Mathematisches Institut der LMU München Summer Semester 2023
Phan Thành Nam, Larry Read 14.7.2023

Harmonic analysis and applications

Homework Sheet 11

(Discussed on 19.7.2023)

E11.1. (Rising sun lemma, a precursor of the Calderón–Zygmund decomposition). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let

$$S = \{x \in [a, b] : \sup_{y>x} f(y) > f(x)\}.$$

(Note that $b \notin S$.) Prove that $E = S \cap (a, b)$ can be decomposed into disjoint open interval $\{(a_k, b_k)\}_k$ such that for all k:

- $f(x) < f(b_k)$ for all $x \in (a_k, b_k)$;
- If $a_k \neq a$, then $f(a_k) = f(b_k)$; otherwise, $a_k = a$ and $f(a) < f(b_k)$.

E11.2. (An alternative version of the Calderón–Zygmund decomposition). Let $f \in L^1(\mathbb{R}^d)$ and $\alpha > 0$. Prove that we can find disjoint cubes $\{Q\} \subset \mathbb{R}^d$ and a decomposition

$$f = g + \sum_Q b_Q$$

satisfying there properties:

- $|g(x)| \leq 2^d\alpha$, for all $x \in \mathbb{R}^d$;
- For every cube Q,
$$\operatorname{supp} b_Q \subset Q, \quad \int_Q b_Q = 0, \quad \frac{1}{|Q|}\int_Q |b_Q| \leq 2^{d+1}\alpha.$$
- $\sum_Q |Q| \leq 2^d\alpha^{-1}\|f\|_{L^1}$.

E11.3. (Littlewood-Paley dyadic decomposition) Let $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. Define

$$\varphi_0(x) = \psi(x), \quad \varphi_n(x) = \psi(2^{-n}x) - \psi(2^{-n+1}x) \quad \forall n = 1, 2, ...$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$. Define $\hat{f}_n = \varphi_n(k)\hat{f}(k)$.

(a) Prove that $\sum_{n=0}^\infty \varphi_n = 1$. Deduce that $f = \sum_{n=0}^\infty f_n$.

(b) Prove that for all $n \geq 0$, $p \in [1, \infty]$, there exists $C > 0$ independent of f such that

$$\sup_n \|f_n\|_{L^p} \leq C\|f\|_{L^p}.$$

(c) Prove that the operator $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined as follows is bounded:

$$S(f)(x) = (\sum_{n=0}^\infty |f_n(x)|^2)^{1/2}$$

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Harmonic analysis and applications

Homework Sheet 10

(Discussed on 12.7.2023)

E10.1. (a) Let $t, \alpha \in (0,\infty)$. Prove the following decay of Bessel's function

$$\left| \int_0^{2\pi} e^{\mathbf{i} t \cos(x)} e^{-\mathbf{i} m x} \mathrm{d}x \right| \le C m t^{-1/2}$$

for a constant $C > 0$ independent of m and t. Here $\mathbf{i}^2 = -1$.

(b) Let $\mathbb{1}_B$ is the indicator function of the unit ball in $\mathbb{R}^d$. Prove that

$$|\hat{\mathbb{1}}_B(k)| \le C_d |k|^{-\frac{d+1}{2}}.$$

E10.2. Let $\alpha \in \mathbb{R}$ be an irrational number and $m \in \mathbb{Z} \setminus \{0\}$. Prove that the sequence $\{\alpha k - [\alpha k]\}_{k=1}^{\infty}$ is equidistributed on $[0,1)$.

E10.3. (a) Prove that

$$e^{-2t} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-y - t^2/y} \frac{\mathrm{d}y}{\sqrt{y}}.$$

(b) Let $f(x) = e^{-2\pi|x|}$, $x \in \mathbb{R}^d$. Prove that

$$\hat{f}(k) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{1}{(1+|k|^2)^{\frac{d+1}{2}}}.$$

(c) Using Poisson's summation formula for $d = 1$ to compute

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2}.$$

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Homework Sheet 9

(Discussed on 7.7.2023)

E9.1. Consider Poisson equation $-\Delta u(x) = f(x)$ in $\mathcal{D}'(\mathbb{R}^d)$. Prove that if $f \in C^\infty(\mathbb{R}^d)$, then $u \in C^\infty(\mathbb{R}^d)$.

Hint: In the lecture we discussed the case when f is compactly supported.

