Harmonic Analysis

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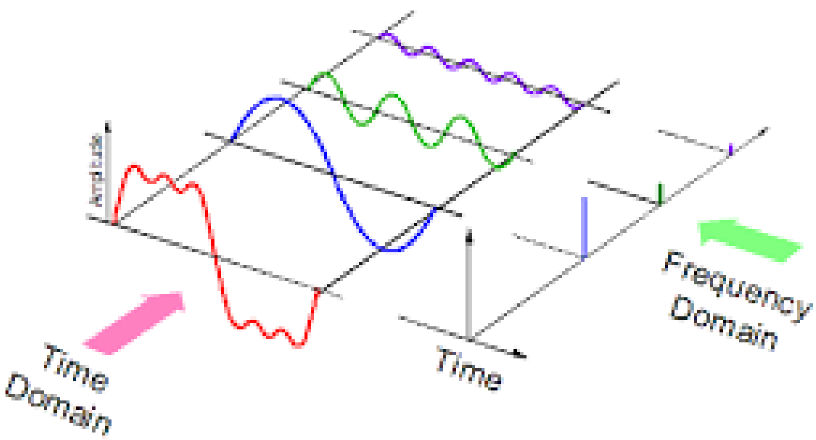

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Introduction

A brief overview over the material of this lecture:

1. Fourier Transform: Let $f : [0,1] \to \mathbb{C}$ and periodic, then $f(x) = \sum_{n\in\mathbb{Z}} = \hat{f}(n)e^{2\pi nix}$, where $\hat{f} = \int_0^1 f(x)e^{-2\pi nix}dx$ This decomposition decomposes f from sth uncountable to sth countable. Fourier transform in general for $f : \mathbb{R}^d \to \mathbb{C}$:

$$\hat{f}(k) = \int_{\mathbb{R}^d} f(x)e^{-2\pi kix}dx$$

Inverse Fourier transform:

$$\hat{f}(x) = \int_{\mathbb{R}^d} \hat{f}(k)e^{2\pi kix}dk$$

2. Convolution form: Fourier transform has the form $\hat{f}(k) = \int f(x)g(kx)dx$

- $f, g : \mathbb{R}^d \to \mathbb{C} : \quad (f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$
  $= \int_{\mathbb{R}^d} f(y)g(x-y)dy = (g \star f)(x)$
- $\widehat{f \star g}(k) = (\hat{f} \cdot \hat{g})(k)$
- If f smooth: $\widehat{\partial x_j f}(k) = (2\pi i k_j)\hat{f}(k)$
- Define operator: $G : \begin{cases} L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \\ \widehat{Gf}(k) = \hat{g}(k)\hat{f}(k) \end{cases}$, then

$$Gf(x) = (g \star f)(x) = \int_{\mathbb{R}^d} g(x-y)f(y)dy$$

  g is called kernel of G
- Example: $-\Delta u = f$ in $\mathbb{R}^d \longleftrightarrow u = (-\Delta)^{-1}f = g \star f$, where $\hat{g}(k) = \frac{1}{|2\pi k|^2}$. If $\mathbb{R}^3$, then $g(x) = \frac{1}{4\pi|x|^2}$ is the Coulomb potential.

3. Hardy-Littlewood maximal function: $f \in L^1(\mathbb{R}^d)$,
$M_f = \sup\limits_{R>0} \fint_{B(x,R)} |f| = \sup\limits_{R>0} \frac{1}{|B(x,R)|} \int\limits_{B(x,R)} |f|$
Then $M_f(x) \geq |f(x)|$ and
$\|M_f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$ for all $p > 1$ ("weak form " if p = 1)

Theorem 0.1 (Lebesgue Differentiation Theorem)**.** If $f \in L^1_{loc}$, then for a.e. x

$$\lim_{r \to 0} \fint_{B(x,r)} f = f(x)$$

To prove this one needs the so called "Covering Lemmas"

4. Theory of interpolation and decomposition:
$T : \begin{cases} X_0 \to Y_0 \\ X_1 \to Y_1 \end{cases} , T : X_\Theta \to Y_\Theta, \Theta \in [0,1]$

5. Applications:
   - Quantum mechanics: *Heisenberg uncertainty* principle, i.e.
     $(p, x) \mapsto \begin{cases} \Psi \in L^2(\mathbb{R}^d) \\ |\Psi(x)|^2 = \text{porbability density in } x \\ |\hat{\Psi}(x)|^2 = \text{porbability density in } p \end{cases}$
     Hardy uncertainty principle: $\widehat{e^{-\pi x^2}} = e^{-\pi k^2}$, if $f(x) \underset{\lim_{|x|\to\infty}}{\lesssim} e^{-\pi \alpha x^2}$
     $\Rightarrow \ \hat{f}(k) \gtrsim e^{-\pi k^2}$ as $|x| \to \infty$
     Sobolev uncertainty principle: $f, g : \mathbb{R}^d \to \mathbb{C}$, operator $f(x)g(x)$ compact, if $f, g \to 0$ at ∞
     $\Rightarrow H^1(\Omega_{bd}) \subset\subset L^2(\Omega)$
   - PDE: $\min \mathcal{E}(u),\ \mathcal{E}(u) = \int\limits_{\mathbb{R}^d} |\nabla u|^2 + V|u|^2$
     $\rightsquigarrow$ Schrödinger eq. $-\Delta u + Vu = Eu$
     Harmonic funciton $\Delta u = 0$
     $\Delta u \geq u \Rightarrow u(x) \leq \fint_{B(x,r)} u$ (max principle) and maximum attained on boundary
   - Potential theory $\rightsquigarrow$ Newton thm: $\int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|} = \frac{\int f \cdot \int g}{d}$
     $\Delta \frac{1}{|x|} = 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$
   - Number theory: $f : \mathbb{R}^d \to \mathbb{C}$
     Poisson formula: if f smooth, decay fast at ∞:

$$\sum_{x \in \mathbb{Z}^d} f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)$$

Lets have a closer look at an example of the applications in number theory:

Gauss circle problem *(still unsolved)*

$$\begin{aligned} S_R &= \#\text{integer points inside circle } B(0,R) \\ &= \#\{(x_1,x_2):\ x_1,x_2 \in \mathbb{Z},\ x_1^2+x_2^2 \leq R\} \end{aligned}$$

Conjecture: $|S_R - \pi R^2| \leq \mathcal{O}(R^{\frac{1}{2}+\varepsilon})$
Gauss: $\mathcal{O}(R)$
Sierpinski: $\mathcal{O}(R^{\frac{2}{3}})$
Huxley (2005): $\mathcal{O}(R^{0.63})$

Theorem 0.2 (Sierpinski)**.** $|S_R - \pi R^2| \leq \mathcal{O}(R^{\frac{2}{3}})$

Proof. The following proof is due to Hugh Montgomery.
Step 1: $S_R = \sum\limits_{x\in\mathbb{Z}^2} \mathbb{1}_{B_R}$. We need to replace the indicator function by some smooth function. Use Convolution. Take $\varphi \in C_c^\infty,\ \varphi \geq 0$ radial $\varphi = 0$ outside B_1 and $\int \varphi = 1$
Define $\varphi_\varepsilon(x) = \varepsilon^{-2}(\frac{x}{\varepsilon}),\ \int \varphi_\varepsilon = 1 \quad f_R(x) = (\varphi_\varepsilon \star \mathbb{1}_{B_R})(x) = \int_{B(y,R)} \varphi(x-y)dy$
Then $f_R \in C_c^\infty(\mathbb{R}^2)$ by Poisson summation formula

$$\sum_{x\in\mathbb{Z}^2} f_R(x) = \sum_{k\in\mathbb{Z}^2} \hat{f}_R(k)$$

We have $\begin{cases} supp\ \varphi_\varepsilon \subset B_\varepsilon \\ supp\ \mathbb{1}_{B_R} \subset B_R \\ supp\ f_R \subset B_{\varepsilon+R} \end{cases}$
Easy to check: $f_R(x) = 1$, if $|x| < R-\varepsilon$, $f_R(x) = 1$, if $|x| < R-\varepsilon$ and
$\tilde{S}_{R-\varepsilon} \leq S_R \leq \tilde{S}_{R+\varepsilon}$

Step 2: Consider $\tilde{S}_R = \sum\limits_{k\in\mathbb{Z}^2} \hat{f_R(k)}$

$$k=0:\ \hat{f}(0) = \int_{\mathbb{R}^2} f_R = \int \varphi_\varepsilon \star \mathbb{1}_{B_R} = (\int \varphi_\varepsilon)(\int \mathbb{1}_{B_R}) = \pi R^2$$

$$k \neq 0:\ \hat{f}_R(k) = \widehat{\varphi_\varepsilon \star \mathbb{1}_{B_R}}(k) = \hat{\varphi}_\varepsilon(k)\hat{\mathbb{1}}_{B_R}(k)$$

We have

$$\varphi_\varepsilon = \int\limits_{\mathbb{R}^2} \varepsilon^{-2}\varphi(\frac{x}{\varepsilon})e^{-2\pi ikx}dx \tag{1}$$

$$\underset{\frac{x}{\varepsilon}=y}{=} \int\limits_{\mathbb{R}^2} \varphi(y)e^{-2\pi ik\varepsilon y}dy \tag{2}$$

$$= \hat{\varphi}(\varepsilon k) \tag{3}$$

$$\Rightarrow |\hat{\varphi}_\varepsilon(k)| = |\hat{\varphi}(\varepsilon k)| \leq \frac{c_l}{|\varepsilon k|^l} \quad \forall l \geq 0$$

(since $\varphi \; inC_c^\infty \Rightarrow \hat{\varphi}(k)$ decays faster than any polynomial)

$$R^{-2}\hat{\mathbb{1}}_{B_R}(k) = \hat{\mathbb{1}}_{B_1}(Rk)$$

Remark. $|\hat{\mathbb{1}}_{B_1}(\xi)| \leq C|\xi|^{-\frac{d+1}{2}}, \; \xi \in \mathbb{R}^d$ is a Bessel function

Exercise. Define $\mathcal{J}_1(t) = \frac{1}{i\pi}\int\limits_0^\pi e^{it\cos(\Theta)}\cos(\Theta)d\Theta$, then $|\mathcal{J}_1(t)| \leq Ct^{-\frac{1}{2}}, \; \forall t > 0$

$\Rightarrow \hat{\mathbb{1}}_{B_R}(k) = R^2|\hat{\mathbb{1}}_{B_1}(Rk)| \leq \frac{CR^2}{(R|k|)^{\frac{2}{3}}}$

In summary: $|\hat{f}_R(k)| \leq \frac{c_l R^{\frac{1}{2}}}{|k^{\frac{3}{2}}|} \cdot \frac{1}{|\varepsilon k|^l}, \; \forall l \geq 0$

$$\begin{aligned} \Rightarrow \sum_{k\neq 0} |\hat{f}_R(k)| &= \sum_{0<|k|<K} |\hat{f}_R(k)| + \sum_{|k|\geq K} |\hat{f}_R(k)| && (4) \\ &\leq \sum_{0<|k|<K} \frac{cR^{\frac{1}{2}}}{|k^{\frac{3}{2}}|} + \sum_{|k|\geq K} \frac{cR^{\frac{1}{2}}}{\varepsilon|k^{\frac{5}{2}}|} && (5) \\ &\leq cR^{\frac{1}{2}}K^{\frac{1}{2}} + \frac{cR^{\frac{1}{2}}}{\varepsilon|K^{\frac{1}{2}}|}, \; \forall K > 0 && (6) \end{aligned}$$

Now, choose best K!

Step 3: $S_R \geq \tilde{S}_{R-\varepsilon} \geq \pi(R-\varepsilon)^2 - C\frac{(R-\varepsilon)^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}$

Thus, $S_R \leq \tilde{S}_{R+\varepsilon} \leq \pi(R+\varepsilon)^2 + C\frac{(R-\varepsilon)^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}$

$\Rightarrow |S_R - \pi R^2| \leq C(R\varepsilon + \frac{R^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}) \underset{\text{opt over } \varepsilon>0}{\rightarrow} CR^{\frac{2}{3}}$

$$R\varepsilon \sim \frac{R^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}} \sim \left(R\varepsilon\left(\frac{R^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\right)\left(\frac{R^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\right)\right)^{\frac{1}{3}} = R^{\frac{2}{3}}$$

$\square$

Chapter 1

L^p-Spaces

1.1 Measures

(Σ, Ω), $\Sigma \subset 2^\Omega = \{\text{subsets of } \Omega\}$

Σ is called a $\sigma-$ algebra, if $\begin{cases} \Omega \in \Sigma \\ A \in \Sigma \Rightarrow A^c \in \Sigma \\ (A_i)_{i=0}^n \subset \Sigma \Rightarrow \bigcap\limits_{i=0}^n A_i \in \Sigma \\ (A_i)_{i\in\mathbb{N}} \ countable \Rightarrow \bigcup\limits_{i\in\mathbb{N}} A_i \in \Sigma \end{cases}$

μ is called a (positive) measure, if $\mu : \Sigma \to [0, \infty]$, s.t. $\mu() = 0$ and if $(A_i)_{i\in\mathbb{N}}$ *countable* and $A_i \cap A_j =$ for all $i \neq j$, then $\mu(\bigcup\limits_{i\in\mathbb{N}} A_i) = \sum\limits_{i\in\mathbb{N}} \mu(A_i)$

Example 1.1 (Dirac-Delta measure)**.** μ_y a measure on $\mathbb{R}^d$, $y \in \mathbb{R}^d$ s.t.
$\mu_y(A) = \begin{cases} 1, & if\ y \in A \\ 0, & else \end{cases}$. Here $\Sigma = 2^{\mathbb{R}^d}$

Example 1.2. (counting measure) $\mu(A) = \#A = |A|$ with $\Omega = \mathbb{R}^d$ and $\Sigma = 2^\Omega$

Remark. This measure is not sigma-finite

Definition 1.3. A measure μ is called $\sigma-$ finite, there is a countable family $(A_i)_{i\in\mathbb{N}}$, s.t. $\bigcup\limits_{i\in\mathbb{N}} A_i = \Omega$ and $\mu A_i < \infty$ for all i

Example 1.4 (Borel measure)**.** $(\mu, \Sigma, \Omega) \rightsquigarrow \Sigma =$ smallest $\sigma-$Algebra s.t. it contains all open and closed sets. In $\mathbb{R}^d$ we have a uniquely defined measure s.t. $\mu([0,1]^d) = 1$ and $\forall x \in \mathbb{R}^d, \; A \in \Sigma : \; \mu(x + A) = \mu(A)$

Theorem 1.5. There is exaclty one Borel measure μ

Example 1.6 (Lebesgue Measure)**.** $(\mu, \Sigma, \Omega) \rightsquigarrow \Sigma =$ Borel $\sigma-$Algebra + "sets of 0-measure"
$\mu(A) = \mu(A_{borel\ measurable} + \text{"sets of 0-measure"}) := \mu(A_{borel\ measurable})$

Definition 1.7. $A \subset \mathbb{R}^d$ is a set of $0-$measure, if for every $\varepsilon > 0$ there is a Borel measurable set A_ε s.t. $A \subset A_\varepsilon$ and $\mu(A_\varepsilon) < \varepsilon$

Remark.

- The advantage of the Lebesgue $\sigma-$Algebra is its completeness, i.e. if $\mu(A) = 0$ for a Borel set A, then for all $B \subset A : \; \mu(B) = 0$.
- The disadvantage of the Lebesgue $\sigma-$Algebra is that we need to be careful with the "product property", i.e. if $A_1 \subset \mathbb{R}$ is not Lebesgue measurable and A_2 is a set of $0-$measure, then $\mu_{\mathbb{R}^2}(A_1 \times A_2) = 0$, but $\mu_{\mathbb{R}^2}(A_1 \times A_2) = \underbrace{\mu_{\mathbb{R}}(A_1)}_{\text{not well defined}} \; \mu_{\mathbb{R}}(A_2) \; \lightning$

  This is not nice when applying Fubini
- For two functions f, g we say that $f = g$ a.e, if there is a Lebesgue set of $0-$measure A, s.t. $\forall x \in \mathbb{R}^d \setminus A : \; f(x) = g(x)$

Theorem 1.8 (Regularity of Lebesgue measure)**.** If $A \subset \mathbb{R}^d$ is Lebesgue measurable, then

1. $|A| := \mu(A) = \inf\{|O| : O \text{ open and } A \subset O\}$ (outer regularity)
2. $|A| := \mu(A) = \sup\{|B| : B \text{ compact and } B \subset A\}$ (inner regularity)

1.2 Integration

(μ, Σ, Ω) measure space. Let $f : \Omega \to \mathbb{R}$

Definition 1.9. f is measurable, if for all $\lambda \in \mathbb{R}$ $\underbrace{x \in \Omega : \ f(x) > \lambda}_{\text{level set}}$ is measurable

In general, $f : \Omega \to \mathbb{C}$ measurable, iff $Re(f)$ and $Im(f)$ are measurable

Definition 1.10. Take $f : \Omega \to [0, \infty]$ measurable.

Define $\underbrace{\int_\Omega f\mu}_{\text{Lebesgue integral}} := \underbrace{\int_0^\infty \mu(\{x \in \Omega : \ f(x) > \lambda\})}_{\text{Riemann integral}}$

This works, since $\lambda \mapsto \mu(\{x \in \Omega : \ f(x) > \lambda\})$ is monotone

Remark. If $g : \mathbb{R} \to \mathbb{R}$ monotone, then g is continuous except on a countable set. More precisely: $\exists$ countable points $\{x_i\}_{i \in I} \subset \mathbb{R}$ s.t. g is continuous on $\mathbb{R} \setminus \{x_i\}_{i \in I}$ and at x_i :

$$g(x_i, -) = \lim_{y \nearrow x_i} g(y) \leq \lim_{y \searrow x_i} g(y) = g(x_i, +)$$

In general, if $f : \Omega \to \mathbb{R}$ and $f = f_+ - f_-$, s.t. $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ and assume $\int f_+$ and $\int f_-$ are finite, then we say that f is summable and define $\int f = \int f_+ - \int f_-$. Also if $f : \Omega \to \mathbb{C}$ and $Re\ f$ and $Im\ f$ are summable, then f is summable and $\int f = \int Re\ f + i \int Im\ f$

Theorem 1.11. (Layer cake representation) For all $1 \leq p < \infty$:

$$\int_\Omega |f(x)|^p d\mu(x) = \int_0^\infty p\lambda^{p-1} \mu(\{x \in \Omega : \ |f(x)| > \lambda\}) d\lambda$$

Remark. Think of Fubini:

$$\int_0^\infty p\lambda^{p-1} \mu(\{x \in \Omega : \ |f(x)| > \lambda\}) d\lambda = \int_0^\infty p\lambda^{p-1} \left(\int_\Omega \mathbb{1}_{\{|f(x)| > \lambda\}} d\mu(x) \right) d\lambda \quad (1.1)$$

$$= \int_\Omega \underbrace{\left(p\lambda^{p-1} \int_0^\infty \mathbb{1}_{\{|f(x)| > \lambda\}} d\lambda \right)}_{= |f(x)|^p} d\mu(x) \quad (1.2)$$

5 fundamental theorems:

1. Monotone c.v.
2. Dominated c.v.
3. Fatou's Lemma
4. Brezis-Lieb Lemma
5. Fubini

1.3 L^p spaces

Definition 1.12. $\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}, & if\ 1 \le p < \infty \\ ess\sup|f|, & if\ p = \infty \end{cases}$
$ess\sup =$"sup up to sets of $0-$measure" $= \inf\{\lambda : \ |f(x)| \le \lambda\ a.e.\}$ $f \in L^p(dom(f)) \Leftrightarrow \|f\|_p < \infty$

Theorem 1.13. $L^p(\Omega)$ is a Banach space for all $p \in [1, \infty]$

Theorem 1.14 (Hölder inequality)**.** If for all $f \in L^p,\ g \in L^q$, $1 \le p, q \le \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|\int_\Omega fg| \le \|f\|_p \|g\|_q$$

Remark. Dual formulation: $\|f\|_p = \sup_{0 \neq g \in L^q} \frac{|\int_\Omega fg|}{\|g\|_q}$

Theorem 1.15 (Riesz-Representation)**.** $(L^p(\Omega))^* = L^q(\Omega)$ for all $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p, q < \infty$. Moreover, if (Ω, μ) is $\sigma-$finite, then $(L^1)^* = L^\infty$, but $(L^\infty)^*$ is in general much bigger than L^1.

