{3}{M}\|\Delta w\|^2_{L^2(\Omega)}+\frac{60(\sqrt{M}+1)}{M}\left(\|w_x\|^2_{L^2(\Omega)}+\|w_{xx}\|^2_{L^2(\Omega)}+\|w_y\|^2_{L^2(\Omega)}+\|w_{yy}\|^2_{L^2(\Omega)}\right)\\
\leq\ &\frac{68}{\sqrt{M}}\|w\|^2_{H^2(\Omega)}
\end{aligned}$$

where we used $M \geq 64$ in the last inequality. Therefore, it follows from (12) that

$$\begin{aligned}
\|\Gamma_M(w) - w\|^2_{H^1(\Omega)} &\leq (1+\pi^2) \sum_{m,n\geq 0, m^2+n^2>M} \left(m^2+n^2\right) \kappa(m,n) \left|F(w)(m\pi, n\pi)\right|^2 \\
&\leq \frac{68(1+\pi^2)}{\pi^4\sqrt{M}} \|w\|^2_{H^2(\Omega)} \leq \frac{8}{\sqrt{M}} \|w\|^2_{H^2(\Omega)}.
\end{aligned}$$

This completes the proof. □

We are ready to prove the main theorem.

Proof of Theorem 2. We shall use the notation $\Gamma_{M_\varepsilon}(f_0)$ as in Lemma 5. In the following $\varepsilon_0 > 0$ and $C_0 > 0$ are constants depending on $(\varphi_0, ||g||_{L^1(\Omega)}, ||f_0||_{L_1(\Omega)}, ||u_0(.,.,0)||_{L_1(\Omega)})$ but independent of ε.

Step 1. Bound on $|F(f_0)(m\pi, n\pi) - F_{\varepsilon,m,n}|$ for $m^2+n^2 \leq M_\varepsilon$.

We first note that $M_\varepsilon \leq C_1\varepsilon^{-1}$ if $0 < \varepsilon \leq C_1^{-7/5}$, where $C_1 = C_1(\varphi_0) > 0$ is given in Lemma 3. Thus for $N_\varepsilon \leq m^2+n^2 \leq M_\varepsilon$ it follows from Lemma 3 that

$$|F(f_0)(m\pi, n\pi) - H(\varphi_\varepsilon, g_\varepsilon)(m\pi, n\pi)| \leq C_0(m^2\pi^2 + n^2\pi^2)^2\varepsilon \leq C_0\pi^2\varepsilon^{5/7}. \tag{14}$$

Now we consider the case $m^2+n^2 < N_\varepsilon$. For each n, applying Lemma 4 to $r = r_\varepsilon$, $w(\alpha) = F(f_0)(\alpha, n\pi)$ and $\widetilde{w}(\alpha) = H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)$ we find that

$$\begin{aligned}
& |F(f_0)(m\pi, n\pi) - F_{\varepsilon,m,n}| = F(f_0)(m\pi, n\pi) - L[A_\varepsilon; H(\varphi_\varepsilon, g_\varepsilon)(., n\pi)](m\pi)| \\
\leq\ & \|f_0\|_{L^1} e^{-r_\varepsilon/2} + r_\varepsilon e^{4r_\varepsilon} \max_{\alpha\in A_{r_\varepsilon}} |F(f_0)(\alpha, n\pi) - H(\varphi_\varepsilon, g_\varepsilon)(\alpha, n\pi)| \\
\leq\ & \|f_0\|_{L^1} e^{-r_\varepsilon/2} + r_\varepsilon e^{4r_\varepsilon}.C_0((5r_\varepsilon)^2 + N_\varepsilon)^2\varepsilon.
\end{aligned}$$

Here we used $\sup_{\alpha\in\mathbb{R}} |w^{(8r_\varepsilon)}(\alpha)| \leq \|f_0\|_{L^1}$ by Lemma 2 in the first inequality, and used Lemma 3 again in the last inequality. Since $e^{-r_\varepsilon/2} = e^{4r_\varepsilon}\varepsilon = \varepsilon^{1/9}$ we conclude that

$$|F(f_0)(m\pi, n\pi) - F_{\varepsilon,m,n}| \leq C_0(1+r_\varepsilon)^5\varepsilon^{1/9} \text{ if } m^2+n^2 < N_\varepsilon \tag{15}$$

Step 2. Bound on $\|\Gamma_{M_\varepsilon}(f_0) - f_\varepsilon\|_{H^1(\Omega)}$.