E9.2. Using the Fourier transform to prove Sobolev's embeding theorem $H^k(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ if $k > d/2$.

E9.3. Let $f \in L^2(\mathbb{R}^d)$. Consider the solution to the heat equation

$$u(t,x) = (e^{t\Delta} f)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y) \mathrm{d}y, \quad t > 0.$$

(a) Prove that for every $t > 0$, we have $u(t,\cdot) \in H^k(\mathbb{R}^d)$ for all $k \geq 1$.

(b) Prove that $\|u(t,\cdot) - f\|_{L^2} \to 0$ when $t \to 0$.

(c) Assume further that $f \in H^1(\mathbb{R}^d)$. Prove that $\|u(t,\cdot) - f\|_{L^2} \leq \|f\|_{H^1} \sqrt{t}$.

E9.4. Let $f \in L^2(\mathbb{R}^d)$ such that $e^{T|2\pi k|^2} \hat{f}(k) \in L^2(\mathbb{R}^d)$.

(a) Prove that the function $u(t,x)$ defined by $\hat{u}(t,k) = e^{(T-t)|2\pi k|^2} \hat{f}(k)$, is the solution to the backward heat equation

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x), \quad t \in (0,T), \quad x \in \mathbb{R}^d, \\ u(T,x) = f(x). \end{cases}$$

(b) Given function $f_\varepsilon \in L^2(\mathbb{R}^d)$ such that $\|f - f_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \varepsilon$. Define $u_\varepsilon(t,x)$ by

$$\hat{u}_\varepsilon(t,k) = e^{(T-t)|2\pi k|^2} \hat{f}_\varepsilon(k) \mathbb{1}_{\{|k| \leq L_\varepsilon\}}, \quad L_\varepsilon = |\log \varepsilon|^{1/4}.$$

Prove that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon(t,\cdot) - u(t,\cdot)\|_{L^2} = 0, \quad \forall t \in [0,T].$$

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Homework Sheet 8

(Discussed on 28.6.2023)

E8.1. Let $f \in L^\infty(\mathbb{R}^d)$ satisfies that $\hat{f}$ is compactly supported. Prove that

$$D^\alpha f \in L^\infty(\mathbb{R}^d)$$

for every multi-index α.

E8.2. Let $N, d \geq 1$ and $\mathbf{i}^2 = -1$

(a) Find the function $f_N : \mathbb{R}^d \to \mathbb{C}$ such that

$$\hat{f}(k) = e^{-\pi |k|^2 (1+\mathbf{i}N)}.$$

(b) Prove that for all $p > 2$ and $R > 0$, we have

$$\lim_{N\to\infty} \frac{\|f_N\|_{L^p(\mathbb{R}^d)}}{\|\hat{f}_N\|_{L^1(B(0,R))}} = 0.$$

(c) Prove that for all $p > 2$, there exists a function $f \in L^p(\mathbb{R}^d)$ such that $\hat{f} \notin L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

E8.3. Let $d \geq 1$. Recall that $\widehat{e^{-\pi|x|^2}} = e^{-\pi|k|^2}$.

(a) Prove that

$$\widehat{e^{-\pi\lambda|x|^2}} = \lambda^{-d/2} e^{-\pi\lambda^{-1}|k|^2}, \quad \forall \lambda > 0.$$

(b) Let $0 < \alpha < d$. Prove that

$$\int_0^\infty e^{-\pi\lambda|x|^2} \lambda^{\alpha/2-1} \mathrm{d}\lambda = \frac{c_\alpha}{|x|^\alpha}, \quad \forall x \in \mathbb{R}^d \backslash \{0\}.$$

where

$$c_\alpha = \pi^{-\alpha/2}\Gamma(\alpha/2) = \pi^{-\alpha/2} \int_0^\infty e^{-\lambda} \lambda^{\alpha/2-1} \mathrm{d}\lambda.$$

(c) Prove that for all $0 < \alpha < d$,

$$c_\alpha \widehat{|x|^{-\alpha}} = c_{d-\alpha} |k|^{-(d-\alpha)}.$$

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Homework Sheet 7

(Discussed on 21.6.2023)

E7.1. Prove that $f(x) = e^x$, $x \in \mathbb{R}$, is not a tempered distribution.