Definition 1.16 (Weak L^p)**.** Take $f : \Omega \to \mathbb{C}$ measurable. We say $f \in L^p_w(\Omega)$, if

$$\sup_{\lambda>0} \lambda |\{x \in \Omega : \ |f(x)| > \lambda\}|^{\frac{1}{p}} < \infty$$

and define

$$\|f\|_{p,w} = \sup_{\lambda>0} \lambda^p |\{x \in \Omega : \ |f(x)| > \lambda\}|$$

Example 1.17. Take $f : x \mapsto \frac{1}{|x|}$ for $x \in \mathbb{R}^d$.

$$|\{f(x) > \lambda\}| = |x : |x| < \frac{1}{\lambda}| = C \cdot \frac{1}{\lambda^d}$$

$$\Rightarrow \sup_{\lambda>0} \lambda^d |\{f(x) > \lambda\}| < \infty$$

$$\Rightarrow \frac{1}{|x|} \in L^p_w(\mathbb{R}^d)$$

Similarly for $\frac{1}{|x|^d} \in L_{1,w}$

Chapter 2

Hardy-Littlewood maximal functions

2.1 Hardy-Littlewood maximal inequalities

Motivation: We want to prove the Lebesgue differentiation theorem for L^1 functions.

Theorem 2.1 (Lebesgue Differentiation Theorem)**.** If $f \in L^1(\mathbb{R}^d)$, then

$$\fint_{B(x,r)} |f - f(x)| \xrightarrow{r \to 0} 0 \ a.e.$$

The same holds, if $f \in L^1_{loc}(\mathbb{R}^d)$ or $f \in L^1_{loc}(\Omega)$ for some $\Omega \subset \mathbb{R}^d$

Definition 2.2 (Maximal function)**.** If $f \in L^1_{loc}(\mathbb{R}^d)$, then define

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f|$$

Remark. 1. By Lebesgue differentiation theorem, $Mf(x) \geq |f(x)|$ a.e.

2. If $f \not\equiv 0$, then $Mf \notin L^1(\mathbb{R}^d)$ even, if $f \in L^1(\mathbb{R}^d)$, since

$$Mf(x) \geq \fint_{B(x,r)} |f| = \frac{1}{|B_1| r^d} \int_{B(x,r)} |f| \tag{2.1}$$

$$\underset{r=2|x|}{\geq} \frac{1}{|B_1|(2|x|)^d} \int_{B(0,|x|)} |f(y)| dy \tag{2.2}$$

$$\geq \frac{\varepsilon}{|x|^d}, \ as \ |x| \to \infty \tag{2.3}$$

for some $\varepsilon > 0$ (ε depends on f but not on x) Thus, $\int_{\mathbb{R}^d} Mf(x) dx \geq \int_{\mathbb{R}^d} \frac{\varepsilon}{|x|^d} dx = \infty$

We can not even expect $Mf \in L^1_{loc}(\mathbb{R}^d)$, if $f \in L^1(\mathbb{R}^d)$

Example 2.3. $f(x) = \frac{1}{|x||\ln x|^2} \in L^1(-1,1)$. Note that $\frac{1}{|x||\ln x|} \notin L^1$

$$Mf(x) \geq \fint_0^{2x} |f| = \frac{1}{2x} \int_0^{2x} |f| \sim \frac{1}{|x||\ln x|} \notin L^1_{loc}(-1,1)$$

We will prove that on the other hand Mf is "as nice as f"

Theorem 2.4 (Hardy-Littlewood inequality - L^1 weak form)**.** If $f \in L^1(\mathbb{R}^d)$, then

$$|\{x: \; Mf(x) > \lambda\}| \leq \frac{C_d}{\lambda} \|f\|_{L^1}$$

Here we can take $C_d = 5^d$ (or 3^d).
Note that the bound is equivalent: $\|Mf\|_{L^1_w} \leq C_d \|f\|_{L^1}$

Proof (Lebesgue differentiation theorem by Hardy-Littlewood maximal inequality 2.1). Wlog assume $f \in L^1_c$, i.e. $f \in L^1$ and f has compact support.
Step 1 (Reformulate the statement): We want to prove

$$\limsup_{n \to 0} \fint_{B(x,r)} |f - f(x)| = 0 \text{ for a.e. } x$$

Define $A_\varepsilon := \{x: \; \limsup_{n \to 0} |f - f(x)| > \varepsilon\}$
We will prove that for all $\varepsilon > 0: \; |A_\varepsilon| = 0$
Then this implies $\bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}} = 0$ and $\bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}} = \{x: \; \limsup_{n \to 0} |f - f(x)| > 0\}$.
This implies our desired condition.

Step 2: Take $\varepsilon > 0$ and prove $|A_\varepsilon| = 0$
If f is continuous, then this is obvious. If $f \in L^1_c(\mathbb{R}^d)$, then $\exists \{f_n\}_{n \in \mathbb{N}} \subset C_c(\mathbb{R}^d)$ s.t. $f_n \to f$ in L^1. Then:

$$A_\varepsilon \subset \{x: \; \limsup_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)| dy > \varepsilon\}$$

By triangle inequality:

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|$$

$$\Rightarrow \fint_{B(x,r)} |f(y) - f(x)| dy \leq \fint_{B(x,r)} |f(y) - f_n(y)| dy + \fint_{B(x,r)} |f_n(y) - f_n(x)| dy + |f_n(x) - f(x)|$$

We have that for all n : $\fint_{B(x,r)} |f_n(y) - f_n(x)|dy \xrightarrow{r\to 0} 0$, since f_n is continuous. Moreover, by the definition of the maximal function:

$$\fint_{B(x,r)} |f(y) - f_n(y)|dy \leq \sup_{r\to 0} \fint_{B(x,r)} |f - f_n|(y)dy = M(f - f_n)(x)$$

Thus

$$A_\varepsilon \subset \{x : M(f - f_n)(x) + |f_n(x) - f(x)| > \varepsilon\}.$$

Note: $a + b > \varepsilon \Rightarrow$ either $a > \frac{\varepsilon}{2}$ or $b > \frac{\varepsilon}{2}$

$$\Rightarrow A_\varepsilon \subset \{x : \ M(f - f_n)(x) > \frac{\varepsilon}{2}\} \cup \{x : \ |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}$$

We have: $|\{x : \ |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}| \leq \int\limits_{\mathbb{R}^d} \frac{|f_n(x) - f(x)|}{\frac{\varepsilon}{2}} \leq \frac{2}{\varepsilon} \|f_n - f\|_{L^1}$

$$|\{x : \ M(f - f_n)(x) > \frac{\varepsilon}{2}\}| \leq \frac{c_d}{\varepsilon} \|f_n - f\|_{L^1} \ \text{(maximal inequality)}$$

Conclusion: $|A_\varepsilon| \leq \frac{\tilde{c}}{\varepsilon} \|f_n - f\|_{L^1} \ \ \forall n \in \mathbb{N}$

$\Rightarrow |A_\varepsilon| \leq \lim_{n\to\infty} \frac{\tilde{c}}{\varepsilon} \|f_n - f\|_{L^1} = 0$ □

Lemma 2.5 (Vitali covering lemma)**.** Consider a collection of balls in $\mathbb{R}^d$, $\{B_i\}_{i\in I}$, s.t. $\sup_i diam\ B_i < \infty$. Then $\exists$ a subcollection $I' \subset I$ s.t. $\{B_i\}_{i\in I'}$ contains only disjoint balls and

$$\bigcup_{i\in I} B_i \subset \bigcup_{i\in I'} 5B_i$$

where $5B(x, r) = B(x, 5r)$. Here 5 can be replaced by any $\nu > 3$

Proof. First, we only consider the case, where I is finite. Choose I' by induction:

1. B_1 is the largest ball in $\{B_i\}_{i\in I}$. Add B_1 to I'
2. Ignore all balls that intersect with B_1. Note that $5B_1$ covers all balls that intersect with B_1
3. Repeat with remaining balls

The remaining set has the desired properties.

□

Proof (Hardy-Littlewood 2.4). We only consider the case where $f \in L^1_c$. Note that this is enough to prove 2.1.
Let $f \in L^1_c(\mathbb{R}^d)$. By the definition of the maximal function we get that for all $x\ in \mathbb{R}^d$ $\exists r_x > 0$ s.t.

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f| \leq 2 \fint_{B(x,r_x)} |f|$$

$\Rightarrow \int\limits_{B(x,r_x)} |f| \geq \frac{1}{2}|B_r|Mf(x)$

<u>Idea:</u> We want to decompose $\{x : \ Mf(x) > \lambda\}$ into disjoint balls $\{B(x,r_x)\}_{x\in I}$. In this step we need that $\sup\limits_x r_x < \infty$. This is guaranteed by $f \in L^1_c$.

$$\Rightarrow \quad \underbrace{|\{x : \ Mf(x) > \lambda\}|}_{|\bigcup_x B(x,r_x)| \leq \sum_x |B_{r_x}| \lesssim \sum_x \frac{1}{\lambda}\int_{B(x,r_x)}|f|} \lesssim \frac{1}{\lambda}\sum_{x\in I}\int\limits_{B(x,r_x)} |f| \leq \frac{1}{\lambda 1}\int\limits_{\mathbb{R}^d}|f| = \frac{1}{\lambda}\|f\|_{L^1}$$

We have $A = \{x : \ Mf(x) > \lambda\} \subset \bigcup_{x\in A} B(x,r_x) \underset{\text{Vitali covery lemma}}{\Rightarrow} \exists A' \subset A$ s.t.

$$\bigcup_{x\in A} B(x,r_x) \subset \bigcup_{x\in A'} 5B(x,r_x)$$

and A' contains only disjoint balls.
For any $x \in A' \subset A$: $\int\limits_{B(x,r_x)} |f| \geq \frac{1}{2}|B(r_x)|\lambda$

$$\sum_{x\in A'}\int\limits_{B(x,r_x)} |f| \geq \sum_{x\in A'}\frac{1}{2}|B(r_x)|\lambda = \sum_{x\in A'}\frac{1}{2\cdot 5^d}|B(5r_x)|\lambda \tag{2.4}$$

$$\geq \frac{1}{2\cdot 5^d}|\bigcup_{x\in A'} B(5r_x)|\lambda \tag{2.5}$$

$$\geq |A| \tag{2.6}$$

$\Rightarrow \ |A| \leq \frac{2\cdot 5^d}{\lambda}\|f\|_{L^1}$

□

Theorem 2.6 (Hardy-Littlewood maximal inequality - L^p strong version)**.** If $f \in L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$, then:

$$\|Mf\|_{L^p} \leq C_{d,p}\|f\|_{L^p}$$

<u>Interpolation idea:</u> $p = \infty$ is trivial. $Mf(x) = \sup\limits_{r>0} \fint_{B(x,r)} |f| \leq \|f\|_\infty$

The lower p is the harder it gets. In fact, we will see that the hardest case $p = 1$ in its weak form implies the theorem.

Remark. We can make $C_{d,p}$ independent of p if p is far away from 1 but $C_{d,p} \to \infty$ if $p = 1$ or $d \to \infty$.

Proof. Let $1 < p < \infty$. We use the layer-cake representation

$$\|Mf\|^p_{L^p} = \int\limits_0^\infty p\lambda^{p-1}|\{Mf > \lambda\}|d\lambda$$

First try: Applying weak L^1 bound form 2.4

$\Rightarrow \quad \|Mf\|_{L^p}^p = c_{d,p} \int_0^\infty \lambda^{p-1} \frac{\|f\|_{L^1}}{\lambda} d\lambda = \infty$, since $\int_0^\infty \lambda^s d\lambda = \infty$ for all $s > 0$

Second try: First show $\{Mf > 0\} \subset \{Mf_{\frac{\lambda}{2}} > \frac{\lambda}{2}\}$ where $f_{\frac{\lambda}{2}} = f \cdot \mathbb{1}_{f > \frac{\lambda}{2}}$:
This holds because $f(x) = f_{\frac{\lambda}{2}}(x) + f(x)\mathbb{1}_{f \leq \frac{\lambda}{2}}$ for all x

$$\Rightarrow Mf(x) = \sup_{r>0} \fint_{B(x,r)} f(y)dy \leq (\sup_{r>0} \fint_{B(x,r)} f_{\frac{\lambda}{2}}(y)dy) + \frac{\lambda}{2} = Mf_{\frac{\lambda}{2}}(x) + \frac{\lambda}{2}$$

$$\Rightarrow \text{If } Mf(x) > \frac{\lambda}{2} \Rightarrow Mf_{\frac{\lambda}{2}} > \frac{\lambda}{2} \Rightarrow \{Mf > 0\} \subset \{Mf_{\frac{\lambda}{2}} > \frac{\lambda}{2}\}$$

By the weak L^1 bound:

$$\begin{aligned} |\{x: \ Mf_{\frac{\lambda}{2}}(x) > \frac{\lambda}{2}\}| &\leq \frac{C_d}{\frac{\lambda}{2}} \left\| f_{\frac{\lambda}{2}} \right\|_{L^1} \\ &= \frac{2C_d}{\lambda} \int_{\mathbb{R}^d} f(x)\mathbb{1}_{f > \frac{\lambda}{2}} dx \end{aligned}$$

Then we conclude from the layer-cake formula:

$$\begin{aligned} \|Mf\|_{L^p}^p &\leq C_{d,p} \int_0^\infty \lambda^{p-1} (\frac{1}{\lambda} \int_{\mathbb{R}^d} f(x)\mathbb{1}_{f > \frac{\lambda}{2}})dx \\ &\overset{\text{Fubini}}{=} C_{d,p} \int_{\mathbb{R}^d} f(x) \quad \underbrace{(\int_0^\infty \lambda^{p-2}\mathbb{1}_{f > \frac{\lambda}{2}})dx}_{= \int_0^{2f(x)} \lambda^{p-2} d\lambda = c(f(x))^{p-1}} \\ &= c \int_{\mathbb{R}^d} f^p \end{aligned}$$

□

Remark. (Interpolation) From the proof, we use 2 inequalities:

$$f(x) = \underbrace{f(x)\mathbb{1}_{f > \frac{\lambda}{2}}}_{\text{weak } L^1} + \underbrace{f(x)\mathbb{1}_{f \leq \frac{\lambda}{2}}}_{\text{strong } L^\infty}$$

The same idea can be used in a much more general setting $\rightsquigarrow$ interpolation inequalities

2.2 Hardy-Littlewood-Sobolev inequality

Theorem 2.7 (Hardy-Littlewood-Sobolev inequality)**.** Let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{d} = 2$

$$|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f(x)g(y)}{|x-y|^\lambda} dxdy| \leq c \, \|f\|_{L^p} \, \|g\|_{L^q}$$

where $c = c_{d,\lambda,p,q} < \infty$ independently of f, g

Remark.

1. This implies the standard Sobolev inequality. E.g. in $3D$: $\int_{\mathbb{R}^3} |\nabla u|^2 \geq c(\int_{\mathbb{R}^3} |u|^6)^{\frac{1}{3}}$
   The latter follows from HLS and $(-\delta)^{-1} f(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} f(y)dy$

2. HLS inequality is also called weak young-ineq.

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \, \|g\|_{L^q}, \ \forall r, p, q > 1 \text{ s.t. } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

$$\Leftrightarrow |\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(x-y)h(y)dxdy| \leq C \, \|f\|_{L^p} \, \|g\|_{L^r} \, \|h\|_{L^q}, \ \forall r, p, q > 1 \text{ s.t. } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$$

3. Scaling argument: Let $f_t(x) := f(tx), \ t > 0$
   LHS:

$$|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f_t(x)g_t(y)}{|x-y|^\lambda} dxdy| \overset{\hat{x}=tx, \ \hat{y}=ty}{=} |\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f(\hat{x})g(\hat{y})}{|\frac{\hat{x}}{t} - \frac{\hat{y}}{t}|^\lambda} d\hat{x}d\hat{y}|$$

$$= t^{\lambda-2d} |\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f_t(x)g_t(y)}{|x-y|^\lambda} dxdy|$$

   LHS:

$$\|f_t\|_{L^p} \, \|g_t\|_{L^q} = t^{-d(\frac{1}{p}+\frac{1}{q})} \, \|f\|_{L^p} \, \|g\|_{L^q}$$

   (just via substitution $\hat{x} = tx$)

   Observation: If $A, B > 0$ and $At^\alpha \leq Bt^\beta \ \forall t > 0$, then $\alpha = \beta$
   HLS ineq $\Rightarrow \ \lambda - 2d = -d(\frac{1}{p} + \frac{1}{q}) \Rightarrow \ \frac{\lambda}{d} - 2 = -(\frac{1}{p} + \frac{1}{q})$. So this is the only reasonable choice for λ, p, q.

Lemma 2.8. (Fefferman-de la Llave) Let $0 < \lambda < d$ and $x, y \in \mathbb{R}^d$ s.t. $x \neq y$. Then there exists a $c = c_{d,\lambda} > 0$ s.t.

$$\frac{1}{|x-y|^\lambda} = c_{d,\lambda} \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{B(t,r)}(x) \mathbb{1}_{B(z,r)}(y) dz \frac{dr}{r^{d+\lambda+1}}$$

Proof. The proof consists of showing that the function on the RHS $RHS(x,y)$ satisfies $RHS(tx,ty) = \frac{1}{t^\lambda} RHS(x,y)$. Then it follows that $RHS(x,y) = c\frac{1}{|x-y|^\lambda}$ for a constant $c > 0$ (exercise).

$$\begin{aligned}
RHS(tx,ty) &= \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{B(x,r)}(tx)\mathbb{1}_{B(z,r)}(ty)dz\frac{dr}{r^{d+\lambda+1}} \\
&= \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{B(tz,tr)}(tx)\mathbb{1}_{B(tz,tr)}(ty)\underbrace{d(tz)}_{=t^d dz}\frac{d(tr)}{(tr)^{d+\lambda+1}} \\
&= (\int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{B(t,r)}(x)\mathbb{1}_{B(z,r)}(y)dz\frac{dr}{r^{d+\lambda+1}})\frac{1}{t^\lambda} \\
&= \frac{1}{t^\lambda}RHS(x,y)
\end{aligned}$$

□

proof of HLS by HL max inequality + FD lemma 2.8.
Assume $\|f\|_{L^p} = \|g\|_{L^q} = 1$ and $f, g \geq 0$. Then:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f(x)g(y)}{|x-y|^\lambda}dxdy \\
= \quad & c_{d,\lambda}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)g(y)\left(\int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{B(t,r)}(x)\mathbb{1}_{B(z,r)}(y)dz\frac{dr}{r^{d+\lambda+1}}\right)dxdy \\
= \quad & c_{d,\lambda}\int_{\mathbb{R}^d}\left[\int_0^\infty (f\star\mathbb{1}_{B(0,r)})(z)(g\star\mathbb{1}_{B(0,r)})(z)\right]dxdy
\end{aligned}$$

We have the following bounds:

1. $f\star\mathbb{1}_{B(0,r)}(z) \leq |B_r|Mf(z) \leq cr^dMf(z)$.
   Identically: $g\star\mathbb{1}_{B(0,r)}(z) \leq cr^dMg(z)$

   $$\Rightarrow F := (f\star\mathbb{1}_{B(0,r)})(z)(g\star\mathbb{1}_{B(0,r)})(z)\frac{1}{r^{d+\lambda+1}} \lesssim r^{d-\lambda+1}Mf(z)Mg(z)\ \forall r>0$$

   (We will only use $r < R$)

2. $$\left\|f\star\mathbb{1}_{B(0,r)}\right\|_{L^\infty} \overset{\text{Young ineq}}{\leq} \|f\|_{L^p}\left\|\mathbb{1}_{B(0,r)}\right\|_{L^{p'}} \lesssim r^{\frac{d}{p'}}$$

for $\frac{1}{p}+\frac{1}{p'}=1$.