Proceeding as in the proof of Lemma 5 (ii), we get

$$\begin{aligned}
& \|\Gamma_{M_\varepsilon}(f_0) - f_\varepsilon\|^2_{H^1(\Omega)} \\
=\ & \sum_{m,n\geq 0, m^2+n^2\leq M_\varepsilon^2} \left(1+\pi^2(m^2+n^2)\right) \kappa(m,n) \left|F(f_0)(m\pi, n\pi) - F_{\varepsilon,m,n}\right|^2 \\
\leq\ & 4(1+\sqrt{N_\varepsilon})^2 \left(1+\pi^2 N_\varepsilon\right)^2 \sup_{m^2+n^2<N_\varepsilon} \left|F(f_0)(m\pi, n\pi) - F_{\varepsilon,m,n}\right|^2 \\
& +4(1+\sqrt{M_\varepsilon})^2 \left(1+\pi^2 M_\varepsilon\right)^2 \sup_{N_\varepsilon\leq m^2+n^2\leq M_\varepsilon} \left|F(f_0)(m\pi, n\pi) - F_{\varepsilon,m,n}\right|^2
\end{aligned}$$

where we employed the fact that

$$\#\left\{(m,n)\in\mathbb{Z}^2 | m^2+n^2 \leq R\right\} \leq (1+\sqrt{R})^2.$$

Substituting (14) and (15) into the above estimate and using that $N_\varepsilon, r_\varepsilon$ are of order $\ln(\varepsilon^{-1})$, we conclude that

$$\|\Gamma_{M_\varepsilon}(f_0) - f_\varepsilon\|_{H^1(\Omega)} \le \varepsilon^{1/10}, \quad \forall \varepsilon \in (0, \varepsilon_0) \tag{16}$$

for some constant $\varepsilon_0 > 0$ depending only on $(\varphi_0, ||g||_{L^1(\Omega)}, ||f_0||_{L_1(\Omega)}, ||u_0(.,.,0)||_{L_1(\Omega)})$.

Step 3. Estimate errors between f_0 and f_ε.
(i) We first consider the case $f_0 \in L^2(\Omega)$. Using the triangle inequality and (16) we find that

$$\begin{aligned} \|f_0 - f_\varepsilon\|_{L^2(\Omega)} &\le \|\Gamma_{M_\varepsilon}(f_0) - f_\varepsilon\|_{L^2(\Omega)} + \|\Gamma_{M_\varepsilon}(f_0) - f_0\|_{L^2(\Omega)} \\ &\le \varepsilon^{1/10} + \|\Gamma_{M_\varepsilon}(f_0) - f_0\|_{L^2(\Omega)} . \end{aligned} \tag{17}$$

Thus $\lim_{\varepsilon\to 0^+} \|f_0 - f_\varepsilon\|_{L^2(\Omega)} = 0$ due to Lemma 5 (i).
(ii) We next consider the case $f_0 \in H^1(\Omega)$. Similarly to (17) we have

$$\|f_0 - f_\varepsilon\|_{H^1(\Omega)} \le \varepsilon^{1/10} + \|\Gamma_{M_\varepsilon}(f_0) - f_0\|_{H^1(\Omega)} \tag{18}$$

and then $\lim_{\varepsilon\to 0^+} \|f_0 - f_\varepsilon\|_{H^1(\Omega)} = 0$ due to the first assertion of Lemma 5 (ii). Moreover, employing Lemma 5 (ii) and (17) we get

$$\|f_0 - f_\varepsilon\|_{L^2(\Omega)} \le \varepsilon^{1/10} + \frac{1}{\pi} \|f_0\|_{H^1(\Omega)} \varepsilon^{1/7}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

(iii) Finally if $f_0 \in H^2(\Omega)$ then it follows from Lemma 5 (iii) and (18) that

$$\|f_0 - f_\varepsilon\|_{H^1(\Omega)} \le \varepsilon^{1/10} + 2\sqrt{2} \|f_0\|_{H^2(\Omega)} \varepsilon^{1/14}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

The proof is completed. □

4 Numerical experiments

In this section we shall examine some numerical examples to see how our method works. For simplicity we fix $T = 1$.