E7.2. Recall that given a vector space V and a countable family of semi-norms $\{\rho_k\}_k$, $(V, \{\rho_k\})$ is called a Fréchet space if V is a complete metric space with respect to the metric

$$d(x,y) = \sum_{k=1} 2^{-k} \frac{\rho_k(x-y)}{1+\rho_k(x-y)}.$$

(a) Verify that d defines a metric on V.

(b) Given any compact set $K \subset \mathbb{R}^d$, show that $(C_0^\infty(K), \{\rho_\beta\})$ with $\rho_\beta(\phi) = \sup_{x\in K} |\partial^\beta \phi|$, $\beta \in \mathbb{N}_0^d$, is a Fréchet space.

(c) Prove that $(\mathcal{S}(\mathbb{R}^d), \{\rho_{\alpha,\beta}\})$ with $\rho_{\alpha,\beta}(\phi) = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta \phi|$, $\alpha, \beta \in \mathbb{N}_0^d$, is a Fréchet space.

E7.3. Recall that the set of finite measures lies in $\mathcal{S}'(\mathbb{R}^d)$, and that one such example is the Dirac delta function $\delta(\phi) = \phi(0)$.

(a) Let μ be a positive measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu}{(1+|x|)^k} < \infty$$

for some $k > 0$. Show that $\mu\colon \phi \mapsto \int_{\mathbb{R}^d} \phi \mathrm{d}\mu$ defines a tempered distribution.

(b) Show that the derivative of the Dirac delta distribution, $\delta' \in \mathcal{S}'(\mathbb{R}^d)$, doesn't come from a measure.

E7.4. Calculate the Fourier transform of the following functions in the sense of tempered distributions in $\mathbb{R}$:

(b) $f(x) = \mathbb{1}_{[a,b]}(x)$ for any $-\infty < a < b < \infty$,

(a) $f(x) = \dfrac{\sin x}{x}$,

(c) $f(x) = \mathrm{sech}(\pi x)$,

(d) $f(x) = 1/|x|^r$ with $r \in (0,1)$.

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Homework Sheet 6

(Discussed on 7.6.2023)

E6.1. Consider the step function $f : \mathbb{R}^d \to [0, \infty)$ given by

$$f(x) = \sum_{i=1}^{N} \alpha_i \mathbb{1}_{A_i}(x)$$

where $\alpha_i \ge 0$ and $A_1 \supsetneq A_2 \supsetneq ... \supsetneq A_N$. Denote $A_0 = \mathbb{R}^d$ and $A_{N+1} = \emptyset$. Prove that

$$f_*(t) = \sum_{j=1}^{N} \left(\sum_{i \le j} \alpha_i \right) \mathbb{1}_{[|A_{j+1}|, |A_j|)}(t), \quad f_{**}(t) = \sum_{i=1}^{N} \alpha_i \min(|A_i|, t).$$

E6.2. Let $1 \le p, q < \infty$ and $d \ge 1$.

(a) Prove that for all $\lambda > 0$,

$$\|f_\lambda\|_{L^{p,q}} = \|f\|_{L^{p,q}}, \quad \text{where } f_\lambda(x) = \lambda^{d/p} f(\lambda x), x \in \mathbb{R}^d.$$

(b) Assume further that $q < d$, and that we have the inequality

$$\|f_\lambda\|_{L^{p,q}} \le C\|f\|_{W^{1,q}} = C(\|\nabla f\|_{L^q}^q + \|f\|_{L^q}^q)^{1/q}, \quad \forall f \in W^{1,q}(\mathbb{R}^d),$$

for a constant $C = C(d, p, q) > 0$ independent of f. Prove that $p \le q^* = dq/(d-q)$ and

$$\|f\|_{L^{q^*,q}} \le C\|\nabla f\|_{L^q}, \quad \forall f \in W^{1,q}(\mathbb{R}^d),$$

with the same constant C.

E6.3. Let $1 \le p, q < \infty$ and $d \ge 1$. Let $f : \mathbb{R}^d \to [0, \infty)$ and denote $c_k = 2^{k/p} f^*(2^k)$, $k \in \mathbb{Z}$. Prove that

$$C^{-1}\|\{c_k\}\|_{\ell^q(\mathbb{Z})} \le \|f\|_{L^{p,q}} \le C\|\{c_k\}\|_{\ell^q(\mathbb{Z})}$$

for a constant $C = C(p, q, d) > 0$ independent of f.