Identically: $\left\|f\star\mathbb{1}_{B(0,r)}\right\|_{L^\infty}\lesssim r^{\frac{d}{q'}}$ for for $\frac{1}{q}+\frac{1}{q'}=1$

$$\Rightarrow F\lesssim r^{\frac{d}{p'}}r^{\frac{d}{q'}}\frac{1}{r^{d+\lambda+1}}=\frac{1}{r^{1+d}}$$

We used that $\frac{d}{p}+\frac{d}{q}=2d-\lambda$ and since we have Hölder conjugates, the same holds for p',q' instead of p,q.

$$\begin{aligned}\Rightarrow\int_0^\infty Fdr &= \underbrace{\int_0^R Fdr}_{\text{use first bound}} + \underbrace{\int_R^\infty Fdr}_{\text{use second bound}}\\ &\lesssim \int_0^R r^{d-\lambda+1}Mf(z)Mg(z)dr+\int_R^\infty\frac{1}{r^{1+d}}dr\\ &\lesssim R^{d-\lambda}Mf(z)Mg(z)+\frac{1}{R^d},\ \forall R>0\end{aligned}$$

Optimize over R :

$$\begin{aligned}\Rightarrow\int_0^\infty Fdr &\lesssim \left[(R^{d-\lambda}Mf(z)Mg(z))^d(\frac{1}{R^d})^{d-\lambda}\right]^{\frac{1}{2d}-\lambda}\\ &= (Mf(z))^{\frac{d}{2d-\lambda}}(Mg(z))^{\frac{d}{2d-\lambda}}\end{aligned}$$

Finally:

$$\begin{aligned}\int_{\mathbb{R}^d}\int_0^\infty Fdrdz &\lesssim \int_{\mathbb{R}^d}(Mf(z))^{\frac{d}{2d-\lambda}}(Mg(z))^{\frac{d}{2d-\lambda}}\\ &\leq \left(\int_{\mathbb{R}^d}(Mf(z))^{\frac{d}{2d-\lambda}\alpha}dz\right)^{\frac{1}{\alpha}}\left(\int_{\mathbb{R}^d}(Mg(z))^{\frac{d}{2d-\lambda}\beta}dz\right)^{\frac{1}{\beta}}\\ &= \|Mf\|_{L^p}^{\frac{p}{\alpha}}\|Mg\|_{L^q}^{\frac{q}{\alpha}}\overset{\text{HL max}}{\lesssim}\|f\|_{L^p}^{\frac{p}{\alpha}}\|g\|_{L^q}^{\frac{q}{\alpha}}\end{aligned}$$

for

1. $\frac{1}{\alpha}+\frac{1}{\beta}=1$

2. $\frac{d}{2d-\lambda}-\alpha=p,\ \frac{d}{2d-\lambda}-\beta=q$

$\Rightarrow\ 1=\frac{1}{\alpha}+\frac{1}{\beta}=\frac{d}{2d-\lambda}\frac{1}{p}+\frac{d}{2d-\lambda}\frac{1}{q}=\frac{1}{2d-\lambda}(\frac{d}{p}+\frac{d}{q})$

$\Rightarrow\ d(\frac{1}{p}+\frac{1}{q}=d(2-\frac{\lambda}{q}))$ □

2.3 Lieb-Oxford inequality

Theorem 2.9. Let $\Psi \in L^2(\mathbb{R}^{dN})$, $\|\Psi\|_{L^2} = 1$ and define the one-body density

$$\rho := \rho_\Psi(x) := \sum_{i=1}^{N} \int\limits_{\mathbb{R}^{d(N-1)}} |\Psi(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_N)|^2 dx_1, ..., dx_{i-1}, x_{i+1}, ..., dx_N$$

Then $\forall 0 < \lambda < d$ we have

$$\int\limits_{\mathbb{R}^{dN}} |\Psi(x_1, ..., x_N)|^2 \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|^\lambda} \ge \frac{1}{2} \int\limits_{\mathbb{R}^{dN}} \frac{\rho_\Psi(x)\rho_\Psi(y)}{|x-y|^\lambda} dxdy - C_{d,\lambda} \int\limits_{\mathbb{R}^d} \rho_\Psi^{1+\frac{\lambda}{d}} dx$$

Motivation: In QM $|\Psi|^2$ is the probability density of N quantum particles.
Density functional theory: Only consider $\rho \in L^1(\mathbb{R}^d)$, $\rho \ge 0$ and $\int\limits_{\mathbb{R}^d} \rho_\Psi dx = N$

If $|\Psi(x_1, ..., x_N)|$ is symmetric, i.e. $|\Psi(x_1, ..., x_N)| = |\Psi(x_{\sigma(1)}, ..., x_{\sigma(N)})|$ for all $\sigma \in S_N$, then the one-body density is simply given by

$$\rho(x) = N \int\limits_{\mathbb{R}^{d(N-1)}} |\Psi(x, x_2, ..., x_N)|^2 dx_2, ..., dx_N$$

Actually, there are two kind of particles:

1. Bosons: $|\Psi(x_1, ..., x_N)| = |\Psi(x_{\sigma(1)}, ..., x_{\sigma(N)})|$ for all $\sigma \in S_N$
2. Fermions: $|\Psi(x_1, ..., x_N)| = (-1)^\sigma |\Psi(x_{\sigma(1)}, ..., x_{\sigma(N)})|$ for all $\sigma \in S_N$

In density functional theory, people try to replace Ψ by ρ in various situations. E.g. if we compute the ground state energy of a system described by a Hamitionian H usually simplified

$$\inf_{\|\Psi\|_{L^2}=1} < \Psi, H\Psi >= \inf_{\substack{0 \le f \in L^2(\mathbb{R}^d) \\ \int f = N}} \underbrace{\inf_{\Psi:\ \rho_{Psi}=f} < \Psi, H\Psi >}_{\substack{\text{In computational physics/chemistry,} \\ \text{one tries to find approximations} \\ \text{for this as a functional } \mathcal{E}(f)}}$$

The L-O inequality suggests that for $H = \sum\limits_{1 \le i < j \le N} \frac{1}{|x_i - x_j|^\lambda}$ ($\lambda = 1$ and $d = 3$ gives Coulomb interaction) we have the approximation

$$\mathcal{E}(f) = \underbrace{\frac{1}{2} \int\limits_{\mathbb{R}^{dN}} \frac{f(x)f(y)}{|x-y|^\lambda} dxdy}_{\substack{\text{direct energy on mean} \\ \text{field energy}}} - \underbrace{C \int\limits_{\mathbb{R}^d} f(x)^{1+\frac{\lambda}{d}} dx}_{\text{exchange energy}}$$

The effective formulas can be obtained by mean-field approximation:

- Bosons: Take $\Psi = u(x_1)...u(x_N)$

$$\int_{\mathbb{R}^{dN}} \frac{|\Psi(x_1,...,x_N)|^2}{|x_i-x_j|^\lambda} = \frac{N(N-1)}{2}\int_{\mathbb{R}^{dN}} \frac{|u(x_1)|^2...|u(x_N)|^2}{|x_1-x_2|}dx_1...dx_N$$

$$= \frac{N(N-1)}{2}\int_{\mathbb{R}^{dN}} \frac{|u(x_1)|^2|u(x_2)|^2}{|x_1-x_2|}dx_1dx_2$$

and $\rho = N|u(x)|^2 \Rightarrow \frac{(N-1)}{2N}\int_{\mathbb{R}^{dN}} \frac{\rho(x)\rho(y)}{|x-y|^\lambda}dxdy$ (leading term as $N\to\infty$)

- Fermions: Take $\Psi(x_1,...,x_N) = \frac{1}{\sqrt{N!}}\det(u_i(x_j))_{i,j}$ for $\{u_i\}_{i=1^N}$ orthonormal family in L^2

Exercise. For the Slater determinant $\rho(x) = \rho_\Psi(x) = \sum_{i=1}^N |u_i(x)|^2$ we have

$$\sum_{i<j}\int |\Psi|^2\frac{1}{|x_i-x_j|^\lambda} = \frac{1}{2}\int_{\mathbb{R}^{Nd}}\frac{\rho(x)\rho(y)}{|x-y|^\lambda} - \frac{1}{2}\int \frac{|\sum_i u_i(x)u_i(y)|^2}{|x-y|^\lambda}dxdy$$

Proof of LO inequality using the HL maximal inequality. We use Fefferman-de la Llave approximation: $\forall 0<\lambda<d$ we have

$$\frac{1}{|x-y|^\lambda} = C_{\lambda,d}\int_0^\infty\int_{\mathbb{R}^d} \mathbb{1}_{B(z,r)}(x)\mathbb{1}_{B(z,r)}(y)dz\frac{dr}{r^{\lambda+d+1}}$$

that can be applied to x_i and x_j s.t.

$$\int_{\mathbb{R}^{dN}} |\Psi|^2\sum_{i<j}\frac{1}{|x_i-x_j|^\lambda} = C_{d,\lambda}\int_{\mathbb{R}^{dN}}\int_0^\infty\int_{\mathbb{R}^d}\sum_{i<j}\mathbb{1}_{B(z,r)}(x)\mathbb{1}_{B(z,r)}(y)|\Psi|^2dz\frac{dr}{r^{\lambda+d+1}}$$

Note that

- $\sum_{i<j} X_iX_j = \frac{1}{2}(\sum_{i<j}X_i)^2 - \frac{1}{2}(\sum_{i<j}X_i^2)$ to be used with $X_i = \mathbb{1}_{B(z,r)}(x_i)$ so that $X_i^2 = X_i$
- $\sum_{i=1}^N\int_{\mathbb{R}^{dN}} |\Psi|^2\mathbb{1}_{B(z,r)}(x_i) = \int_{\mathbb{R}^d}\rho 1_{B(z,r)}(x)dx = \rho\star\mathbb{1}_{B(0,r)}(z)$
- $\int_{\mathbb{R}^{dN}} |\Psi|^2(\sum_{i=1}^N\mathbb{1}_{B(z,r)}(x_i))^2 \geq (\int_{\mathbb{R}^{dN}}|\Psi|^2\sum_{i=1}^N\mathbb{1}_{B(z,r)}(x_i))^2$
(using that in general $(\int|\Psi|F^2)(\int|\Psi|^2)\geq(\int|\Psi|^2F)^2$ and $\int|\Psi|^2=1$)

In summary, we get that

$$\int_{\mathbb{R}^{dN}} |\Psi|^2\sum_{i<j}\frac{1}{|x_i-x_j|^\lambda} \geq \frac{C_{d,\lambda}}{2}\int_{\mathbb{R}^d}\int_0^\infty\left[(\rho\star\mathbb{1}_{B(0,r)})^2(z) - (\rho\star\mathbb{1}_{B(0,r)})(z)\right]dz\frac{dr}{r^{d+\lambda+1}}$$

Claim:

$$C_{\frac{d,\lambda}{2}} \int_{\mathbb{R}^d} \int_0^\infty (\rho \star \mathbb{1}_{B(0,r)})^2(z) dz \frac{dr}{r^{d+\lambda+1}} = \frac{1}{2} \int_{\mathbb{R}^{2d}} \frac{\rho(x)\rho(y)}{|x-y|^\lambda} dxdy$$

(using Fefferman-de laLLave (exercise))
Consider the error term: by the HL maximal function:

$$(\rho \star \mathbb{1}_{B(0,r)})(z) = \int_{B(z,r)} \rho(x) dx \leq |B(z,r)| M\rho(z) = Cr^d M\rho(z)$$

$$\int_0^\infty (\rho \star \mathbb{1}_{B(0,r)})(z) \frac{dr}{r^{d+\lambda+1}} \leq C \int_0^\infty M\rho(z) \frac{dr}{r^{\lambda+1}}$$

BUT $\frac{1}{r^{\lambda+1}}$ is not integrable. We use the trivial fact that we can put a positive part

$$\left[(\rho \star \mathbb{1}_{B(0,r)})^2(z) - (\rho \star \mathbb{1}_{B(0,r)})(z)\right]_+ = (\rho \star \mathbb{1}_{B(0,r)})^2(z) - \min\{(\rho \star \mathbb{1}_{B(0,r)})(z), (\rho \star \mathbb{1}_{B(0,r)})^2(z)\}$$

and by the maximal function again:

$$\int_0^\infty \min\{...\} \frac{dr}{r^{d+\lambda+1}} = \int_0^L ... + \int_L^\infty ... \leq \int_0^L (M\rho)^2 \frac{r^{2d}}{r^{d+\lambda+1}} dr + \int_L^\infty M\rho \frac{r^d}{r^{d+\lambda+1}} dr$$

$$\lesssim (M\rho)^2 L^{d-\lambda} + M\rho(z) L^{d-\lambda} + M\rho(z) L^{-\lambda} \lesssim (M\rho(z))^{1+\frac{\lambda}{d}}$$

The last inequality follows from $M^2 L^{d-\lambda} \sim ML^{-\lambda}$. Optimizing over L yields:

$$\int_{\mathbb{R}}^d \int_0^\infty \min\{...\} \frac{dr}{r^{d+\lambda+1}} dz \lesssim \int_{\mathbb{R}^d} (M\rho(z))^{1+\frac{\lambda}{d}} dz \underset{\text{HL max ineq}}{\lesssim} \int_{\mathbb{R}^d} \rho(z)^{1+\frac{\lambda}{d}} dz$$

□

Remark. L-O: $\int |\Psi|^2 \sum_{i<j} \frac{1}{|x_i - x_j|} \geq \frac{1}{2} \int \frac{\rho(x)\rho(y)}{|x-y|^\lambda} - C_{L-O} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{\lambda}{d}} dx$ The most interesting case is $d = 3$ and $\lambda = 1$
→ 1979 Lieb: using Newton's theorem for Coulomb ⇝ $C_{L-O} \simeq 8$
→ Lieb-Oxford 1980 ⇝ $C_{L-O} \simeq 1.68$
Numerically, the expected best constant is $C_{L-O} \simeq 1.45$
(2022 Lewin-Lieb-Seiringer $C_{L-O} \simeq 1.58$)
→ Our proof is due to Lieb-Solovej-Yngvason. This gives a very bad constant! ≥ 100

Open Problem: Find best constant for HL maximal inequality.
Only solved for $d = p = 1$: $|x \in \mathbb{R} : Mf(x) > \lambda| \leq \frac{C}{\lambda} \|f\|_{L^1}$. By Melas (2003) finding: $C = \frac{11+\sqrt{61}}{12} \simeq 1.567$

Dependence on d: $\|Mf\|_{L^p} \leq C_{d,p} \|f\|_{L^p}$ $\forall 1 < p \leq \infty$ In our proof (using Vitali's covering lemma) we get that $d \mapsto C_{d,p}$ grows exponentially.

By Stein's theorem one can see that $C_{d,p} = C_p < \infty$ and by Stein's lemma: $C_{s,d+1,p} \leq C_{s,d,p}$ for all $d \in \mathbb{N}$

Chapter 3

Interpolation Theory

Motivation:

Exercise.

$$\left.\begin{array}{l} 0 \le A_0 \le B_0 \\ 0 \le A_1 \le B_1 \end{array}\right\} \Rightarrow A_\theta \le B_\theta \;\forall \theta \in [0,1]$$

with $A_\theta = A_0^{1-\theta} A_1^\theta$, $B_\theta = B_0^{1-\theta} B_1^\theta$

Exercise (Hölder)**.**

$$|\int fg| \le \|f\|_p \|g\|_{p'} \qquad \frac{1}{p} + \frac{1}{p'} = 1,\; 1 \le p \le \infty$$

Interpolation:

$$\|f\|_r \le \|f\|_p^\theta \|g\|_q^{1-\theta} \;\; \forall p < r < q \exists \theta \in (0,1)$$

Thus: $L^p \cap L^q \subset L^r$
Reversely: $L^r \subset L^p + L^q$ as $f \in L^r \Rightarrow f = \underbrace{f\mathbb{1}_{|f|>\varepsilon}}_{\in L^p \cap L^r} + \underbrace{f\mathbb{1}_{|f|\le\varepsilon}}_{\in L^q \cap L^r}$,

since $\int |f|^p \mathbb{1}_{|f|>\varepsilon} \le \int \frac{|f|^r}{\varepsilon^{r-p}} < \infty$

Exercise (Young inequality)**.**

$$\|f \star g\|_{L^r} \le \|f\|_p \|g\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

proof 1.
Duality: $\|f \star g\|_{L^r} = \sup_{\|h\|_{r'}=1} |\int (f\star g)h| = \sup_{\|h\|_{r'}=1} |\int f(x)g(y-x)h(y)dxdy| \le \|f\|_p \|g\|_q \underbrace{\|h\|_r}_{=1}$

□

proof 2.
Fix g s.t.$\|g\|_q = 1$ Define $T : f \mapsto f \star g$. We prove that $\|Tf\|_q \le \|f\|_p \;\Leftrightarrow\; \|T\|_{L^p \to L^r} \le 1$.
We have two cases:

- $p = 1,\; r = q$

$$\|Tf\|_q \le \|f\|_1 \;\Leftrightarrow\; \|f \star g\|_{L^q} \le \|f\|_1 \|g\|_q$$

- $p = q', \ r = \infty$

$$\|Tf\|_\infty \leq \|f\|_{q'} \quad \Leftrightarrow \quad \|f \star g\|_\infty \leq \|f\|_{q'} \|g\|_q$$

Desired claim: if $\|T\|_{L^1 \to L^q} \leq 1, \ \|T\|_{L^{q'} \to L^\infty} \leq 1 \overset{(?)}{\Rightarrow} \ \|T\|_{L^p \to L^r} \leq 1$ ☐

Exercise (Fourier Transform)**.** $f \in L^1, \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-2\pi ikx} dx$
$\Rightarrow \ \hat{f} \in L^\infty, \ \left\|\hat{f}\right\|_\infty \leq \|f\|_1$

Plancherel: $\|\mathcal{F}f\|_2 = \left\|\hat{f}\right\|_2 = \|f\|_2$

3.1 Complex Interpolation

Theorem 3.1 (Hausdorff-Young)**.** The Fourier transform $\mathcal{F}$ can be defined on $L^1 + L^2$ and it satisfies

$$\|\mathcal{F}f\|_{p'} \leq \|f\|_p, \qquad \forall 1 \leq p \leq 2, \frac{1}{p} + \frac{1}{p'} = 1$$

"Formal proof": $\|\mathcal{F}\|_{L^1 \to L^\infty} \leq 1, \quad \|\mathcal{F}\|_{L^2 \to L^2} = 1$
$\Rightarrow \quad \|\mathcal{F}\|_{L^p \to L^{p'}} \leq 1, \quad \forall p \in [1, 2]$ (by Riesz-Thorin)

Theorem 3.2 (Riesz-Thorin)**.** Let $(X, \mu), \ (Y, \sigma)$ be two measure spaces, both sigma finite. Let $T : L^{p_0}(X) + L^{p_1}(X) \to L^{q_0}(Y) + L^{q_1}(Y)$ be a linear operator, where $1 \leq p_0 < p_1 \leq \infty, \ 1 \leq q_0, q_1 \leq \infty$ s.t.

$$\|T\|_{L^{p_0} \to L^{q_0}} \leq 1, \quad \|T\|_{L^{p_1} \to L^{q_1}} \leq 1$$

Then:

$$\|T\|_{L^{p_\theta} \to L^{q_\theta}} \leq 1$$

for any $\theta \in (0, 1)$,

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}$$

We use the maximum principle for analytic functions in complex planes!