Example 1. Let us consider the exact data

$$\varphi_0(t) = \pi^2 e^{\pi^2 (t-1)}, \; g_0(x, y) = (1 + \cos(\pi x)) \cos(\pi y).$$

Then system (1) has the exact solution

$$\begin{aligned} u_0(x, y, t) &= e^{\pi^2 (t-1)} (1 + \cos(\pi x)) \cos(\pi y), \\ f_0(x, y) &= 2\cos(\pi y) + 3\cos(\pi x)\cos(\pi y). \end{aligned}$$

For each $n = 1, 2, ...$, corresponding to the disturbed data

$$\varphi_n(t) = \varphi_0(t),\ g_n(x, y) = g_0(x, y) + \frac{\pi}{n} (\sin(\pi x))^2 \cos(n\pi y),$$

system (1) has the disturbed solution

$$\begin{aligned} u_n(x, y, t) &= u_0(x, y, t) + \frac{\pi}{n} e^{\pi^2(t-1)}(\sin(\pi x))^2 \cos(n\pi y), \\ f_n(x, y, t) &= f_0(x, y, t) + \frac{\pi}{n} \big((n^2 + 5)(\sin(\pi x))^2 - 2\big) \cos(n\pi y). \end{aligned}$$

It is straightforward to see that

$$\|g_n - g_0\|_{L^1(\Omega)} = \frac{1}{n} \to 0,\ \ \|f_n - f_0\|_{L^2(\Omega)} = \frac{\pi}{n}\sqrt{27 + 14n^2 + 3n^4} \to \infty$$

as $n \to \infty$. Thus for large n then a small error of data may cause a large error of solutions. Therefore, the problem is ill-posed and a regularization is necessary.

Using the regularization scheme in Theorem 2 with respects to $\varepsilon = n^{-1} = 10^{-2}$, we obtain the regularized solution

$$f_\varepsilon(x, y) = \frac{2}{1 - e^{-2\pi^2}} \cos(\pi y) + \frac{3}{1 - e^{-3\pi^2}} \cos(\pi y) \cos(\pi y).$$

with the errors

$$\|f_\varepsilon - f_0\|_{L^2(\Omega)} \approx 3.783 \times 10^{-9},\ \ \|f_\varepsilon - f_0\|_{H^1(\Omega)} \approx 1.247 \times 10^{-8}.$$

The approximation in this case is very good because our regularization is particularly suitable for the case that f_0 is already a truncated Fourier series.

Example 2. In the second example we examine a more complicated situation. Let us consider the exact data

$$\varphi_0(t) = e^{t-1},\ g_0(x, y) = (1 + \cos(\pi x))(2y^3 - 3y^2)$$

which give the following exact solution to system (1),

$$\begin{aligned} u_0(x, y, t) &= e^{t-1}(1 + \cos(\pi x))(2y^3 - 3y^2), \\ f_0(x, y) &= (1 + \cos(\pi x))(2y^3 - 3y^2 - 12y + 6) + \pi^2 \cos(\pi x)(2y^3 - 3y^2). \end{aligned}$$

On the other hand, for each $n = 1, 2, ...$, the disturbed data

$$\begin{aligned} \varphi_n(t) &= \varphi_0(t), \\ g_n(x, y) &= g_0(x, y) + \frac{\pi}{n}(\sin(n\pi x))^2 \cos(2\pi y). \end{aligned}$$

produce the disturbed solution

$$\begin{aligned} \widetilde{u}_n(x, y, t) &= u_0(x, y, t) + \frac{\pi}{n} e^{t-1}(\sin(n\pi x))^2 \cos(2\pi y), \\ \widetilde{f}_n(x, y) &= f_0(x, y) + \pi \cos(2\pi y). \left(2\pi^2 n \cos(2n\pi x) - \frac{4\pi^2 + 1}{n}(sin(n\pi x))^2\right). \end{aligned}$$

In this case we also encounter the instability since

$$\begin{aligned}\|g_n - g_0\|_{L^1(\Omega)} &= \frac{1}{n} \to 0,\\ \left\|\widetilde{f}_n - f_0\right\|_{L^2(\Omega)} &= \frac{\pi}{4}\sqrt{16\pi^4 n^2 + 32\pi^4 + 8\pi^2 + \frac{48\pi^4 + 24\pi^2 + 3}{n^2}} \to \infty\end{aligned}$$

as $n \to \infty$.