E6.4. (Keel-Tao's Atomic decomposition) Let $1 \le p, q < \infty$ and $d \ge 1$. Prove that $f \in L^{p,q}(\mathbb{R}^d)$ if and only if it satisfies the pointwise inequality

$$|f(x)| \le \sum_{k \in \mathbb{Z}} 2^{-k/p} c_k \mathbb{1}_{A_k}(x)$$

where $|A_k| \le 2^k$ for all $k \in \mathbb{Z}$ and $\{c_k\}_k \in \ell^q(\mathbb{Z})$.

Hint: Consider $c_k = 2^{k/p} f^*(2^k)$ and $A_k = \{x \in \mathbb{R}^d : f_*(2^{k+1}) < |f(x)| \le f_*(2^k)\}$.

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Homework Sheet 5

(Discussed on 31.5.2023)

E5.1. For $f : \mathbb{R}^d \to \mathbb{C}$, recall that $f_* : [0,\infty) \to [0,\infty]$ is the decreasing arrangement such that $|\{|f| > t\}| = |\{f_* > t\}|$ for all $t > 0$. Prove the Hardy–Littlewood rearrangement inequality

$$\left| \int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x \right| \le \int_0^\infty f_*(t)g_*(t)\mathrm{d}t.$$

(Note: In the lecture we proved a similar statement for $f^* : \mathbb{R}^d \to [0,\infty)$.)

E5.2. Let $f, f_n : \mathbb{R}^d \to [0,\infty)$ be measurable such that when $n \to \infty$ we have $f_n(x) \uparrow f(x)$ for a.e. $x \in \mathbb{R}^d$.

(a) Prove that $(f_n)_*(t) \uparrow f_*(t)$ for all $t > 0$.

(b) Prove that if $f \in L^{p,q}(\mathbb{R}^d)$ with $1 \le p, q < \infty$, then $f_n \to f$ in $f \in L^{p,q}(\mathbb{R}^d)$.

E5.3. Let $1 \le p_0 < p < p_1 \le \infty$ and $0 < q \le \infty$. Prove that

$$(L^{p_0,\infty} \cap L^{p_1,\infty}) \subset L^{p,q}.$$

(Note: We proved this result before when $q = p$.)

E5.4. Let $1 \le p < \infty$. Recall that we proved $L^{p,q} \subset L^{p,r}$ for all $q > r$. Now we prove that the inclusion is strict.

(a) Consider $f(x) = |x|^{-d/p}$. Prove that $f \in L^{p,\infty}(\mathbb{R}^d)$ but $f \notin L^{p,q}(\mathbb{R}^d)$ for any $q < \infty$.

(b) Let $q < \infty$. Consider

$$g(x) = |x|^{-d/p} |\log|x||^{-1/q} \mathbb{1}_{B(0,1/2)}(x), \quad x \in \mathbb{R}^d.$$

Prove that $g \in L^{p,r}(\mathbb{R}^d)$ for all $r > q$, but $g \notin L^{p,q}(\mathbb{R}^d)$.

E5.5. Let $|\Omega| < \infty$ and $1 \le p_0 < p_1 \le \infty$ and $0 < q_0, q_1 \le \infty$. Prove that

$$L^{p_1,q_1}(\Omega) \subset L^{p_0,q_0}(\Omega).$$

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Homework Sheet 4

(Discussed on 24.5.2023)

E4.1. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfy

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)|^2 \mathrm{d}x \le 1, \quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x,y)| \mathrm{d}y \le 1.$$

Let $1 \le p \le 2$ and $1/p + 1/p' = 1$. Prove that $T : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ defined by

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x,y) f(y) \mathrm{d}y.$$

is a linear bounded operator with operator norm $\|T\|_{L^p \to L^{p'}} \le 1$.

E4.2. Recall the Marcinkiewicz Interpolation Theorem: "If T is quasi-linear and $\|T\|_{L^p \to L^p_{\mathrm{w}}} \le 1$, $\|T\|_{L^q \to L^q_{\mathrm{w}}} \le 1$, then $\|T\|_{L^r \to L^r} \le C$ for all $1 \le p < r < q \le \infty$." In the lecture, we have proved this theorem for $q < \infty$. Now complete the proof for $q = \infty$.