Maximum principle: $f : \Omega \overset{\text{bounded, open}}{\subset} \mathbb{C} \to \mathbb{C}$, analytic in Ω, continuous on $\overline{\Omega}$. Then:

$$\sup_{z \in \overline{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|$$

Lemma 3.3 (Hadamard's three lines theorem)**.** Let f be analytic on $S = \{0 < Re\ z < 1\}$ and f continuous on $\overline{S}$ and $|f(z)| \leq Ce^{|z|}$ for all $z \in S$ Then:

$$\sup_{z \in S} |f(z)| = \sup_{z \in \overline{S},\ Re\ z \in \{0,1\}} |f(z)|$$

Proof. Step 1: Assume additionally $|f(z)| \to 0$ as $|z| \to \infty$. Take $l_n \to \infty$.
Then $\overline{\sup_{Im\ z = \pm l_n}} |f(z)| \to 0$ as $n \to \infty$.
By the maximum principle:

$$\begin{aligned} \sup_{z \in S_n} |f(z)| &\leq \sup_{z \in \partial S_n} |f(z)| \\ &\leq \max\{\sup_{Re\ z \in \{0,1\}} |f(x)|, \sup_{Im\ z \in \{\pm l_n\}} |f(x)|\} \end{aligned}$$

Take $n \to \infty$.

$$\sup_{z \in S} |f(z)| = \sup_{z \in \overline{S},\ Re\ z \in \{0,1\}} |f(z)|$$

Step 2: In general, if we only know $|f(z)| \leq Ce^{|z|}$, then we define

$$f_n(z) = f(z)e^{\frac{z^2 - 1}{n}} \qquad \forall z \in S\ \forall n \geq 1$$

Then:

- $|f_n(z)| = |f_n(x + iy)| \leq Ce^{|z|}e^{-\frac{y^2}{n}} \leq Ce^{-\frac{y^2}{n} + y + 1} \to 0$, as $|y| \to \infty$ or equivalently $|z| \to \infty$, where $z = x + iy$
- $|f_n(z)| \leq |f(z)|$ for all $z \in S$

By step 1, $\sup_{z \in S} |f_n(z)| \leq \sup_{Re\ z \in \{0,1\}} |f_n(z)| \leq \sup_{Re\ z \in \{0,1\}} |f(z)|$.
Take $n \to \infty$, use $f_n(z) \to f(z)$ pointwise $\Rightarrow$ desired bound for f □

Remark. The H-3-lines still holds if $\forall \varepsilon > 0 :\ e^{-\varepsilon |z|^2} f(z) \to 0$ as $|z| \to \infty$

proof for Hölder inequality using H3L. Assume $\|f\|_{L^p} = 1,\ \|g\|_{L^q} = 1$. We prove $\|f\|_{L^r} = 1,\ p < r < q$ First try:

$$F(z) = \int |f|^{p_z} \qquad p_z = (1 - z)p + zq$$

If $|f| > 0$ s.t. $|f|^{p_z}$ is well defined and $|f|^{p_z}$ is analytic, then $F(z)$ is also analytic and by H3L:

$$\begin{aligned} \int |f|^r \leq \sup_{z \in S} |F(z)| &\leq \sup_{Re\ z \in \{0,1\}} |F(z)| \\ &\leq \max(\int |f|^p, \int |f|^q) = 1 \end{aligned}$$

There are two restrictions:

1. $|f|^{p_z}$ might not be well defined if $f = 0$
2. the center $|F(z)| \leq Ce^{|z|}$ is somewhere nontrivial

Second try: Assume f is a step function, namely $f(x) = \sum_{n=1}^{M} a_m e^{i\alpha_m} \mathbb{1}_{A_m}, \quad a_m > 0,\ \alpha_m \in \mathbb{R},\ A_m \subset \Omega,\ (A_m)$ disjoint. Define

$$F(z) := \sum_{m=1}^{M} a_m^{P_z} |A_m|, \quad P_z = p(1-z) + qz \ \ \forall z \in \mathbb{C}$$

$$\overset{H3L}{\Rightarrow} \quad \int |f|^r \leq \max(\int |f|^p, \int |f|^q)$$

for all step functions f. To conclude, we use the density argument, {step functions} is dense in $L^r(\Omega)$ if $\begin{cases} \Omega \text{ is sigma-finite} \\ r < \infty \end{cases}$ □

Motivated by the proof above we want to use our new tools to prove Riesz-Thorin:

proof of Riesz-Thorin.
By duality: $\|Tf\|_{p_\Theta} \leq \|f\|_{p_\Theta} \quad \forall f \in L^{p_\Theta}$

$$\Leftrightarrow \ |\int_Y (Tf)(y) g(y) dy| \leq \|f\|_{L^{p_\Theta}} \|g\|_{L^{q_\Theta}}$$

We will prove the statement for step functions and then use a density argument.
Step 1: Take f, g step functions.

$$f(x) := \sum_{m=1}^{M} a_m e^{i\alpha_m} \mathbb{1}_{A_m}(x)$$

and

$$g(y) := \sum_{n=1}^{N} b_n e^{i\beta_n} \mathbb{1}_{B_n}(y)$$

for $a_m, b_m > 0, \alpha_m, \beta_m \in \mathbb{R}, \{A_m\}_m$ disjoint sets of finite measure, $\{B_n\}_n$ disjoint sets of finite measure.
For all $z \in \overline{S} = \{0 \leq Re\ z \leq 1\}$ define

$$f_z(x) = \sum_{m=1}^{M} a_m^{P_z} e^{i\alpha_m} \mathbb{1}_{A_m}(x)$$

for $P_z = p_\Theta(\frac{1-z}{p_0} + \frac{z}{p_1})$ (chosen s.t. $P_\Theta = 1$)

$$g_z(y) = \sum_{n=1}^{M} b_n^{Q_z} e^{i\beta_n} \mathbb{1}_{B_n}(y)$$

for $Q_z = q'_\Theta(\frac{1-z}{q'_0} + \frac{z}{q'_1})$ (chosen s.t. $Q_\Theta = 1$)
Define: $F(z) := \int_Y (Tf_z)(y) g_z(y) dy$. One can prove that F is analytic in S, $|F(z)| \leq e^{c|z|}$ and that F is continuous in $\overline{S}$.
By H3L Lemma:

$$|\int (Tf)g| = |F(\Theta)| \leq \sup_{Re\ z \in \{0,1\}} |F(z)|$$

So now we have to check the cases where $Re\ z = 0$ and $Re\ z = 1$.
<u>$Re\ z = 0$:</u>

$$\begin{aligned}
f_z(x) &= \sum_{m=1}^{M} a_m^{P_z} e^{i\alpha_m} \mathbb{1}_{A_m}(x) \\
\Rightarrow \int |f_z(x)|^p dx &= \sum_{m=1}^{M} |a_m^{P_z}|^p |A_m| \\
&= \sum_{m=1}^{M} |a_m^{Re\ P_z}|^p |A_m| \\
&= \sum_{m=1}^{M} a_m^{p(\frac{p_\Theta}{p_0} + Re\ z(\frac{p_\Theta}{p_1} - \frac{p_\Theta}{p_0}))} |A_m| \\
&\overset{Re\ z=0}{=} \sum_{m=1}^{M} a_m^{p\frac{p_\Theta}{p_0}} |A_m|
\end{aligned}$$

Analogously:

$$\int |g_z(y)|^p dy = \sum_{n=1}^{N} b_n^{p\frac{q'_\Theta}{q'_0}} |B_n|$$

$$\begin{aligned}
|F(z)| = |\int_Y (Tf_z) g_z| &\leq \|Tf_z\|_{L^{q_0}} \|g_z\|_{L^{q'_0}} \\
&\leq \|f_z\|_{L^{p_0}} \|g_z\|_{q'_0} \\
&= \underbrace{(\sum_{m=1}^{M} a_m^{p_\Theta} |A_m|)}_{=\|f\|_{L^{p_\Theta}}^{p_\Theta} = 1}^{\frac{1}{p_0}} \underbrace{(\sum_{n=1}^{N} b_n^{q'_\Theta} |B_n|)}_{=\|g\|_{L^{q'_\Theta}}^{q'_\Theta} = 1}^{\frac{1}{q'_0}} \\
&= 1
\end{aligned}$$

<u>$Re\ z = 1$:</u>

$$\int |f_z(x)|^p dx = \sum_{m=1}^{M} a_m^{p\frac{p_\Theta}{p_1}} |A_m|$$

$$\int |g_z(y)|^p dy = \sum_{n=1}^{N} b_n^{p\frac{q'_\Theta}{q'_1}} |B_n|$$

$$\begin{aligned}\Rightarrow\ |F(z)| = |\int_Y (Tf_z)g_z| &\le \|Tf_z\|_{L^{q_1}} \|g_z\|_{L^{q_1'}} \\ &\le \|f_z\|_{L^{p_1}} \|g_z\|_{L^{q_1'}} \\ &= (\sum_{m=1}^{M} a_m^{p_\Theta}|A_m|)^{\frac{1}{p_1}} (\sum_{n=1}^{N} b_n^{q_\Theta'}|B_n|)^{\frac{1}{q_1'}} \\ &= 1\end{aligned}$$

Step 2 (Density Argument): We proved the statement for the case that f, g are step functions. By a standard density argument, this extends to all $g \in L^{q_\theta'}$, as

$$\sup_{g \text{ step funciton}} \frac{|\int hg|}{\|g\|_{L^{q'}}} = \|h\|_{L^q} \quad \forall 1 \le q \le \infty \text{ (exercise)}$$

Thus: $\|Tf\|_{L^{q_\theta}} \le \|f\|_{L_{p_\Theta}}$ for all f step functions.
We want to extend this to all $f \in L^{p_\Theta}$.
Easy case: Assume $f \in L^{p_\Theta} \cap L^{p_0}$ and $p_0 < p_\Theta < \infty$. Then find $\{f_n\}_{n\in\mathbb{N}}$ step functions s.t. $f_n \to f$ in L^{p_Θ} and L^{p_0} ($|f_n| \le |f|$).
By assumption: $\|Tf_n - Tf\|_{L^{q_0}} \le \|f_n - f\|_{L^{p_0}} \to 0$.
By step 1: $\|Tf_n - Tf\|_{L^{p_\Theta}} \le \|f_n - f\|_{L^{p_\Theta}} \to 0$

$$\text{Thus:} \quad \left.\begin{array}{l} Tf_n \to Tf \text{ in } L^{L^{p_0}} \\ \{Tf_n\} \text{ Cauchy sequence in } L^{p_\Theta} \end{array}\right\} \Rightarrow Tf_n \to Tf \text{ in } L^{p_\Theta}$$

Consequently, by step 1 again:

$$\|Tf\|_{L^{q_\Theta}} = \lim_{n\to\infty} \|Tf_n\|_{L^{q_\Theta}} \le \lim_{n\to\infty} \|f_n\|_{L^{p_\Theta}} = \|f\|_{L^{p_\Theta}}$$

More general case: Assume $f \in L^{p_\Theta}, \quad p_0 < p_\Theta < p_1$
Decompose: $f = \underbrace{f\mathbb{1}_{\{|f|>\varepsilon\}}}_{=:f_\varepsilon \in L^{p_0}\cap L^{p_\Theta}} + \underbrace{f\mathbb{1}_{\{|f|<\varepsilon\}}}_{\in L^{p_1}}$
Then: $Tf = Tf_\varepsilon + T\tilde{f}_\varepsilon$

$$\|Tf_\varepsilon\|_{L^{q_\Theta}} \le \|f_\varepsilon\|_{L^{p_\Theta}} \le \|f\|_{L^{p_\Theta}} \qquad \forall \varepsilon > 0$$

$$\left\|T\tilde{f}_\varepsilon\right\|_{L^{q_1}} \le \left\|\tilde{f}_\varepsilon\right\|_{L^{p_1}} \to 0 \quad as\ \varepsilon \to 0$$

□

We used:
Q: If $F = F_\varepsilon + G_\varepsilon$ and $\|F_\varepsilon\|_{L^q} \le 1$ for all $\varepsilon > 0$ and $\|G_\varepsilon\|_{L^r} \to 0$ as $\varepsilon \to 0$. How can we conclude that $\|F\|_{L^q} \le 1$?

Idea: If we only need an upper bound, then Fatou is helpful!
Apply: From $\|G_\varepsilon\|_{L^r} \to 0 \;\Rightarrow$ up to a subseq. $G_\varepsilon \to 0$ a.e.
$\Rightarrow \; F_\varepsilon \to F$ a.e.
$\Rightarrow \; \|F\|_{L^q} \leq \limsup \varepsilon \to 0 \, \|F_\varepsilon\|_{L^q} \leq 1$

In general: If $g_n \to g$ in L^r for $r < \infty$, then $\exists$ subseq:

$$\begin{cases} g_n(x) \to g(x) a.e. \\ |g_n(x)| \leq G(x) \in L^r \end{cases}$$

Note: If $p = p_0 = p_1$, then we cannot use $p_0 < p_\Theta < p_1$ but we can simply use Hölder:

$$\|Tf\|_{L^{q_\Theta}} \leq \max\{\|Tf\|_{L^{q_0}}, \|Tf\|_{L^{q_1}}\} \leq \|f\|_{L^p}$$

3.2 Real Interpolation

Theorem 3.4 (Marcinkiewicz Interpolation Theorem)**.** Let X be a sigma finite measure space. Take $1 \leq p_0 < p_1 \leq \infty$. Assume T is a quasi linear map of measurable functions from X to measurable functions in X, i.e.

$$\begin{cases} |T(\lambda f)(y)| \leq |\lambda||Tf(y)| \quad \forall \lambda \in \mathbb{R} \\ |T(f+g)(y)| \leq K(|Tf(y)| + |Tg(y)|) \end{cases}$$

for all $y \in X$. And

$$\begin{cases} \|T\|_{L^{p_0} \to L^{p_0}_w} \leq 1 \\ \|T\|_{L^{p_1} \to L^{p_1}_w} \leq 1 \end{cases}$$

Then:

$$\|T\|_{L^p \to L^p} \leq C_{p,K} \|Tf\|_{L^p} \leq C_p \|f\|_{L^p}$$

for all $p_0 < p < p_1$.

<u>Motivation:</u> HL max ineq: $\left.\begin{matrix} \|Mf\|_{L^1_w} \leq C\|f\|_{L^1} \\ \|Mf\|_{L^\infty} \leq \|f\|_{L^\infty} \end{matrix}\right\} \Rightarrow \quad \|Mf\|_{L^p} \leq \|f\|_{L^p}$

<u>Recall:</u> $f \in L^p_w \;\Leftrightarrow\; \|f\|_{L^p_w} = \sup_{\lambda > 0} \lambda |\{|f| > \lambda\}|^{\frac{1}{p}} < \infty$
Here $\|\cdot\|_{L^p_w}$ is a quasi norm, i.e.

$$\|f+g\|_{L^p_w} \leq K(\|f\|_{L^p_w} + \|g\|_{L^p_w})$$

Remark. We can take $C_{p,K} = 2K(\frac{p}{p-p_0} + \frac{p}{p_1-p})^{\frac{1}{p}} \to \infty$ as $p \to p_1$ or $p \to p_0$
In comparison to the Riesz-Thorin thm, the constant here is not too good but the thm applies to non-linear mappings!

proof of Marcinkiewicz Interpolation Theorem. Start with the layer-cake representation ($p_0 < p < p_1 < \infty$)

$$\|Tf\|_{L^p}^p = p\int_0^\infty \lambda^{p-1}|\{|Tf| > \lambda\}|d\lambda$$

Decompose: $f = \underbrace{f\mathbb{1}_{\{|f|>\frac{\lambda}{2}\}}}_{f_\lambda^>} + \underbrace{f\mathbb{1}_{\{|f|<\frac{\lambda}{2}\}}}_{f_\lambda^<}$
$\Rightarrow\ |Tf| \leq K(|Tf_\lambda^>| + |Tf_\lambda^<|)$

$$\begin{aligned}
|\{|Tf| > \lambda\}| &\leq |\{|Tf_\lambda^>| > \frac{\lambda}{2K}\}| + |\{|Tf_\lambda^<| > \frac{\lambda}{2K}\}| \\
&\lesssim \|Tf_\lambda^>\|_{L_w^{p_0}}^{p_0} + \frac{1}{\lambda^{p_1}}\|f_\lambda^<\|_{L^{p_1}}^{p_1} \\
&\lesssim \frac{1}{\lambda^{p_0}}\|f_\lambda^>\|_{L^{p_0}}^{p_0} + \frac{1}{\lambda^{p_1}}\|f_\lambda^<\|_{L^{p_1}}^{p_1} \\
&= \frac{1}{\lambda^{p_0}}\int |f|^{p_0}\mathbb{1}_{\{|f|>\frac{\lambda}{2}\}} + \frac{1}{\lambda^{p_1}}\int |f|^{p_1}\mathbb{1}_{\{|f|<\frac{\lambda}{2}\}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ \|Tf\|_{L^p}^p &\lesssim \int_0^\infty d\lambda\left[\frac{1}{\lambda^{p_0}}\int |f|^{p_0}\mathbb{1}_{\{|f|>\frac{\lambda}{2}\}} + \frac{1}{\lambda^{p_1}}\int |f|^{p_1}\mathbb{1}_{\{|f|<\frac{\lambda}{2}\}}\right] \\
&= \int |f|^{p_0}\underbrace{\int_0^\infty \lambda^{p-p_0-1}\mathbb{1}_{\{|f|>\frac{\lambda}{2}\}}d\lambda}_{|f|^{p-p_0}} + \int |f|^{p_1}\underbrace{\int_0^\infty \lambda^{p-p_1-1}\mathbb{1}_{\{|f|<\frac{\lambda}{2}\}}d\lambda}_{|f|^{p-p_0}} \\
&\lesssim \int |f|^p
\end{aligned}$$

case $p_1 = \infty$ (exercise) □

Chapter 4

Lorentz spaces

Motivation: X_0, X_1 2 Banach spaces compatible (i.e. $X_0 \cap X_1$ and $X_0 + X_1$ can be defined, $X_0, X_1 \subset Z$ vs)

$$X_0 \cup X_1 \subset X \subset X_0 + X_1$$

X is intermediate between $X_0 \cap X_1$ and $X_0 + X_1$.
With Lorentz spaces $L_{p,q}$ we want to study what's inbetween $L^{p,p} = L^p$ and $L^{p,\infty} = L^p_w$ if $p < q$. If $p > q$ then $L^{p,q}$ has a stronger norm than L^p.

Definition 4.1 (Lorentz space)**.** Let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable. $f \in L^{p,q} \Leftrightarrow \|f\|_{L^{p,q}} < \infty$

- $\|f\|_{L^{p,q}} := p^{\frac{1}{q}} \left\| \lambda |\{|f| > \lambda\}|^{\frac{1}{p}} \right\|_{\mathbb{R}_+, \frac{d\lambda}{\lambda}} = \left(\int_0^\infty \lambda^{q-1} |\{|f| > \lambda\}|^{\frac{q}{p}} d\lambda \right)^{\frac{1}{q}}$
- $\|f\|_{L^{p,\infty}} := \|f\|_{L^p_w}$

Definition 4.2 (Quasi-normed vector space)**.** Let V be a vector space with field $\mathbb{C}$. We say that $(V \, \|\cdot\|)$ is a quasi normed vector space, if $\|\cdot\| : V \to [0, \infty)$ satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \, \|x\|$ for all $\lambda \in \mathbb{C}$, $x \in V$
- $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in V$, $C \geq 1$ independent of x, y

Remark. It's easy to see that $\|\cdot\|_{L^{p,q}}$ is a quasi norm. (to see this use $|\{|f + g| > t\}| \leq |\{|f| > \frac{t}{2}\}| + |\{|g| > \frac{t}{2}\}|$)

Example 4.3. $L^{1,\infty} = L^1_w$ is a quasi normed space with $C = 2$. Recall that $\|f\|_{L^{1,\infty}} = \sup\limits_{\lambda>0} \lambda|\{|f| > \lambda\}|$.
Take $f, g \in L^{1,\infty}$. Consider $|\{|f+g| > \lambda\}| \leq |\{|f| > \frac{\lambda}{2}\}| + |\{|g| > \frac{\lambda}{2}\}|$

$$\begin{aligned} \lambda|\{|f+g| > \lambda\}| &\leq 2(\frac{\lambda}{2}|\{|f| > \frac{\lambda}{2}\}| + \frac{\lambda}{2}|\{|f| > \frac{\lambda}{2}\}|) \\ &\overset{\sup_\lambda}{\leq} 2(\|f\|_{L^{1,\infty}} + \|g\|_{L^{1,\infty}}) \end{aligned}$$

Remark. There exists no norm $||| \cdot |||$ on $L^{1,\infty}$ which is equivalent to $\|\cdot\|_{L^{1,\infty}}$, i.e. there are no $C_1, C_2 > 0$ s.t. for all $f \in L^{1,\infty}$:

$$C_1 \|f\|_{L^{1,\infty}} \leq |||f||| \leq C_2 \|f\|_{L^{1,\infty}}$$

Example 4.4. Define $f_{N,R}(x) = \frac{1}{\log N} \frac{1}{|x - \frac{R}{N}|}$ for $N \geq 1,\ R \in \{1, 2, ..., N\}$. Easy to check: $f_{N,R} \in L^{1,\infty}(\mathbb{R})$ and:

$$\begin{aligned} |\{f_{N,R} > \lambda\}| &= |\{x \mid \frac{1}{\log N} \frac{1}{|x - \frac{R}{N}|} > \lambda\}| \\ &= |\{x \mid \frac{1}{\log N \lambda} > |x - \frac{R}{N}|\}| \\ &= \frac{2}{\log(N)\lambda} \end{aligned}$$

$$\Rightarrow \quad \|f_{N,R}\|_{L^{1,\infty}} = \sup_{\lambda>0} \lambda|f_{N,R} > \lambda| = \frac{2}{logN} \to 0 \quad (as\ N \to 0)$$

However, $F_N(0) = \frac{1}{N} \sum\limits_{R=1}^{N} f_{N,R}(0) = \frac{1}{\log N} \sum\limits_{R=1}^{N} \frac{1}{R} \geq 1 > 0\ \forall N$
Exercise: similarly $\|F_N\|_{L^{1,\infty}} \not\to 0$ as $N \to \infty$. This shows: $\nexists$ norm equivalent to $\|\cdot\|_{L^{1,\infty}}$. ($L^{1,\infty}$ is not locally convex, i.e. $\nexists$ local basis of neighbourhood of 0 which consists of convex sets)

4.1 Aoki-Robewicz theorem

Theorem 4.5 (Aoki-Robewicz)**.** Let $(V; \|\cdot\|)$ be a quasi normed vector space. Then it is metrizable, i.e. $\exists$ a metric $d(x, y) = \Lambda(x - y)$ for a function $\Lambda : V \to [0, \infty]$ s.t.