Using the regularization scheme in Theorem 2 with $\varepsilon = n^{-1}$ we get the following regularized solutions f_{ε_k} corresponding to $\varepsilon = \varepsilon_k := 10^{-k}$,

$$\begin{aligned}f_{\varepsilon_1}(x,y) &= -0.6429040080 - 5.434905616\cos(\pi x) + 5.356285882\cos(\pi y),\\ f_{\varepsilon_2}(x,y) &= -0.5150600756 - 5.434905616\cos(\pi x) + 5.356285882\cos(\pi y)\\ &\quad +10.21960079\cos(\pi x)\cos(\pi y),\\ f_{\varepsilon_4}(x,y) &= -0.5024461774 - 5.434905616\cos(\pi x) + 5.356285882\cos(\pi y)\\ &\quad +10.21960078\cos(\pi x)\cos(\pi y) + 0.006358334970\cos(2\pi y)\\ &\quad +0.5464631910\cos(3\pi y) + 0.6065053740\cos(\pi x)\cos(3\pi y).\end{aligned}$$

The (relative) errors between the regularized solutions and the exact solution in the second example are given in Table 1. Figure 1, Figure 2 and Figure 3 represent, respectively, the disturbed solution, the regularized solution (corresponding to $\varepsilon = 10^{-2}$) and the exact solution for a visual comparison.

$\varepsilon = \frac{1}{n}$	$\frac{\|f_\varepsilon - f_0\|_{L^2}}{\|f_0\|_{L^2}}$	$\frac{\|f_\varepsilon - f_0\|_{H^1}}{\|f_0\|_{H^1}}$
10^{-1}	0.09217686999	0.02681665374
10^{-2}	0.009558836387	0.007396833224
10^{-4}	0.003701017794	0.005197014371
10^{-6}	0.001347817742	0.003666997806
10^{-8}	0.000587555769	0.002739639346

Table 1. Errors between the regularized solutions and the exact solution.

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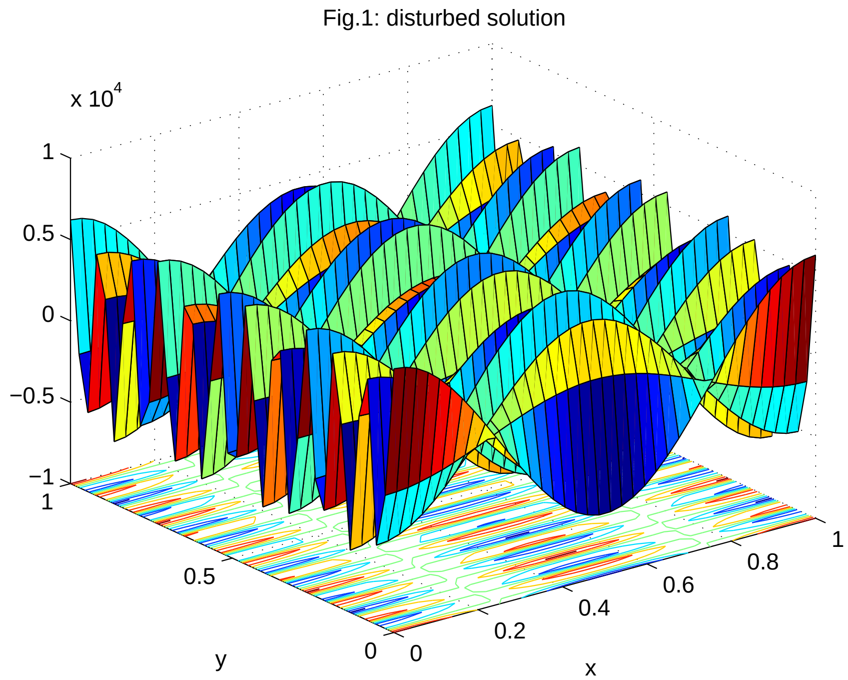

Figure 1. The disturbed solution with $\varepsilon = 10^{-2}$.