E4.3. Let $d \ge 1$, $1 < p < \infty$ and $1/p + 1/p' = 1$. Prove that $f \in L^{p,\infty}(\mathbb{R}^d)$ if and only if

$$\|f\| = \sup \left\{ \frac{1}{|E|^{1/p'}} \int_E |f| \,\middle|\, E \subset \mathbb{R}^d, 0 < |E| < \infty \right\} < \infty.$$

Moreover, prove that $\|f\|$ defines a norm which is equivalent to the quasi-norm $L^{p,\infty}$.

E4.4. Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ satisfy

$$\sup_{y\in\mathbb{R}^d} \|K(\cdot, y)\|_{L^{2,\infty}} \le 1.$$

Prove that $T : L^1(\mathbb{R}^d) \to L^{2,\infty}(\mathbb{R}^d)$ defined by

$$(Tf)(x) = \int_{\mathbb{R}^d} K(x,y) f(y) \mathrm{d}y.$$

is a linear bounded operator.

E4.5. For every $N \ge 1$ define $F_N : \mathbb{R} \to \mathbb{R}$ by

$$F_N(x) = \frac{1}{N \ln N} \sum_{k=1}^{N} |x - kN^{-1}|^{-1}$$

Prove that F_N does not converge to 0 in $L^{1,\infty}$ when $N \to \infty$.

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Homework Sheet 3

(Discussed on 17.5.2023)

In the following, we always assume let (Ω, μ) be a sigma-finite measure space.

E3.1. Consider Young's inequality

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^p}, \quad \forall f \in L^p, g \in L^q, \quad 1 \leq p, q, r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Assume that we have proved Young's inequality for $p = 1$ and $p = q'$. Use the Riesz–Thorin interpolation inequality to conclude Young's inequality in the general case $1 \leq p \leq q'$.

E3.2. Prove that for every $1 \leq p \leq \infty$ and $f \in L^p(\Omega)$, we have

$$\|f\|_{L^p} = \sup \left\{ \int_\Omega |fg| \,|\, g \text{ a step function}, \|g\|_{L^{p'}} = 1 \right\}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

You can use the fact that step functions are dense in $L^r(\Omega)$ with $1 \leq r < \infty$.

E3.3. Let $1 \leq p, q \leq \infty$ and $f_n, f \in L^p_{\mathrm{w}}(\Omega)$.

(a) Prove that if $f_n \to f$ in L^p, then $f_n \to f$ in L^p_{w}.

(b) Prove that if $f_n \to f$ in L^p_{w}, then up to a subsequence, $f_n \to f$ pointwise.

(c) Prove that if $f_n \to f$ in L^p_{w}, and $\|f_n\|_{L^q} \leq 1$, then $f \in L^q$ and $\|f\|_{L^q} \leq 1$.

Recall that the weak-L^p norm is defined as

$$\|f\|_{L^p_{\mathrm{w}}} = \sup_{\lambda > 0} \lambda |\{x : |f(x)| > \lambda\}|^{1/p}.$$

E3.4. Let $1 \leq p < r < q \leq \infty$.

(a) Prove that $L^r_{\mathrm{w}} \subset L^p + L^q$.

(b) Prove that if $L^r \subset L^p_{\mathrm{w}} \cap L^q_{\mathrm{w}}$, and

$$\|f\|_{L^r} \leq C \|f\|^{\theta}_{L^p_{\mathrm{w}}} \|f\|^{1-\theta}_{L^q_{\mathrm{w}}}$$

for constants $\theta \in (0, 1)$ and $C > 0$ depending only on p, q, r (but independent of f).

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Homework Sheet 2

(Discussed on 10.5.2023)

E2.1 (A Vitali type covering lemma leading to constant 2^d in L^1_w maximal inequality, formulated by Rupert Frank). Let X be a metric space and let $\{B_i\}_{i=I}$ be a finite collection of open balls in X. Prove that there exists a sub-collection $\{B_i\}_{i=I'}$ of disjoint balls such that for every $\varepsilon > 0$,

$$\bigcup_{i\in I}(\varepsilon B_i) \subset \bigcup_{i\in I'}\Big((2+\varepsilon)B_i\Big)$$

where we denoted $\lambda B(x,r) = B(x,\lambda r)$.

E2.2 (Lebesgue differentiation theorem) In the lecture we already proved that for all $f \in L^1_c(\mathbb{R}^d)$,

$$\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy = f(x), \quad a.e.\ x.$$

Prove that this holds for all $f \in L^1_{\rm loc}(\mathbb{R}^d)$.