- $\Lambda(x) = 0 \;\Leftrightarrow\; x = 0$
- $\Lambda(x) = \Lambda(-x)$
- $\Lambda(x+y) \leq \Lambda(x) + \Lambda(y)$

and

$$x_n \to x \text{ w.r.t. the metric } \Leftrightarrow \|x_n - x\| \to 0$$

Lemma 4.6. Let V be a quasi normed vector space and C be the constant of the quasi norm $\|\cdot\|$ and $\alpha > 0$ s.t. $(2C) = 2^\alpha$. Then $\forall n \in \mathbb{N}$, $\forall x_1, ..., x_n \in V$:

$$\|x_1 + x_2 + ... + x_n\|^{\frac{1}{\alpha}} \leq 4(\|x_1\|^{\frac{1}{\alpha}} + (\|x_2\|^{\frac{1}{\alpha}} + ... + (\|x_n\|^{\frac{1}{\alpha}})$$

Remark. If we use the assumption $\|x+y\| \leq C(\|x\| + \|y\|)$, then

$$\begin{aligned}\|x_1 + x_2 + ... + x_n\| &\leq C\|x_1\| + C\|x_2 + ... + x_n\| \\ &\leq C\|x_1\| + C^2\|x_2\| + C^2\|x_3 + ... + x_n\| \\ &\leq C\|x_1\| + C^2\|x_2\| + ... + C^{N-1}\|x_n\|\end{aligned}$$

proof of lemma. Define $H : V \to [0,\infty)$, $H(x) := \begin{cases} 0, \; if \; x = 0 \\ 2^{j\alpha}, \; if 2^{(j-1)\alpha} \leq \|x\| < 2^{j\alpha} \quad \forall j \in \mathbb{Z} \end{cases}$
(Dyadic decomposition)
Then: $\|x\| \leq \|H(x)\| \leq 2^\alpha \|x\| \quad \forall x \in V$
Claim: $\forall n \forall x_1, ..., x_n : \;\; \|x_1, ..., x_n\|^{\frac{1}{\alpha}} \leq 2(H(x_1)^{\frac{1}{\alpha}} + ... + H(x_n)^{\frac{1}{\alpha}})$
We prove this by induction in n:
$n = 1 : \|x\|_1 \leq H(x_1) \;\Rightarrow\; \|x\|_1^{\frac{1}{\alpha}} \leq H(x_1)^{\frac{1}{\alpha}}$
$(n-1) \rightsquigarrow n$: Wlog, assume $\|x\|_1 \geq \|x\|_2 \geq ... \geq \|x\|_n \;\Rightarrow\; H(x_1) \geq H(x_2) \geq ... \geq H(x_n)$

<u>Case 1:</u> $\exists i_0 \in \{1, ..., n-1\}$ s.t. $H(x_{i_0}) = H(x_{i_0+1}) = 2^{j_0\alpha}$
Then: $\|x_{i_0} + x_{i_0+1}\| \leq C(\|x_{i_0}\| + \|x_{i_0+1}\|) \leq \underbrace{H(x_{i_0}) + H(x_{i_0+1})}_{2C2^{j_0\alpha}} \underset{2C=2^\alpha}{=} 2^{(j_0+1)\alpha}$

$\Rightarrow \;\; H(x_{i_0} + x_{i_0+1}) \leq 2^{(j_0+1)\alpha}$
$\Rightarrow \;\; H(x_{i_0} + x_{i_0+1})^{\frac{1}{\alpha}} \leq 2^{j_0+1} = H(x_{i_0})^{\frac{1}{\alpha}} + H(x_{i_0+1})^{\frac{1}{\alpha}}$ By induction assumption:

$$\begin{aligned}\|x_1 + ... + x_n\| &= \|x_1 + ... + (x_{i_0} + x_{i_0+1}) + ... + x_n\| \\ &\leq 2(H(x_1)^{\frac{1}{\alpha}} + ... + \underbrace{H(x_{i_0} + x_{i_0+1})^{\frac{1}{\alpha}}}_{\leq H(x_{i_0})^{\frac{1}{\alpha}} + H(x_{i_0+1})^{\frac{1}{\alpha}}} + ... + H(x_n)^{\frac{1}{\alpha}})\end{aligned}$$

Case 2: $H(x_1) > H(x_2) > ... > H(x_n)$
$\Rightarrow \ H(x_i) \leq H(x_{i-1})2^{\alpha} \ \forall i \ \ \Rightarrow \ H(x_i) \leq 2^{-(i-1)\alpha}H(x_1) \ \ \forall i$

$$\begin{aligned}
\Rightarrow \ \ \|x_1 + ... + x_n\| &\leq C(\|x_1\| + \|x_2 + ... + x_n\|) \\
&\leq \max(2C\,\|x_1\|, 2C\,\|x_2 + ... + x_n\|) \\
&\leq ... \leq \max(\underbrace{2C}_{=2^{\alpha}}, (2C)^2\,\|x_2\|, ..., (2C)^{n-1}\,\|x_n\|) \\
&\leq \max(2^{\alpha}H(x_1), 2^{2\alpha}H(x_2), ..., 2^{(n-1)\alpha}H(x_n)) \\
&= 2^{\alpha}H(x_1)
\end{aligned}$$

$$\Rightarrow \ \ \|x_1 + x_2 + ... + x_n\|^{\frac{1}{\alpha}}\, 2H(x_1)^{\frac{1}{\alpha}} \leq 2(H(x_1)^{\frac{1}{\alpha}} + ... + H(x_n)^{\frac{1}{\alpha}})$$

Thus the claim is correct and this implies the conclusion.

$$\begin{aligned}
\|x_1 + x_2 + ... + x_n\|^{\frac{1}{\alpha}} &\leq 2(H(x_1)^{\frac{1}{\alpha}} + ... + H(x_n)^{\frac{1}{\alpha}}) \\
&\leq 4(\|x_1\|^{\frac{1}{\alpha}} + ... + \|x_n\|^{\frac{1}{\alpha}})
\end{aligned}$$

□

Remark. The difficulty is related to L^p spaces with $0 < p < 1$.

Proof of the Aoki-Robewicz theorem.
Define $\Lambda : V \to [0, \infty]$ by $\Lambda(x) = \inf\{\sum_{i=1}^{n} \|x_i\|^{\frac{1}{\alpha}} : \ \forall n \ \forall \{x_i\} : \ x = \sum_{i=1}^{n} \|x_i\|\}$,
where $2C = 2^{\alpha}$ and C is the constant from the quasi-norm.
Then:

$$\|x\|^{\frac{1}{\alpha}} \geq \Lambda(x) \geq \frac{1}{4}\,\|x\|^{\frac{1}{\alpha}}$$

(the second bound comes from lemma 4.6)
This function satisfies all desired properties:

- $\|f\|_{L^{p,q}} := p^{\frac{1}{q}} \left\| \lambda |\{|f| > \lambda\}|^{\frac{1}{p}} \right\|_{\mathbb{R}_+, \frac{d\lambda}{\lambda}} = \left(\int_0^{\infty} \lambda^{q-1} |\{|f| > \lambda\}|^{\frac{q}{p}} d\lambda \right)^{\frac{1}{q}}$
- $\|f\|_{L^{p,\infty}} := \|f\|_{L^p_w}$. This is an easy consequence from the definition. In fact $\forall \varepsilon > 0 :$ $\exists \{x_i\}, \{y_j\}$ s.t. $x = \sum_i \|x_i\|, y = \sum_j \|y_j\|$
  $\Lambda(x) \geq \sum_i \|x_i\|^{\frac{1}{\alpha}} - \varepsilon$
  $\Lambda(y) \geq \sum_j \|y_j\|^{\frac{1}{\alpha}} - \varepsilon$

$$\begin{aligned}
\Lambda(x) + \Lambda(y) &\geq \sum_i \|x_i\|^{\frac{1}{\alpha}} - \varepsilon + \Lambda(y) \geq \sum_j \|y_j\|^{\frac{1}{\alpha}} - \varepsilon \\
&\geq \Lambda(\sum_i x_i + \sum_j y_j) - 2\varepsilon = \Lambda(x + y) - 2\varepsilon
\end{aligned}$$

  then $\varepsilon \to 0$

This allows us to define the distance function $d(x, y) = \Lambda(x - y)$, which satisfies all requirements of a metric $\Rightarrow$ $(V, \|\cdot\|)$ is a metric space.

Further, we have that

$$x_n \to x \text{ in } (V, d) \;\Leftrightarrow\; d(x, y) \to 0 \;\Leftrightarrow \|x_n - x\| \to 0$$

Thus quasi-norm and metric are compatible.

□

Remark. This topology, i.e. (V, d) is the unique topology which is compatible with the quasi-norm.
To be precise, if we are given the quasi-norm $\|\cdot\|$, then the only topology should come from the local basis of the neighbourhood of 0 given by $B_r(0) = \{x : \ \|x\| < r\}$ ($B_r(0)$ might not be open in (V, d))
Here the open sets are defined by

$$U \subset V \text{ is open iff } \forall x \in U \exists r_x > 0 \; s.t. \; B_{r_x}(x) = \{y : \ |x - y| < r_x\} \subset U$$

Exercise. The $L^{1,\infty}$ quasi norm is not equivalent to any other norm.

4.2 Normability of Lorentz spaces

Q: For which $L^{p,q}$ do we have a norm which is equivalent to the quasi norm?

Theorem 4.7. $\forall 1 < p \leq \infty$ and $1 \leq q \leq \infty$. Then $\exists$ norm in $L^{p,q}$ that is equivalent to the quasi-norm. We will construct $|||\cdot|||$ s.t.

$$\|f\|_{L^{p,q}} \leq |||f||| \leq p' \|f\|_{L^{p,q}}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

To prove this, we need a rearranged version of f! First, let us consider the basic rearrangement inequality:

Theorem 4.8. If we have real numbers $a_1 \geq a_2 \geq ... \geq a_n, \ b_1 \geq b_2 \geq ... \geq b_n$, then:

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\sigma(i)} \qquad \forall \sigma \in S_n$$

[Basic rearrangement inequality/Chebychev sum inequality] E.g. $a_1 b_1 + a_2 b_2 \geq a_1 b_2 + a_2 b_1 \Leftrightarrow (a_1 - a_2)(b_1 - b_2) \geq 0$

Definition 4.9. Let $f : \mathbb{R}^d \to \mathbb{C}$. We define $f^* : \mathbb{R}^d \to [0, \infty]$ s.t.

1. f^* is radially symmetric decreasing, i.e. $f^*(x)$ depends only on $|x|$ and $|x| \mapsto f^*(x)$ is decreasing
2. $|\{f^* > t\}| = |\{|f| > t\}|$, for all $t > 0$.

Equivalently, we can use the layer cake representation to define f^*.

$$|f(x)| = \int_0^\infty \mathbb{1}_{\{|f|>t\}}(x)dt$$

$$\rightsquigarrow \; f^* := \int_0^\infty \mathbb{1}_{\{|f|>t\}^*}$$

Where for a set $\Omega \subset \mathbb{R}^d$, $\Omega^* =$ ball in $\mathbb{R}^d$ centered at 0, with $|\Omega| = |\Omega^*|$

Theorem 4.10 (Hardy-Littlewood rearrangement inequality)**.** If $f, g : \mathbb{R}^d \to \mathbb{C}$, then

$$|\int_{\mathbb{R}^d} fg| \leq \int_{\mathbb{R}^d} f^* g^*$$

Proof. Consider $f, g \geq 0$. Then by layer cake

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)g(x) &= \int_{\mathbb{R}^d} (\int_0^\infty \mathbb{1}_{\{f>t\}}(x)dt)(\int_0^\infty \mathbb{1}_{\{g>t\}}(x)dt)dx \\
&= \int_{\mathbb{R}^d} (\int_0^\infty \mathbb{1}_{\{f>t\}}(x)dt)(\int_0^\infty \mathbb{1}_{\{g>s\}}(x)ds)dx \\
&\leq \int_{\mathbb{R}^d} (\int_0^\infty \mathbb{1}_{\{f>t\}^*}(x)dt)(\int_0^\infty \mathbb{1}_{\{g>s\}^*}(x)ds)dx \\
&= \int_{\mathbb{R}^d} f^*(x)g^*(x)dx
\end{aligned}$$

We used: $\int_{\mathbb{R}^d} \mathbb{1}_A \mathbb{1}_B = |A \cap B| = |A^* \cap B^*| = \int_{\mathbb{R}^d} \mathbb{1}_{A^*} \mathbb{1}_{B^*}$
And $|A \cap B| \leq \min(|A|, |B|) = \min(|A^*|, |B^*|) = |A^* \cap B^*|$ □

Definition 4.11 (Decreasing rearrangement)**.** Let $f : \mathbb{R}^d \to \mathbb{C}$. Define $f_* : [0, \infty) \to [0, \infty]$ s.t.

- f_* is decreasing
- $|\{f_* > t\}|_{\mathbb{R}_+} = |\{|f| > t\}|_{\mathbb{R}^d}$

Example 4.12. If $f(x) = \frac{1}{|x|^p}$ for $x \in \mathbb{R}^d$, then $f^*(x) = f(x) = \frac{1}{|x|^p}$ and $f_*(t) = \frac{1}{(\frac{t}{|B_1|})^{\frac{p}{d}}}$ where $t \in \mathbb{R}_+$ and $|B_1|$ = volume of unit ball in $\mathbb{R}^d$ as:

$$|\{f > \lambda\}| = |\{\frac{1}{|x|^p} > \lambda\}| = |\{|x| < \lambda^{-\frac{1}{p}}\}|_{\mathbb{R}^d} = \lambda^{-\frac{d}{p}}|B_1|$$

$$|\{f_* > \lambda\}|_{\mathbb{R}_+} = |\{(\frac{|B_1|}{t})^{\frac{p}{d}} > \lambda\}|_{\mathbb{R}_+} = |B_1|\lambda^{\frac{-d}{p}}$$

Theorem 4.13 (Alternative definition of $\|\cdot\|_{L^{p,q}}$)**.**
$p, q < \infty$:

$$\begin{aligned} \|f\|_{L^{p,q}} &\overset{def}{=} (p\int_0^\infty t^{q-1}|\{|f| > t\}|^{\frac{q}{p}}dt)^{\frac{1}{q}} \\ &\overset{new}{=} (\int_0^\infty \lambda^{\frac{q}{p}-1}f_*(\lambda)^q d\lambda)^{\frac{1}{q}} \\ &= \left\|\lambda^{\frac{1}{p}}f_*(\lambda)\right\|_{L^q(\mathbb{R}_+, \frac{d\lambda}{\lambda})} \end{aligned}$$

If $q = \infty$:

$$\begin{aligned} \|f\|_{L^{p,\infty}} &\overset{def}{=} \sup_{t>0} t|\{|f| > t\}|^{\frac{1}{p}} \\ &\overset{new}{=} \left\|\lambda^{\frac{1}{p}}f_*(\lambda)\right\|_{L^\infty(\mathbb{R}_+)} \end{aligned}$$

Proof. Case $q < \infty$:
From the def:

$$\begin{aligned} \|f\|_{L^{p,q}}^q &= p\int_0^\infty t^{q-1}|\{|f| > t\}|^{\frac{q}{p}}dt \\ &= p\int_0^\infty t^{q-1}(\frac{q}{q}\int_0^\infty \lambda^{\frac{q}{p}-1}\mathbb{1}_{\{f_*(\lambda)>t\}}d\lambda)dt \end{aligned}$$

$$f_*(\lambda) > t \Leftrightarrow |\{|f| > t\}| > \lambda \;\; \text{Duality formula (exercise)}$$

$$\begin{aligned} &\overset{\text{Fubini}}{=} \int_0^\infty (q\int_0^\infty t^{q-1}\mathbb{1}_{\{f_*(\lambda)>t\}}dt)\lambda^{\frac{q}{p}-1}d\lambda) \\ &= \int_0^\infty f_*(\lambda^{\frac{q}{p}-1}d\lambda) \end{aligned}$$

case $q = \infty$ (exercise) $\square$

Lemma 4.14.

$$\int_0^t f_*(s)ds = \sup\{\int_E |f| : \ |E| \leq t\}$$

Proof. Consider step functions $f(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(x)$ with $a_1 > a_2 > ... > a_n$ and $\{A_i\}$ disjoint.