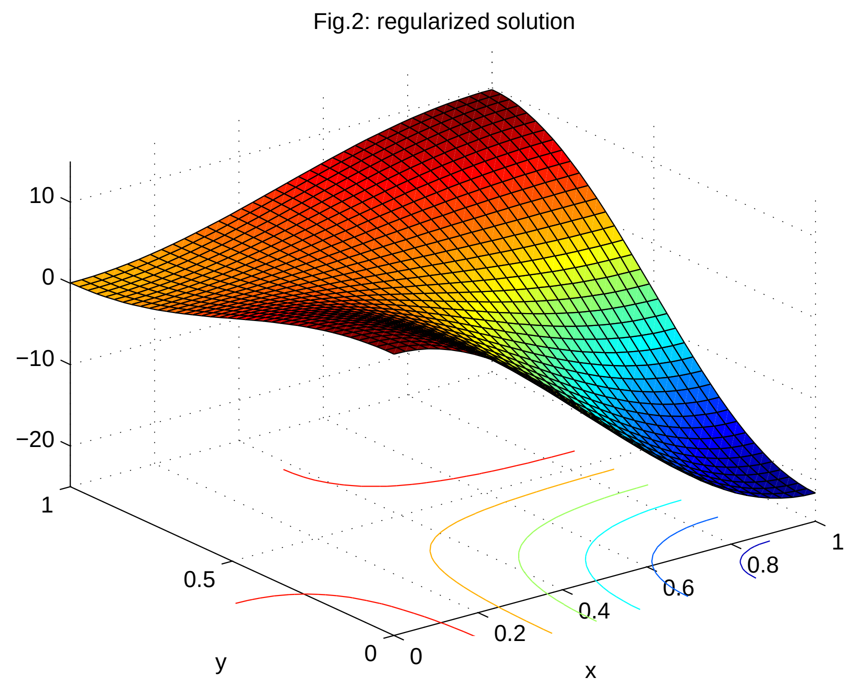

Figure 2. The regularized solution with $\varepsilon = 10^{-2}$.

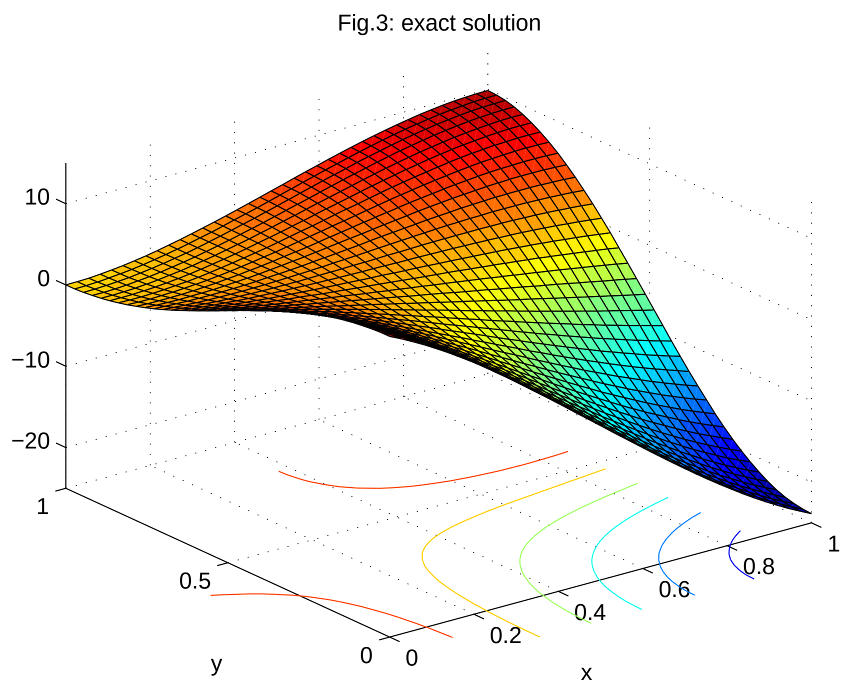

Figure 3. The exact solution.

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