E2.3 Consider the function $f(x) = \mathbb{1}_{B(0,1)(x)}$, $x \in \mathbb{R}^d$.

(a) Prove that

$$\frac{1}{(|x|+1)^d} \le Mf(x) \le \frac{C_d}{(|x|+1)^d}, \quad \forall x \in \mathbb{R}^d.$$

(b) Prove that

$$M(Mf)(x) \le \frac{C_d \log(e+|x|)}{(1+|x|)^d}$$

Argue directly that Mf and $M(Mf)$ belong to $L^p(\mathbb{R}^d)$ for all $p > 1$.

E2.4 (A generalization of maximal function) Let $h \in C_c(\mathbb{R}^d)$ be a non-negative radially symmetric descreasing function. Define $h_r(x) = r^d h(rx)$. Prove that for all $0 \le f \in L^1_{\rm loc}(\mathbb{R}^d)$

$$\sup_{r>0}(f * h_r)(x) \le \|h\|_{L^1} Mf(x).$$

E2.5 (A generalization of Fefferman - de la Llave decomposition) Let $h \in C_c(\mathbb{R}^d)$ be a non-negative radial function. Prove that for every $0 < \lambda < d$, there exist $s = s(d,\lambda) \in \mathbb{R}$ and $C = C(h,d,\lambda) > 0$ such that for all $x,y \in \mathbb{R}^d$ and $x \ne y$ we have

$$\frac{1}{|x-y|^\lambda} = C\int_0^\infty \int_{\mathbb{R}^d} h(t(z-x))h(t(z-y))r^s dz dr.$$

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Harmonic analysis and applications

Homework Sheet 1

(Discussed on 3.5.2023)

E1.1 (Counterexample for Fubini theorem when the $\sigma-$finiteness is missing). Let $\Omega_1 = \Omega_2 = (0,1)$, let μ_1 be the Lebesgue measure and let μ_2 be the counting measure. Take $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ with

$$f(x,y) = \mathbb{1}_{\{x=y\}} = \begin{cases} 1 \text{ if } x = y, \\ 0 \text{ otherwise.} \end{cases}$$

Prove that

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) \mathrm{d}\mu_2(y) \right) \mathrm{d}\mu_1(x) \neq \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y) \mathrm{d}\mu_1(x) \right) \mathrm{d}\mu_2(y).$$

E1.2 (The Brezis-Lieb lemma). Let $1 < p < \infty$.

(a) Prove that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ depending only on ε and p such that

$$||a|^p - |b|^p - |a-b|^p| \le \varepsilon |a|^p + C_\varepsilon |b|^p, \quad \forall a, b \in \mathbb{C}.$$

(b) Let (Ω, μ) be a measure space. Let $f_n \in L^p(\Omega)$ such that $\|f_n\|_p \le C$ for all n. Prove that if $f_n \to f$ a.e. as $n \to \infty$ then

$$\int_\Omega ||f_n(x)|^p - |f(x)|^p - |f_n(x) - f(x)|^p| \mathrm{d}\mu(x) \to 0.$$

E1.3 (Dual version of Hölder's inequality) Let $1 \le p, q \le \infty$, $1/p + 1/q = 1$, and $f \in L^p(\Omega)$. Prove that

$$\|f\|_p = \sup_{g \in L^q(\Omega), g \neq 0} \frac{\left|\int_\Omega fg\right|}{\|g\|_q} = \sup_{\|g\|_q = 1} \left| \int_\Omega fg \right|$$

(You can use Hölder's inequality $\left|\int_\Omega fg\right| \le \|f\|_p \|g\|_q$.)

E1.4 (Dual space of L^1). Recall the Riesz representation theorem: $(L^p(\Omega))^* = L^q(\Omega)$ for all $1 < p, q < \infty$, $1/p + 1/q = 1$.

(a) Let (Ω, μ) be a measurable space such that $\mu(\Omega) < \infty$. Prove that for every $p > 1$, $L^p(\Omega)$ is a dense subset of $L^1(\Omega)$.

(b) Use (a) and the Riesz representation theorem for $p > 1$ to show that $(L^1(\Omega))^* = L^\infty(\Omega)$.

(c) Prove that if (Ω, μ) is $\sigma-$finite, then $(L^1(\Omega))^* = L^\infty(\Omega)$.