- case $t < \sum_{i=1}^n |A_i|$ and we stop at some $k < n$ s.t.

$$\sup_{|E|\leq t} \int_E |f| = \sum_{i=1}^{k} + a_{k+1}(t - \sum_{i=1} |A_i|)$$

- else: $k = n$ and $a_{k+1} = 0$

(bathtub principle)
$f_*(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{[B_{i-1},B_i]}(s)$ where $B_i = \sum_{j=1}^{i} |A_j|$
$\Rightarrow \ \int_0^t f(s)ds = \sum_{i=1}^{k} a_i(B_i - B_{i-1}) + a_{k+1}(t - B_k)$

Still we need to pass the limit for the general case $f \in L^{p,q}$. In the general case, we can choose a sequence of step functions $\{f_n\}$ s.t. $f_n \uparrow f$ point wise a.e.
$\Rightarrow \ (f_n)_*(t) \uparrow f_*(t) \ \ \forall t \ \ (exercise)$
$\Rightarrow \ \int_0^t (f_n)_*(s)ds \to \int_0^t f_*(s)ds$ by monotone convergence
Moreover: $\sup_{|E|\leq t} \int_E |f| \uparrow \sup_{|E|\leq t} \int_E |f|$ by monotone convergence ◻

Lemma 4.15 (Hardy inequality)**.**
If $g \in C^1(\mathbb{R}_+), \ g(0) = 0$, then :

$$p' \|g\|_{L^p(\mathbb{R}_+)} \geq \left\| \frac{g}{|x|} \right\|_{L^p(\mathbb{R}_+)}$$

More generally:

$$p'(\int_0^\infty x^{\frac{q}{p}-1} |g'(x)|^q)^{\frac{1}{q}} \geq (\int_0^\infty x^{\frac{q}{p}-1} |\frac{g(x)}{x}|^q dx)^{\frac{1}{q}}$$

Proof. Duality $(L^q(X))^* = L^{q'}(X)$ i.e. $\|G\|_{L^q(X)} = \sup_{\|\varphi\|_{L^{q'}(X)}=1} |\int G\varphi|$, $X = \mathbb{R}_+$, $d\mu(t) =$

$t^{\frac{q}{p}-1}dt$

$$
\begin{aligned}
(\int_0^\infty t^{\frac{q}{p}-1}|\frac{1}{t}\int h(s)ds|^q dt)^{\frac{1}{q}} = \left\|\frac{1}{t}\int_0^t h(s)ds\right\|_{L^q(X,d\mu)} &= \sup_{\varphi_{L^{q'}}=1} |\int_0 1\infty \frac{1}{t}\int_0^t h(s)ds\varphi(t)dt| \\
&= |\int_0^\infty \frac{1}{t}\int_0^t h(s)ds\varphi(t)d\mu(t)| \\
&= |\int_0^\infty \int_0^1 h(t\xi)d\xi\varphi(t)d\mu(t)| \\
&\leq \int_0^1 \|h(\xi\cdot)\|_{L^q(X,d\mu)} \|\varphi\|_{L^{q'}(X,d\mu)} d\xi \\
&\leq \int_0^1 \|h\|_{L^q(X)} \xi^{\frac{-1}{p}} d\xi \\
&= \|h\|_{L^q(X)} \left[\frac{\xi^{1-\frac{1}{p}}}{1-\frac{1}{p}}\right]_0^1 \\
&= p' \|h\|_{L^q(X)}
\end{aligned}
$$

□

proof of the normality of $L^{p,q}$ for $p > 1$.
Define $f_{**}(t) = \frac{1}{t}\int_0^t f_*(s)ds$ and

$$
\|f\| = \begin{cases} (\int_0^\infty \lambda^{\frac{q}{p}-1} f_{**}(\lambda)^q d\lambda)^{\frac{1}{q}} & if\ q < \infty \\ \left\|\lambda^{\frac{1}{p}} f_{**}(\lambda)\right\|_{L^\infty} & if\ q = \infty \end{cases}
$$

Then we claim that $\|f\|$ is a norm in $L^{p,q}$ and

$$
\|f\|_{L^{p,q}} \leq |||\cdot||| \leq p' \|f\|_{L^{p,q}}
$$

Step 1: (triangle inequality)

$$
(f+g)_{**}(\lambda) \leq f_{**}(\lambda) + g_{**}(\lambda),\ \forall \lambda > 0
$$

Remark: We only have $(f+g)_*(t) \leq f_*(\frac{t}{2}) + g_*(\frac{t}{2})$ (exercise)

From lemma 4.14:

$$
\begin{aligned}
(f+g)_{**} &= \frac{1}{t}\int_0^t (f+g)(s)ds = \sup\{\int_E |f+g| :\ |E| \leq t\} \\
&\leq \sup\{\int_E |f| + \int_E |g| :\ ...\} \\
&\leq \sup\{\int_E |f| :\ |E| \leq t\} + \sup\{\int_E |g| :\ |E| \leq t\}
\end{aligned}
$$

As a consequence:

$$\begin{aligned} |||f+g||| &= \left(\int_0^\infty \lambda^{\frac{q}{p}-1}(f+g)_{**}(\lambda)^q d\lambda\right)^{\frac{1}{q}} \\ &= \left\|\lambda^{\frac{1}{p}}(f+g)_{**}(\lambda)\right\|_{L^q(\mathbb{R}_+,\frac{d\lambda}{\lambda})} \\ &\leq \left\|\lambda^{\frac{1}{p}}(f_{**}(\lambda)+g_{**}(\lambda))\right\|_{L^q(\mathbb{R}_+,\frac{d\lambda}{\lambda})} \\ &\leq \left\|\lambda^{\frac{1}{p}}f_{**}(\lambda)\right\|_{L^q(\mathbb{R}_+,\frac{d\lambda}{\lambda})} + \left\|\lambda^{\frac{1}{p}}f_{**}(\lambda)\right\|_{L^q(\mathbb{R}_+,\frac{d\lambda}{\lambda})} \end{aligned}$$

Step 2: We still have to prove that

$$\|f\|_{L^{p,q}} \leq |||\cdot||| \leq p' \|f\|_{L^{p,q}}$$

It follows directly from the definition that $|||f||| \geq \|f\|_{L^{p,q}}$.
The other side follows from Hardy's inequality:

$$g(x) = \int_0^x h(t)dt, \quad g(0)=0, \quad g'(x)=h(x)$$

$$\Rightarrow \; p'(\int_0^\infty t^{\frac{p}{q}-1}|h(t)|^q)^{\frac{1}{q}} \geq (\int_0^\infty t^{\frac{p}{q}-1}|\frac{1}{t}\int_0^t h(s)ds|^q)^{\frac{1}{q}}$$

$$\overset{h\rightsquigarrow f_*}{\Rightarrow} p'\|f\|_{L^{p,q}} \geq |||f|||$$

□

4.3 Some functional inequalities for Lorentz norms

Theorem 4.16 (Monotonicity)**.** If $q<r$, then $\exists C>0$ s.t. $\forall f \in L^{p,q}$:

$$C\|f\|_{L^{p,q}} \geq \|f\|_{L^{p,r}}$$

Remark. If $|\Omega| < \infty$, then $L^{p_1} \subset L^{p_2}$ if $p_1 > p_2$. We get the same for all q,r with $L^{p_1,q} \subset L^{p_2,r}$ (exercise)

Proof.
$r=\infty$:

$$\begin{aligned}
\|f\|_{L^{p,q}} &= (\int_0^\infty r^{\frac{q}{p}-1} f_*(t)^q dt)^{\frac{1}{q}} \\
&\geq (\int_0^\lambda r^{\frac{q}{p}-1} \underbrace{f_*(t)}_{\geq f_*(\lambda)}{}^{q}\, dt)^{\frac{1}{q}} \\
&\geq (\int_0^\lambda t^{\frac{q}{p}-1} dt)^{\frac{1}{q}} f_*(\lambda) \\
&= (\frac{p}{q}\lambda^{\frac{q}{p}})^{\frac{1}{q}} f_*(\lambda) \sim \lambda^{\frac{1}{p}} f_*(\lambda), \quad \forall \lambda > 0
\end{aligned}$$

$$\Rightarrow \quad \|f\|_{L^{p,q}} \geq \sup_{\lambda>0} \lambda^{\frac{1}{p}} f_*(\lambda) = \|f\|_{L^{p,\infty}}$$

<u>$r < \infty$:</u>

$$\begin{aligned}
\|f\|_{L^{p,r}} &= (\int_0^\infty r^{\frac{r}{p}-1} f_*(t)^r dt)\frac{1}{r} \\
&\leq \left(\int_0^\infty t^{\frac{q}{p}-1} f_*^q(t) dt \;(\sup_{\lambda>0} \lambda^{\frac{r-q}{p}} f_*^{r-q}(\lambda))\right)^{\frac{1}{r}} \\
&= (\|f\|_{L^{p,q}}^q \, \|f\|_{L^{p,\infty}}^{r-q})^{\frac{1}{r}} \lesssim \|f\|_{L^{p,q}}
\end{aligned}$$

(We used $\|f\|_{L^{p,\infty}} \lesssim \|f\|_{L^{p,q}}$ version $r = \infty$)

□

Theorem 4.17 (Hölder)**.** If $1 < p_1, p_2, p < \infty$, $1 \leq q_1, q_2, q \leq \infty$ s.t.

$$\frac{1}{p_1} + \frac{1}{p_2} = 1, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$$

then:

$$\|fg\|_{L^{p,q}} \lesssim \|fg\|_{L^{p_1,q_1}} \, \|fg\|_{L^{p_2,q_2}}$$

Remark. 1. We can replace $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ by $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{q}$

2. $p = q = 1 \;\Rightarrow\; \|fg\|_{L^1} \lesssim \|f\|_{L^{p_1,q_1}} \|f\|_{L^{p_2,q_2}}$
$\Rightarrow\; (L^{p_1,q_1})^* = L^{p_2,q_2}, \quad \forall 1 < p_1 < \infty\; 1 \leq q_1 < \infty$

Proof. <u>Claim:</u> $(fg)_*(t_1 + t_2) \leq f_*(t_1) g_*(t_2)$. To see this, we use the density relation:

$$\begin{aligned}
& |\{|f| > \lambda\}| > t \quad &\Leftrightarrow \quad & f_*(t) > \lambda \\
\rightsquigarrow\; & |\{|f| > \lambda\}| \leq t \quad &\Leftrightarrow \quad & f_*(t) \leq \lambda \\
\rightsquigarrow\; & |\{|f| > f_*(t)\}| \leq t \quad , & \forall t > 0 &
\end{aligned}$$

Consequentially:

$$|\{|fg| > f(t_1)g(t_2)\}| \le |\{|f| > f_*(t_1)\}| + |\{|g| > g_*(t_2)\}| \le t_1 + t_2$$

$$\Rightarrow \ (fg)_*(t_1+t_2) \le f_*(t_1)g_*(t_2)$$

To conclude:

$$\begin{aligned}
\|fg\|_{L^{p,q}} &= \left(\int_0^\infty t^{\frac{q}{p}-1}(fg)_*^q(\underbrace{t}_{=\frac{t}{2}+\frac{t}{2}})dt\right)^{\frac{1}{q}} \\
&\le \left(\int_0^\infty t^{\frac{q}{p}-1} f_*^q(\frac{t}{2}) g_*^q(\frac{t}{2})dt\right)^{\frac{1}{q}} \\
&\le \left(\left(\int_0^\infty t^{\frac{q_1}{p_1}-1} f_*^{q_1}(\frac{t}{2})dt\right)^{\frac{q}{q_1}} \left(\int_0^\infty t^{\frac{q_2}{p_2}-1} g_*^{q_2}(\frac{t}{2})dt\right)^{\frac{q}{q_2}}\right)^{\frac{1}{q}} \\
&\overset{\frac{t}{2}\rightsquigarrow t}{\le} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}
\end{aligned}$$

□

Theorem 4.18 (Improved Sobolev inequality)**.**
If $1 \le p < d$, $p^* = \frac{dp}{d-p}$, then

$$\|f\|_{L^{p^*,p}} \lesssim \|\nabla f\|_{L^p}$$

Remark. Since $p < p^*$, this is an improvement of the standard Sobolev inequality.
This can be proven in two ways. 1) Weak-Young inequality 2) Dyadic decomposition.
We will have a look at 1) first. The proof of the improved Sobolev inequality with weak-young is motivated by the proof of the standard Sobolev inequality with weak young.

Weak-Young: $\|g \star h\|_{L^p} \lesssim \|g\|_{L^{q,\infty}} \|h\|_{L^r}$ if $\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}$ for $1 < p, q, r < \infty$

proof of standard Sobolev inequality by weak-young. Note that the equation $-\Delta f = g$ in $\mathbb{R}^d$ can be written as

$$f(x) = (-\Delta)^{-1}g(x) = (G(x) \star g)(x) = \int_{\mathbb{R}^d} G(x-y)g(y)dy$$

and the Green function

$$\begin{cases} const.ln(x), \ if \ d = 2 \\ const.\frac{1}{|x|^{d-2}}, \ if \ d \ge 3 \end{cases}$$

(Formally: $(-\Delta)^{-1}g(k) = \frac{1}{|2\pi k|^2}\hat{g}(k) \overset{(?)}{=} \widehat{G \star g(k)} = \hat{G}(k)\hat{g}(k) \Rightarrow \hat{G}(g) = \frac{1}{|2\pi k|^2} \Rightarrow \ G(x)$ above) We will discuss the Fourier transform later to make sense of all of this.

Put differently, $f = G \star (-\Delta f) = (-\Delta)(G \star f) = (-\nabla G) \star (\nabla f)$ i.e.

$$f(x) = -\sum_{i=1}^{d} (\partial_{x_i} G \star \partial_{x_i} f)(x)$$

if f regular enough! (e.g. $f \in C_c^\infty$)
Hence:

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} = \|(-\nabla G) \star (\nabla f)\|_{L^{p^*}(\mathbb{R}^d)} \overset{weak-young}{\lesssim} \|\nabla G\|_{L^{q,\infty}} \|\nabla f\|_{L^p}$$

provided that: $\frac{1}{q} + \frac{1}{p} = 1 + \frac{1}{p^*}$
Here $|\nabla G(x)| = const. \frac{1}{|x|^{d-1}}$ for $d \geq 2$
$\Rightarrow \quad \|\nabla G\|_{L^{1,\infty}} < \infty$ with $q = \frac{d}{d-1}$
$\Leftrightarrow \; \frac{1}{p^*} = \frac{1}{q} + \frac{1}{p} - 1 = \frac{1}{\frac{d}{d-1}} + \frac{1}{p} - 1 = \frac{d-1}{d} + \frac{1}{p} - 1 = \frac{1}{p} - \frac{1}{d}$
$\Leftrightarrow \; p^* = \frac{dp}{d-p}$ □

Inspired by this, we want to prove a Young inequality for Lorentz spaces s.t. we can conclude the improved Sobolev inequality. For that we first need the following lemma.

Lemma 4.19. If $f, g : \mathbb{R}^d \to [0, \infty]$, then:

$$(f \star g)_{**}(t) \leq t f_{**}(t) g_{**}(t) + \int_0^\infty f_*(s) g_*(s) ds \quad \forall\, t > 0$$

Proof. Step 1: $f = \mathbb{1}_A, \quad A \subset \mathbb{R}^d, \quad |A| < \infty$.
Easy to check:

$$\begin{aligned} f_*(s) &= \mathbb{1}(s \leq |A|) = \mathbb{1}_{(0,|A|]}(s) \\ f_{**} &= \frac{1}{t}\int_0^t f_*(s) ds = \frac{1}{t}\int_0^t \mathbb{1}_{\{s \leq |A|\}} ds = \frac{1}{t}\min(|A|, t) \end{aligned}$$

RHS:

$$t f_{**}(t) g_{**}(t) + \int_t^\infty f_*(s) g(s) ds = \min(|A|, t) g_{**}(t) + \int_t^\infty \mathbb{1}_{\{s \leq |A|\}} g_{**}(s) ds$$

$$\begin{cases} \overset{|A| \geq t}{=} \int_0^t g_*(s) ds + \int_t^{|A|} g_*(s) ds = \int_0^{|A|} g_*(s) ds = |A| g_{**}(|A|) = |A| g_{**}(\max(t, |A|)) \\ \overset{|A| < t}{=} |A| g_{**}(t) + 0 = |A| \min(g_{**}(t), g_{**}(|A|)) \end{cases}$$

Why:

$$(\mathbb{1}_A \star g)_{**}(t) \leq |A| \min(g_{**}(t), g_{**}(|A|))$$

Recall:

$$h_{**}(t) = \frac{1}{t} \sup\{\int_B |h| : \; \forall B \subset \mathbb{R}^d \; s.t. \; |B| = t\}$$

(we used this in the proof of theorem 4.7)
Take any $B \subset \mathbb{R}^d,\ |B| = t$. Then:

$$\begin{aligned}\frac{1}{t}\int_B |\mathbb{1}_A \star g| &\leq \frac{1}{t}\int_B (\int_A |g(x-y)|dy)dx \\ &= \frac{1}{t}\int_B (\int \underbrace{X-A}_{|X-A|=A} |g(y)|dy)dx \\ &\leq \frac{1}{t}\int_B (|A|g_{**}(|A|))dx \\ &= |A|g_{**}(|A|)\end{aligned}$$

and

$$\frac{1}{t}\int_B |\mathbb{1}_A \star g| \overset{Fubini}{\leq} \frac{1}{t}\int_A (\int_B |g(x-y)|dx)dy = \frac{1}{t}\int_A (\int_{\underbrace{B-y}_{|B-y|=|B|=t}} |g(x)|dx)dy \leq \frac{1}{t}\int_A (tg_{**}(t))dy = |A|g_{**}(t)$$

$$\Rightarrow \frac{1}{t}\int_B |\mathbb{1}_A \star g| \leq |A| \min(g_{**}(|A|), g_{**}(t)) \Rightarrow \textit{ optimizing over } B$$

Step 2: f as step function:

$$f(x) = \sum_{i=1}^{N} \alpha_i \mathbb{1}_{A_i}(x),\ \forall x \in \mathbb{R}^d$$

with $A_1 \supsetneq A_2 \supsetneq ... \supsetneq A_N$
Why can we write it like that?
Assume $f = \alpha_1 \mathbb{1}_{B_1} + \alpha_2 \mathbb{1}_{B_2}(x),\ B_1 \cap B_2 =$
In this case, we can write $f(x) = \alpha_1 B_1 \cup B_2(x) + (\alpha_2 - \alpha_1)\mathbb{1}_{B_2}(x)$ with $A_1 = B_1 \cup B_2$ and $A_1 = B_2$

Claim: If $f = \sum\limits_{i=1}^{N} \alpha_i \mathbb{1}_{A_i},\quad \alpha_i \geq 0,\quad \mathbb{R}^d = A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq ... \supsetneq A_N \supsetneq A_{N+1} =$
Then:

$$f_*(s) = \sum_{i \leq j} \alpha_i \mathbb{1}_{(|A_{j+1}|,|A_j|)}(s), \quad s > 0 \quad (\textit{exercise})$$

Come back to the lemma with this f:

$$(f+g)_{**}(t) = (\sum_{i=1}^{N} \alpha_i(\mathbb{1}_{A_i} \star g))_{**}(t) = \sum_{i=1}^{N} \alpha_i(\mathbb{1}_{A_i} \star g)_{**}(t) = \sum_{i=1}^{N} \alpha_i |A_i| g(\max(t, |A_i|))$$

Assume $\exists i_0$ s.t. $|A_{i_0}| = t$ (we can take α_{i_0} if we want)

$$\Rightarrow\ (f \star g)_{**}(t) \leq \underbrace{\sum_{i<i_0} \alpha_i |A_i| g(|A_i|)}_{(I)} + \underbrace{\sum_{i \geq i_0} \alpha_i |A_i| g(|A_i|)}_{(II)}$$

$$
\begin{aligned}
(I) &= \sum_{i\le i_0}\alpha_i\int_0^{|A_i|} g_*(s)ds = \sum_{i<i_0}\alpha_i\sum_{j\ge i}\int_{|A_{j+1}|}^{|A_j|} g_*(s)ds \\
&= \underbrace{\sum_{i<i_0\le j}\alpha_i\int_{|A_{j+1}|}^{|A_j|} g_*(s)ds}_{(I')} + \underbrace{\sum_{i\le j<i_0}\alpha_i\int_{|A_{j+1}|}^{|A_j|} g_*(s)ds}_{(I'')}
\end{aligned}
$$

$$
\begin{aligned}
(I') = \sum_{i<i_0}\sum_{j\ge i_0}\alpha_i\int_{|A_{j+1}|}^{|A_j|} g_*(s)ds \quad &= \quad \sum_{i<i_0}\sum_{j\ge i_0}\sum_{j\ge i}\int_0^\infty \alpha_i \mathbb{1}_{[|A_{j+1}|,|A_j|]} g_*(s)ds \\
&= \sum_{i<i_0}\int_0^\infty \alpha_i \mathbb{1}_{[0,t]} g_*(s)ds \\
&= \sum_{i<i_0}\alpha_i\int_0^t g_*(s)ds \\
&= \sum_{i<i_0}\alpha_i t g_{**}(t)
\end{aligned}
$$

$$
\begin{aligned}
(I'') = \sum i\le j<i_0\alpha_i\int_{|A_{j+1}|}^{|A_j|} g_*(s)ds \quad &= \quad \sum j<i_0\int_{|A_{j+1}|}^{|A_j|}(\sum_{i\le j}\alpha_i)g_*(s)ds \\
&= \sum_{j<i_0}\int_{|A_{j+1}|}^{|A_j|} f_*(s)g_*(s)ds \\
&= \int_t^\infty f_*(s)g_*(s)
\end{aligned}
$$

$$
\Rightarrow \ (I') + (II) = (\sum_{i<i_0} + \sum_{i\ge i_0})\alpha_i \min(|A_i|,t)g_{**}(t) = \sum_{i=1}^N \alpha_i \min(|A_i|,t)g_{**}(t) = tf_{**}(t)g_{**}(t)
$$

□

Theorem 4.20 (Young inequality for Lorentz Spaces)**.**

$$
\|g\star h\|_{L^{p,q}} \lesssim \|g\|_{L^{p_1,q_1}}\|h\|_{L^{p_2,q_2}}
$$

with

$$
\begin{cases} \frac{1}{p_1}+\frac{1}{p_2} = 1+\frac{1}{p} & 1<p_1,p_2,p<\infty \\ \frac{1}{q_1}+\frac{1}{q_2} = \frac{1}{q} & 1<q_1,q_2,q<\infty \end{cases}
$$

Remark. 1. The previous weak Young inequality corresponds to the case $p_1 = q,\ q_1 = \infty,\ p_2 = q_2 = r$

$\Rightarrow \frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q}+\frac{1}{r}=1+\frac{1}{p}$ and
$\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{\infty}+\frac{1}{r}=\frac{1}{r}$
$\Rightarrow q=r \overset{Y-Lorentz}{\Rightarrow} \|g\|_{L^{q,\infty}}\|h\|_{L^{r,r}} \gtrsim \|g\star h\|_{L^{p,r}} \gtrsim \|g\star h\|_{L^{p,p}}$ since $r\le p \Leftrightarrow \frac{1}{r}>\frac{1}{p}$
since $\underbrace{\frac{1}{q}}_{<1}+\frac{1}{r}=1+\frac{1}{p}$

2. Stronger result: $\|g\|_{L^{p_1}}\|h\|_{L^{p_2}} \gtrsim \|g\star h\|_{L^{p,q}}$ where
$\frac{1}{p_1}+\frac{1}{p_1}=1+\frac{1}{p},\quad \frac{1}{q}\frac{1}{p_1}+\frac{1}{p_1}=1+\frac{1}{p}$
$\Rightarrow\ q<p\ \Rightarrow\ \|g\star h\|_{L^{p,q}} \gtrsim \|g\star h\|_{L^p}$

3. The improved Sobolev inequality follows easily from the Y-Lorentz spaces:

$$\|f\|_{L^{p^*,p}}=\|\nabla G\star\nabla f\|_{L^{p^*,p}} \lesssim \|\nabla G\|_{L^{\frac{d}{d-1}-\infty}}\|\nabla f\|_{L^{p,p}} \lesssim \|\nabla f\|_{L^p}$$

i.e. $p_1=\frac{d}{d-1},\ q_1=\infty,\ p_2=q_2=p$
$\Rightarrow\ \frac{1}{p^*}+1=\frac{1}{p_1}+\frac{1}{p_2},\ \frac{1}{p}=\frac{1}{q_1}+\frac{1}{q_2}$

proof of the Y-Lorentz inequality (by Niel). Recall

$$\|f\|_{L^{p,q}}=(\int_0^\infty \lambda^{\frac{q}{p}-1}f_*(\lambda)^q d\lambda)^{\frac{1}{q}} \sim (\int_0^\infty \lambda^{\frac{q}{p}-1}f_{**}(\lambda)^q d\lambda)^{\frac{1}{q}}$$

with f_* decreasing rearrangement, $f_{**}(t)=\frac{1}{t}\int_0^t f_*(s)ds \ge f_*(t)$

$$\begin{aligned}
\|g\star h\|_{L^{p,q}} &\sim (\int_0^\infty t^{\frac{q}{p}-1}(g\star h)^q_{**}(t)dt)^{\frac{1}{q}}\\
&= \left\|t^{\frac{1}{p}-\frac{1}{q}}(g\star h)(t)\right\|_{L^1(\mathbb{R}_+)}\\
&\overset{lemma 4.19}{\le} \left\|t^{\frac{1}{p}-\frac{1}{q}+1}g_{**}(t)h_{**}(t)+t^{\frac{1}{p}+\frac{1}{q}}\int_t^\infty g_*(s)h_*(s)ds\right\|_{L^q(\mathbb{R}_+,dt)}\\
&\le \left\|t^{\frac{1}{p}-\frac{1}{q}+1}g_{**}(t)h_{**}(t)\right\|+\left\|t^{\frac{1}{p}+\frac{1}{q}}\int_t^\infty g_*(s)h_*(s)ds\right\|_{L^q(\mathbb{R}_+,dt)}=(I)+(II)
\end{aligned}$$

$$\begin{aligned}
(I) &= \left\|t^{\frac{1}{p}-\frac{1}{q}+1}g_{**}(t)h_{**}(t)\right\|_{L^q(\mathbb{R}_+)} \overset{\text{Hölder}}{\le} \left\|t^{\frac{1}{p_1}-\frac{1}{q_1}}g_{**}(t)\right\|_{L^{q_1}(\mathbb{R}_+)}\left\|t^{\frac{1}{p_2}-\frac{1}{q_2}}h_{**}(t)\right\|_{L^{q_2}(\mathbb{R}_+)}\\
&\le \|g\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}
\end{aligned}$$

Here for Hölder:

$$(\int_0^\infty t^{\frac{q}{p}-1+q}g^q_{**}(t)h^q_{**}(t)dt) \le (\int_0^\infty t^{\frac{q_1}{p_1}-1}g^{q_1}_{**}(t)dt)^{\frac{q}{q_1}}(\int_0^\infty t^{\frac{q_2}{p_2}-1}h^{q_2}_{**}(t)dt)^{\frac{q}{q_2}}$$

$$RHS \ge (\int_0^\infty t^{\frac{q_1}{p_1}-1}g^{q_1}_{**}(t)dt)^{\frac{q}{q_1}}t^{\frac{q_2}{p_2}-1}h^{q_2}_{**}(t)dt)^{\frac{q}{q_2}}=\int_0^\infty t^{\frac{q}{p_1}-\frac{q}{q_1}+\frac{q_2}{p_2}-\frac{q}{q_2}}g^q_{**}(t)h^q_{**}(t)dt$$

So we get

$$\frac{q}{p} - 1 + q = \frac{q}{p_1} - \frac{q}{q_1} + \frac{q_2}{p_2} - \frac{q}{q_2} = q(1 + \frac{1}{p}) - 1$$

(II)

$$\begin{aligned}
\left\| t^{\frac{\frac{1}{q}-1}{p}} \int_t^\infty g_*(s) h_*(s) ds \right\|_{L^q(\mathbb{R}_+)} &= \left(\int_0^\infty t^{\frac{q}{p}-1} (\int_t^\infty g_*(s) h_*(s) ds)^q dt \right)^{\frac{1}{q}} \\
&\overset{t=\frac{1}{u}, s=\frac{1}{v}}{=} \left(\int_0^\infty (\frac{1}{u})^{\frac{q}{p}-1} (\int_0^u g_*(\frac{1}{v}) h_*(\frac{1}{v}) \frac{dv}{v^2})^q \frac{du}{u^2} \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty (\frac{1}{u})^{\frac{q}{p}+1} (\underbrace{\int_0^u g_*(\frac{1}{v}) h_*(\frac{1}{v}) \frac{dv}{v^2}}_{f(u):=})^q du \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \frac{|f(u)|^q}{u^{\frac{q}{p}+1}} du \right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty u^{\frac{q}{p'}-1} \left| \frac{f(u)}{u} \right|^q du \right)^{\frac{1}{q}} \\
&\overset{Hardy}{\lesssim} \left(\int_0^\infty u^{\frac{q}{p'}-1} |f'(u)|^q du \right)^{\frac{1}{q}} \\
&= (\int_0^\infty u^{\frac{q}{p'}-1} |g_*(\frac{1}{u}) h_*(\frac{1}{u}) \frac{1}{u^2}|^q du)^{\frac{1}{q}} \\
&= (\int_0^\infty \frac{1}{u^{\frac{q}{p}+1+q}} |g_*(\frac{1}{u}) h_*(\frac{1}{u})|^q du)^{\frac{1}{q}} \\
&\overset{t=\frac{1}{u}}{=} (\int_0^\infty t^{\frac{q}{p}+1+q} |g_*(t) h_*(t)|^q \frac{dt}{t^2})^{\frac{1}{q}} \\
&= (\int_0^\infty t^{\frac{q}{p}-1+q} |g_*(t) h_*(t)|^q dt)^{\frac{1}{q}} \\
&= \left\| t^{\frac{1}{p}+1-\frac{1}{q}} g_*(t) h_*(t) \right\|_{L^q(\mathbb{R}_+, dt)} \\
&\leq \|g\|_{L^{p_1,q_1}} \|h\|_{L^{p_2,q_2}}
\end{aligned}$$

The last inequality works the same as in (I) except that we use $g_* h_*$ instead of $g_{**} h_{**}$

□

4.4 Dyadic decomposition

We will have a look at another proof of the improved Sobolev inequality. For this we will use the dyadic decomposition. The basic idea comes from deviding the positive real number line in a smart, disjoint way, i.e. $(0,\infty) = \bigcup_{j\in\mathbb{Z}} [2^j, 2^{j+1})$. We can use this for the values of functions.

Example 4.21. $f \in L^p$, $\|f\|_{L^p}^p = p\int \lambda^{p-1}|\{|f|>\lambda\}|d\lambda$
We can decompose $f = \sum_i \underbrace{f\mathbb{1}_{\{|f|\in[2^j,2^{j+1})\}}}_{f_i} = \sum_i f_i \Rightarrow$ all $\{f_i\}$ disjoint support and $supp\ f_i \subset \{|f| \in [2^j, 2^{j+1})\}$

$$\Rightarrow \quad \|f\|_{L^p}^p = \sum_i \|f_i\|_{L^p}^p \ \ as\ |\sum_i f_i|^p = \sum_i |f_i|^p$$

We will extend this to a version with smooth cut-off s.t. f smooth gives us a set of $\{f_i\}$ smooth functions.

Lemma 4.22. $\exists \varphi \in C^\infty(\mathbb{R},\mathbb{R})$ s.t. $\varphi(x) = 0$ if $|x| \notin [\frac{1}{2}, 1]$ s.t.

$$\sum_{j\in\mathbb{Z}} \varphi(2^j t) = 1, \quad \forall t \neq 0$$

Proof. $\varphi(t) = \Psi(t) - \Psi(2t)$

$$\Rightarrow \sum_{j\in\mathbb{Z}} \varphi(2^j t) = \sum_j (\Psi(2^j t) - \Psi(2^{j+1}t)) = \Psi(0) = 1$$

if $\Psi \in C^\infty$ *s.t.* $\Psi(x) = 1$ if $|x| \leq \frac{1}{2}$, $\Psi(x) = 0$ if $|x| \geq 1$ ☐

Lemma 4.23. For any f, we decompose

$$f = \sum_{j\in\mathbb{Z}} f_j \ \ where\ f_j(x) = f(x)\varphi(2^{-j}|f|)$$

In particular, we have $supp\ f_j \subset \{2^{j-1} \leq |f| \leq 2^j\}$.
Then:

$$\|f\|_{L^{p,q}} \lesssim (\sum_j \underbrace{\|f_j\|_{L^p}^q}_{\|\|f_j\|_{L^p}\|_{l^q(\mathbb{Z})}})^{\frac{1}{q}} \lesssim \|f\|_{L^{p,q}} \quad \forall q \leq p$$

Remark.

$$(\sum_{j\in\mathbb{Z}} \|f_j + g_j\|_{L^p}^q)^{\frac{1}{q}} \le (\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q)^{\frac{1}{q}} (\sum_{j\in\mathbb{Z}} \|g_j\|_{L^p}^q)^{\frac{1}{q}}$$

but $f \mapsto (\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q)^{\frac{1}{q}}$ is not a norm, since it is non linear.

Proof. Note that

$$f_j \sim 2^j \mathbb{1}_{\{|f|\in[2^{j-1},2^j]\}}$$

Thus:

$$\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q \sim \sum_{j\in\mathbb{Z}} 2^{jq} |\{|f| \in [2^{j-1}, 2^j)]\}|^{\frac{q}{p}} \le \sum_j 2^{jq} |\{|f| > 2^{j-1}\}|^{\frac{q}{p}}$$
$$\le \int_0^\infty \lambda^{q-1} |\{|f| > \lambda\}|^{\frac{q}{p}} d\lambda = \|f\|_{L^{p,q}}^q$$

On the other hand:

$$\begin{aligned}
\|f\|_{L^{p,q}}^q &= \int_0^\infty \lambda^{q-1} |\{|f| > \lambda\}|^{\frac{q}{p}} d\lambda = \sum_j \int_{2^j}^{2^{j+1}} \lambda^{q-1} \underbrace{|\{|f| > \lambda\}|}_{\lambda\sim 2^j,\ \le |\{|f|\ge 2^j\}|} \\
&\le \sum_j \int_{2^j}^{2^{j+1}} \lambda^{q-1} (\sum_{k\ge j} |\{2^k < |f| < 2^{k+1}\}|)^{\frac{q}{p}} d\lambda \\
&\le \sum_j \sum_k \int_{2^j}^{2^{j+1}} \lambda^{q-1} |\{2^k < |f| < 2^{k+1}\}|^{\frac{q}{p}} d\lambda \\
&\sim \sum_j \sum_k 2^j 2^{j(q-1)} |\{2^k < |f| < 2^{k+1}\}|^{\frac{q}{p}} \\
&= \sum_k \underbrace{\sum_{j\le k}}_{2^{q(k+1)}} |\{2^k < |f| < 2^{k+1}\}|^{\frac{q}{p}} \\
&\sim \sum_k \underbrace{2^{qk} |\{2^k < |f| < 2^{k+1}\}|^{\frac{q}{p}}}_{\||f_k|+|f_{k+1}|\|_{L^p}^q \lesssim \|f_k\|_{L^p}^q + \|f_{k+1}\|_{L^p}^q}
\end{aligned}$$

The last step used that

$$|f_k| + |f_{k+1}| \sim 2^k \mathbb{1}_{\{2^k<|f|<2^{k+1}\}}$$

□

Another proof of the improved Sobolev inequality using the standard Sobolev inequality.

$$\|f\|_{L^{p,q}}^p \lesssim \sum_j \|f_j\|_{L^{p^*}}^p \overset{\text{Standard Sobolev}}{\lesssim} \sum_j \|\nabla f_j\|_{L^p}^p$$
$$= \int_{\mathbb{R}^d} \sum_j |\nabla f_j|^p \lesssim \int_{\mathbb{R}^d} |\nabla f|^p = \|\nabla f\|_{L^p}^p$$

$$\lesssim \int_{\mathbb{R}^d} (|\nabla f|^p |f|^p) \leq \|\nabla f\|_{L^p}^p + \|f\|_{L^p}^p = \|f\|_{W^{1,p}}^p$$

We used the pointwise bound:

$$|\underbrace{\nabla f_j(x)}_{=\nabla(f\varphi(2^{-j}f))=\nabla f\cdot\varphi+f\nabla\varphi}| \lesssim |\nabla f(x)| \mathbb{1}_{\{2^{j-1}\leq|f|\leq 2^{j+1}\}}$$

By Scaling argument: If $\|f\|_{L^{p^*,p}} \lesssim \|\nabla f\|_{L^p} + \|f\|_{L^p}$, $\forall f$

$$\Rightarrow \quad \|f\|_{L^{p^*,p}} \lesssim \|\nabla f\|_{L^p}$$

□

Remark. Scaling argument in 3D:

$$(1) \;\; \|f\|_{L^6} \lesssim (\int |\nabla f|^2 + \int |f|^2)^{\frac{1}{2}}, \;\; \forall f$$

$$(2) \;\; \Rightarrow \;\; \|f\|_{L^6} \leq (\int |\nabla f|^2)^{\frac{1}{2}}$$

Use (1) $f_\lambda(x) = \lambda^{\frac{3}{2}} f(\lambda x) \;\; \Rightarrow \;\; \int_{\mathbb{R}^3} |f\lambda|^2 = \int_{\mathbb{R}^3} |f|^2$

$$\int |\nabla f_\lambda|^2 = \lambda^2 \int |\nabla f|^2, \;\; \|f\|_{L^6}^2 = \lambda^2 \|f\|_{L^6}^2$$

Chapter 5

Fourier Transform

We will define the Fourier transform for tempered distributions, which can be viewed as a generalization of L^p functions. The space of tempered distributions is constructed as the dual space of the Schwartz class $C_c^\infty S(\mathbb{R}^d) \subset C^\infty$.
First, let's start with the Fourier transform in L^p.

Definition 5.1 (Fourier Transform in L^1)**.** Let $f \in L^1(\mathbb{R}^d)$, then the Fourier transform of f is defined as

$$\hat{f} = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\xi x}dx$$

Remark. 1. $\left\|\hat{f}\right\|_{L^\infty} \leq \|f\|_{L^1}$

2. $\hat{f}$ is uniformly continuous: follows from

$$\lim_{\xi\to 0}\int_{\mathbb{R}^d} |f(x)|e^{2\pi i\xi x}dx$$

by dominated convergence

3. $\phi(x) = e^{-\pi|x|^2} \;\Rightarrow\; \hat{\phi}(\xi) = e^{-\pi|x^2|}$. To see this, observe

$$\partial_{\xi_n}\hat{\phi}(\xi) = \int_{\mathbb{R}^d} -2\pi i x_n e^{-\pi|x|^2}e^{-2\pi i\xi x}dx = i\int_{\mathbb{R}^d}\partial_{x_n}(e^{-\pi|x|^2})e^{-2\pi i\xi x}dx$$

$$= -i\int_{\mathbb{R}^d} e^{-\pi|x|}(\partial_{x_n}e^{-2\pi\xi x})dx = -2\pi\xi_n\hat{\phi}(\xi)$$

So $\partial_{\xi_n}\hat{\phi}(\xi) = -2\pi\xi_n\hat{\phi}(\xi)$ and the claim follows from solving this ODE.

4. $g(x) = f(\lambda x) \;\Rightarrow\; \hat{g}(x) = \frac{1}{\lambda^d}\hat{f}(\frac{\xi}{\lambda})$

5. $\tau_a f(x) = f(x-a) \;\Rightarrow\; \widehat{\tau_a f}(k) = e^{-2\pi i\xi a}\hat{f}(\xi)$

Lemma 5.2 (Riemann-Lebesgue)**.** If $f \in L^1(\mathbb{R}^d)$, then

$$\lim_{\xi \to \infty} \hat{f}(\xi) = 0$$

Proof.

$$\begin{aligned} \hat{f}(\xi) &= \frac{1}{2}(f(\xi) + e^{2\pi\xi a}\widehat{\tau_a f}(\xi)) \\ &= \frac{1}{2}\int_{\mathbb{R}^d} (f(x) - \tau_a f(x)) e^{-2\pi i \xi x} dx \end{aligned}$$

$$\Rightarrow \ |\hat{f}(\xi)| \leq \frac{1}{2}\int_{\mathbb{R}^d} |f(x) - f(x-a)| dx$$

and now take $a(\xi) = \frac{\xi}{2|\xi|^2}$.
If f is continuous and compactly supported in $B(0, R)$, then for every $\varepsilon > 0$ we can choose $\delta > 0$ s.t.

$$\|f - \tau_a f\|_{L^1} \leq \varepsilon$$

if $|a| < \delta$ and *supp* $\tau_a f \subset B(0, R)$
$\Rightarrow \ |\hat{f}(\xi)| \leq \frac{\varepsilon}{2}$. Now conclude with approximation argument. □

Lemma 5.3. Let $f, g \in L^1(\mathbb{R}^d)$, then

$$\widehat{f \star g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

Proof.

$$\widehat{f \star g}(\xi) = \int_{\mathbb{R}^d} \int t_{\mathbb{R}^d} f(x-y) g(y) e^{-2\pi i \xi x} dy dx$$

$$\overset{\text{Fubini and change of variable}}{=} \left(\int_{\mathbb{R}^d} f(x-y) e^{-2\pi i \xi (x-y)} dx \right) \left(\int_{\mathbb{R}^d} g(y) e^{-2\pi i \xi (y)} dy \right) = \hat{f}(\xi)\hat{g}(\xi)$$

□

Theorem 5.4. If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^1$, then

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

Proof. The problem is that we want to consider

$$\int\int f(y)e^{2\pi i x\xi}d\xi dy$$

BUT $(\xi, y) \mapsto f(y)e^{2\pi i\xi(x-y)} \notin L^1(\mathbb{R}^d \times \mathbb{R}^d)$.
So we define

$$A_\varepsilon(x) = \int_{\mathbb{R}^d} \hat{f}(\xi)e^{2\pi i x\xi}e^{-\varepsilon|x|^2}d\xi$$

and observe that

$$\lim_{\varepsilon\to 0} A_\varepsilon = \mathcal{F}^{-1}(f)(x)$$

again by DCT and obsereve

$$A_\varepsilon(x) = \int\int \underbrace{f(y)e^{2\pi i\xi(x-y)}e^{\varepsilon|x|^2}}_{\in L^1(\mathbb{R}^d\times\mathbb{R}^d)} d\xi dy = \int f(y)\Phi_\varepsilon(x-y)dy$$

where $\Phi_\varepsilon(y) = (\frac{\pi}{\varepsilon})^{\frac{d}{2}}e^{-\frac{\pi^2}{\varepsilon}|y|^2}$.

$$(f\star\Phi_\varepsilon)(x) - f(x) = \int (f(x-y) - f(x))\Phi_\varepsilon(y)dy$$

$$\Rightarrow \quad \|(f\star\Phi_\varepsilon) - f\|_{L^1} \leq \int \left\|f(x-\sqrt{\varepsilon}y) - f(x)\right\|_{L^1}\Phi_1(y)dy \xrightarrow{\varepsilon\to 0} 0$$

by DCT
so we can take a subsequence $\varepsilon_k \to 0$ s.t. $A_{\varepsilon_k} \to 0$ s.t. $A_{\varepsilon_k}(x) \to f(x)$ pointwise □

Lemma 5.5 (Uncertainty Principle)**.** Let $f \in L^1$. If both $f, \hat{f}$ have compact support, then $f = 0$ a.e.

Proof. (sketch) $\hat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}dx$. Consider it as a function if $\xi \in \mathbb{C}^d$. f compact support $\Rightarrow \hat{f}$ analytic and since it has compact support, then $\hat{f} = 0$ a.e. and thus $f = 0$ a.e. □

Lemma 5.6. Let $f \in L^1 \cap L^2$. Then $\hat{f} \in L^1 \cap L^2$ and $\left\|\hat{f}\right\|_{L^2} = \|f\|_{L^2}$

Proof. First one proves that

$$\int f(x)\hat{g}(x)dx = \int\int f(x)g(y)^{-2\pi ixy}dy\ dx = \int \hat{f}(y)g(y)dy$$

and just take $g = \mathcal{F}^{-1}(\overline{f}) = \overline{\mathcal{F}^{-1}(f)}$. Thus $\left\|\hat{f}\right\|_{L^2} = \|f\|_{L^2}$ □

Now we have $\|\mathcal{F}\|_{L^1\to L^\infty} \leq 1$ and $\|\mathcal{F}\|_{L^2\to L^2} = 1$. We can use interpolation!

Theorem 5.7 (Hausdorff-Young)**.** Let $[1,2]$, then $\|\mathcal{F}\|_{L^p\to L^{p'}} \leq 1$, i.e. $\|\mathcal{F}f\|_{L^{p'}} \leq \|f\|_{L^p}$ where $\frac{1}{p} + \frac{1}{p'} = 1$

Proof. This is a direct application of Riesz-Thorin with $T : L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ where we choose $T = \mathcal{F}$, $p_0 = 1,\ q_0 = \infty,\ p_1 = q_1 = 2$ □

5.1 Schwartzt class

Definition 5.8. Let $\varphi \in C^\infty(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{N}_0^d$. Define

$$\rho_{\alpha,\beta}(\varphi) = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta \varphi|$$

The Schwarzt class is then defined as

$$S(\mathbb{R}^d) = \{\varphi \in C^\infty :\ \forall \alpha, \beta \in \mathbb{N}_0^d : \rho_{\alpha,\beta}(\varphi) < \infty\}$$

Remark.

1. $C_c^\infty(\mathbb{R}^d) \subset S(\mathbb{R}^d)$
2. $\varphi \in S \iff \forall \alpha \in \mathbb{N}_0^d, N \in \mathbb{N}\ \exists C_{\alpha,N}$ s.t. $|\partial^\alpha f(x)| \leq \frac{C_{\alpha,N}}{(1+|x|^2)^N}$
3. $S() \subset L^p$ for every $p \geq 1$
4. $S()$ is closed under differentiation and multiplication by polynomials

Proposition 5.9. Let $f \in S()$ then

1. $\left\|\hat{f}\right\|_{L^\infty} \leq \|f\|_{L^1}$
2. $\left\|\hat{f}\right\|_{L^2} = \|f\|_{L^2}$
3. $\widehat{f \star g} = \hat{f}\hat{g}$
4. $\partial^\alpha \hat{f}(\xi) = (\widehat{(-2\pi x)^\alpha f(x)})(\xi)$
5. $\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$
6. $\hat{f} \in S(\mathbb{R}^d)$

Proof. (1)-(3) already done.

(4)-(5) just use DCT: for example $\partial_{\xi_k}\hat{f}(\xi) = \partial_{\xi_k}\int f(x)e^{-2\pi i\xi x}dx = -2\pi i\xi_k\hat{f}(\xi)$

(6) To show that $\hat{f} \in S$, observe that

$$\rho_{\alpha,\beta}(\hat{f}) = \left\|\xi^\alpha\partial^\beta\hat{f}\right\|_{L^\infty} = \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}}\left\|\widehat{x\partial^\beta f}\right\|_{L^\infty} \le (2\pi)^{|\beta|-|\alpha|}\left\|x^\alpha\partial^\beta f\right\|_{L^1} < \infty$$

□

Definition 5.10. Let X be a vector space. A function $\rho : X \to [0,\infty)$ is called a seminorm, if

1. $\rho(\lambda x) = |\lambda|\rho(x)\ \forall\lambda \in \mathbb{C}, x \in X$
2. $\rho(x+y) \le \rho(x) + \rho(y)\ \ \forall x, y \in X$

Definition 5.11. Let X be a vector space with seminorms $\{\rho_n\}_{n\in\mathbb{N}}$ s.t.

$$\rho_n(x) = 0\forall n \in \mathbb{N} \iff x = 0$$

We call X a Fréchet space if (X,d) is a complete metric space for

$$d(x,y) := \sum_{n\in\mathbb{N}}\frac{1}{2^n}\frac{\rho_n(x-y)}{1+\rho_n(x-y)}$$

Proposition 5.12. Let X be a Fréchet space, then

1. $f_n \xrightarrow{d} f \iff \forall m \in \mathbb{N}\ \rho_m(f_n - f) \xrightarrow{n\to\infty} 0$
2. $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in d iff it is Cauchy in each ρ_m
3. ρ_m are continuous wrt d

Proof. (1) $f_n \to f$ in $d \Rightarrow \rho_m(f_n - f)$ for each $m \in \mathbb{N}$.
Now assume $\rho_m(f_n - f) \to 0$ for each m, then we can choose $M \in \mathbb{N}$ s.t.

$$\sum_{n=M+}^{\infty}\frac{1}{2^n} < \frac{\varepsilon}{2}$$

and choose $N \in \mathbb{N}$ s.t. $\rho_m(f_n - f) < \frac{\varepsilon}{2}\ \forall m = 1, ..., M, n \geq N$

$$\Rightarrow \quad d(f_n, f) \leq \sum_{n=0}^{M} \frac{\rho_m(f_n - f)}{1 + \rho_m(f_n - f)} \frac{1}{2^m} + \frac{\varepsilon}{2} < \varepsilon$$

(2) and (3) are similar □

Theorem 5.13. $S(\mathbb{R}^d)$ with the seminorms $\{\rho_{\alpha,\beta}\}_{\alpha,\beta \in \mathbb{N}_0^d}$ is a Fréchet space.

Proof. (exercise) □

Proposition 5.14. $\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ is a linear homeomorphism, i.e. it is continuous and has a continuous inverse

Proof.

$$\rho_{\alpha,\beta}(\hat{\phi}) = \left\| \xi^\alpha \partial^\beta \hat{\phi} \right\|_{Ł^\infty} = \frac{(2\pi)^{|\beta|}}{(2\pi)^{|\alpha|}} \left\| \widehat{\partial^\alpha x^\beta \phi} \right\|_{L^\infty} \leq C_{\alpha,\beta} \left\| \partial^\alpha x^\beta \phi \right\|_{L^1}$$

And using Leibnitz rule:

$$\partial^\alpha (fg) = \sum_{\gamma| \leq \alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \partial^{\alpha-\gamma} f \partial^\gamma g$$

we get

$$\rho_{\alpha,\beta}(\hat{\phi}) \leq C_{\alpha,\beta} \sum_{\gamma \leq \alpha} c_{\alpha,\beta,\gamma} \left\| x^{\beta-\alpha+\gamma} \check{\phi} \right\|_{L^1}$$

and observe that

$$\|\rho\|_{L^1} = \int \frac{(1+|x|^{2d})}{(1+|x|^{2d})} |\phi| dx \leq C_d \left\| (1+|x|^d)\phi \right\|_{L^\infty}$$

$$\rho_{\alpha,\beta}(\hat{\phi}) \leq \check{C}_{alpha,\beta,\gamma,d} \sum_{|\nu| \leq N, |\kappa| \leq N} \rho_{\nu,\kappa}(\phi)$$

where $N = \max(|\alpha|, |\beta|) + 2d$. Thus if $\phi_n \to 0\ in\ S(\mathbb{R}^d)$ we also get $\mathcal{F}(\phi_n) \to 0$ in $S(\mathbb{R}^d)$ □

5.2 Tempered Distributions

Definition 5.15. The space $S'(\mathbb{R}^d)$ of tempered distributions is the set of linear continuous functionals $T: \ S(\mathbb{R}^d) \to \mathbb{C}$.
Notation: $T \in S'(\mathbb{R}^d)$ and $\phi \in S(\mathbb{R}^d) \ \Rightarrow \ T(\phi) =< T, \phi >$

Proposition 5.16. A linear map $T : S(\mathbb{R}^d) \to \mathbb{C}$ belongs to $S'(\mathbb{R}^d)$ iff $\exists c < \infty$ and $N \in \mathbb{N}_0$ s.t.

$$| < T, \phi > | \leq c \sum_{|\alpha| \leq N, \ |\beta| \leq N} \rho_{\alpha,\beta}(\phi) \ \ \forall \phi \in S(\mathbb{R}^d)$$

Proof. " $\Leftarrow$ " is clear, since $\phi_n \to 0$ in $S(\mathbb{R}^d) \ \Rightarrow \ < T, \phi > \to 0$ in $\mathbb{C}$
" $\Rightarrow$ " By way of contradiction. Suppose for every $N \in \mathbb{N}_0$ we can find $\varphi_n \in S$ s.t. $< T, phi > \geq N \sum_{|\alpha| \leq N, \ |\beta| \leq N} \rho_{\alpha,\beta}(\phi_N) =: N\mathcal{N}_N(\phi_N)$ and define $\Psi_N := \frac{\phi_N}{\sqrt{N} \sum_{|\alpha| \leq N, \ |\beta| \leq N} \rho_{\alpha,\beta}(\phi_N)} \Rightarrow$ for every $|\alpha|, |\beta| \leq K : \ \rho_{\alpha,\beta}(\Psi_N) \leq \frac{\rho_{\alpha,\beta}(\phi_N)}{\sqrt{N} \sum_{|\alpha| \leq N, \ |\beta| \leq N} \rho_{\alpha,\beta}(\phi_N)} \leq \frac{1}{\sqrt{N}}$
i.e. $\Psi_N \to 0$ in S. Thus by continuity, $< T, \Psi_N > \to 0$ in $\mathbb{C}$ BUT $| < T, \Psi_N > | \geq \frac{|< T, \phi_N >|}{\sqrt{N} \sum_{|\alpha| \leq N, \ |\beta| \leq N} \rho_{\alpha,\beta}(\phi_N) \geq \sqrt{N}}$ which is a contradiction. ☐

Example 5.17. 1. $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$

$$T_f : \phi \mapsto \int_{\mathbb{R}^d} f\phi$$

is a tempered distribution. Indeed,

$$| < T_f, \phi > | \leq \|f\|_{L^p} \|\phi\|_{L^{p'}} = \|f\|_{L^p} \left\| |\phi|^{\frac{1}{p'}} |\phi|^{1-\frac{1}{p'}} \right\| \leq \|f\|_{L^p} \|\phi\|_{L^1}^{\frac{1}{p'}} \|\phi\|_{L^\infty}^{1-\frac{1}{p'}}$$

$$\leq \|f\|_{L^p} \sum_{|\alpha|, |\beta| \leq K} \rho_{\alpha,\beta}(\phi)$$

in the last step we used $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$

2. Polynomials are in $S'(\mathbb{R}^d)$

3. Dirac-delta in $S'(\mathbb{R}^d)$: Define $< \delta_a, \phi >= \phi(a)$, then clearly $| < \delta_a, \phi > | \leq \rho_{0,0}(\phi)$ and one usually writes $\phi \mapsto \int \phi d\mu_a = \phi(a) = \int \phi(x)\delta(x-a)dx$

4. Given a finite measure μ on $\mathbb{R}^d$, then $\phi \mapsto \int_{\mathbb{R}^d} \phi d\mu \in S'$ since $| \int_{\mathbb{R}^d} \phi d\mu| \leq \|\phi\|_{L^\infty} \mu(\mathbb{R}^d)$

Definition 5.18. $\{T_n\}_{n\in\mathbb{N}} \subset S'(\mathbb{R}^d)$ converges to $T \in S'(\mathbb{R}^d)$ if

$$\forall\phi \in S(\mathbb{R}^d): \quad < T_n, \phi > \to < T, \phi > \ in\ \mathbb{C}$$

One usually writes $T_n \rightharpoonup T$

Remark. Given $g \in C_c^\infty(\mathbb{R}^d)$ with $\|g\|_{L^1} = 1$, define $g_k(x) = 2^{-kd}g(2^k x)$. Then $g_k = T_{g_k} \rightharpoonup \delta_0$ in S'

Observe that for $\phi, \psi \in S$ we have the integration by parts formula:

$$\int_{\mathbb{R}^d} (\partial^\alpha \phi)\psi = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \phi(\partial^\alpha \psi)$$

Definition 5.19. For $T \int S'$ and $\alpha \in \mathbb{N}_0^d$, we define $\partial^\alpha T \in S'$ as

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi) \ \ \forall\phi \in S$$

Remark. To show that $\partial^\alpha T \in S'$ we have to check $\exists C, K$ s.t. $| < \partial^\alpha T, \phi > | \leq C\mathcal{N}_K(\phi) \ \ \forall\phi \in S$ and this is true, since $| < \partial^\alpha T, \phi > | = |T, \partial^\alpha \phi| \leq C\mathcal{N}_{K+|\alpha|}(\phi)$

Example 5.20.

1. $\partial^\alpha \delta_a = (-1)^{|\alpha|}\partial^\alpha\phi(0)$ by definition
2. In $d = 1$ define $g(x) = \begin{cases} x, \ if\ x \geq 0 \\ 0, \ else \end{cases}$ so that $| \int g(x)\phi(x)| \leq \|x\phi\|_{L^1} \leq C\mathcal{N}_K(\phi)$ and $< \partial g, \phi >= -\int_0^\infty x\phi'(x)dx = \int_0^\infty \phi(x)dx =< \Theta, \phi >$ where $\partial g = \Theta = \begin{cases} 1, \ if\ x \geq 0 \\ 0, \ else \end{cases}$ the Heavyside function

$$< \partial\Theta, \phi >= -\int_0^\infty \phi'(x)dx = \phi(0) = \delta_0(\phi) \ \Rightarrow \ \partial\Theta = \delta_0$$

Definition 5.21. The space of slowly increasing functions is given by

$$\mathcal{O}(\mathbb{R}^d) := \{\phi \in C^\infty(\mathbb{R}^d): \ \forall\alpha \in \mathbb{N}_0^d \ \exists C_\alpha, \ m_\alpha \in \mathbb{N} \ s.t. \ |\partial^\alpha\phi| \leq C_\alpha(1 + |x|)^{m_\alpha}\}$$

Remark. 1. $\mathcal{O} \hookrightarrow S'$ in the sense of integration

2. Given $\psi \in \mathcal{O}$: $M_\psi : \phi \in S \mapsto \psi\phi \in S$ is continuous

Definition 5.22. Given $T \in S'$ and $g \in \mathcal{O}$ we can define $< gT, \phi >=< T, g\phi > \ \forall \phi \in S$

Definition 5.23. Let $T \in S'$. Define its Fourier transform $\hat{T} = \mathcal{F}(T)$ as

$$< \hat{T}, \phi >=< T, \hat{\phi} >, \ \forall \phi \in S$$

Remark. $\hat{T} \in S'$ because $\mathcal{F} : S \to S$ is a linear homeomorphism. This also allows us to define the inverse Fourier transform for tempered distributions:

$$< \check{T}, \phi >=< T, \check{\phi} >$$

Example 5.24. 1. $< \hat{\delta}_a, \phi >=< \delta_a, \hat{\phi} >= \hat{\phi}(a) = \int \phi(x)e^{-2\pi iax}dx$ i.e. $\hat{\delta}_a = e^{-2\pi iax}$ and in particular $\hat{\delta}_0 = 1$

2. Similarly $< \hat{1}, \phi >=< 1, \hat{\phi} >= \int \hat{\phi}(\xi)d\xi = \phi(0)$, i.e. $\hat{1} = \delta_0$

3. $< \widehat{\partial^\alpha \delta_a}, \phi >=< \partial^\alpha \delta_a, \hat{\phi} >= (-1)^{|\alpha|} < \delta_a, \partial^\alpha \hat{\phi} >= (-1)^{|\alpha|} < \delta_a, \widehat{(2\pi ix)^\alpha \phi} >= (2\pi ix)^\alpha e^{-2\pi iax}$

Proposition 5.25. Let $T \in S'$ and $\alpha \in \mathbb{N}_0^d$, then we have

$$\partial^\alpha \hat{T} = (\widehat{-2\pi ix)^\alpha T}$$

and

$$\widehat{\partial^\alpha T} = (-2\pi i\xi)^\alpha \hat{T}$$

in the sense of distributions

Proof. (exercise) □

Definition 5.26. Let $T \in S'$ and $\Omega \subset \mathbb{R}^d$ open. We say that T vanishes in Ω if $\forall \phi \in S$ with *supp* $(\phi) \subset \Omega$ we have

$$< T, \phi >= 0$$

The support of T is the complement of the largest open set in which T vanishes.

Proposition 5.27. Suppose $T \in S'$ has a support at the origin. Then $\exists K \in \mathbb{N}$ s.t.

$$T = \sum_{|\alpha| \leq K} c_\alpha \partial^\alpha \delta_0$$

Proof. Since $T \in S'$, we can find $c > 0$ and $K \in \mathbb{N}$ s.t.

$$| < T, \phi > | \leq c \sum_{|\alpha|,|\beta| \leq K} \rho_{\alpha,\beta}(\phi) = c\mathcal{N}_K(\phi)$$

We use Taylor's formula for ϕ at the origin:

$$\phi(x) = \sum_{|\gamma| \leq K} a_\gamma \partial^\gamma \phi(0) x^\gamma + R_K(x)$$

where $R_K(x) = \mathcal{O}(|x|^\gamma)$ as $|x| \to 0$ and thus $\lim_{|x| \to 0} \frac{|\partial^\gamma R_K(x)|}{|x|^{K-|\gamma|}} = 0$ for $|\gamma| \leq K$.
Let $\chi \in \mathbb{C}_c^\infty$ with support $B(0,1)$ and $\chi = 1$ in $B(0, \frac{1}{2})$ s.t.
$< T, \phi >=< T, \chi\phi + (1-\chi)\phi >=< T, \chi\phi >= \sum_{|\gamma \leq K|} a_\gamma \partial^\gamma \phi(0) < T, \chi x^\gamma > + < T, \chi R_K(x) >$.
So it is enough to show that the second term is equal to 0.

Let $\varepsilon < 1$. Define $\chi_\varepsilon := \chi(\frac{x}{\varepsilon})$ s.t.

$< T, \chi(x)R_K(x) >=< T, \chi(x)R_K(x)\chi_\varepsilon + \chi(x)R_K(x)(1-\chi_\varepsilon) >=< T, \chi R_K(x)\chi_\varepsilon(x) >=< T, \chi_\varepsilon R_K(x) >$

and by continuity of T

$$| < T, \chi_\varepsilon R_K > | \leq C\mathcal{N}_K(\chi_\varepsilon R_K) \leq C \sum_{|\alpha|,|\beta| \leq K} \sum_{\gamma \leq \beta} c_{\gamma,\beta} \left\| x^\alpha \partial^{\beta-\gamma} R_K(x) \partial^\gamma \chi_\varepsilon \right\|_{L^\infty}$$

$$\leq \tilde{C} \sum_{|\beta| \leq K} \left\| \partial^{\beta-\gamma} R_K(x) \partial^\gamma \chi_\varepsilon(x) \right\|_{L^\infty}$$

Now observe that $|\partial^\gamma \chi_\varepsilon(x)| \leq \frac{C_\gamma}{|\varepsilon|^\gamma} \chi_{B(0,\varepsilon)}(x)$ and $\left\| \partial^{\beta-\gamma} R_K(x) \chi_{B(0,\varepsilon)}(x) \right\|_{L^\infty} = \mathcal{O}(\varepsilon^{K-|\beta-\gamma|})$ by Taylor.

Thus all in all: $| < T, \chi_\varepsilon R_K > | = \mathcal{O}(\sum_{|\alpha|,|\beta| \leq K} \sum_{\gamma \leq \beta} \varepsilon^{K-|\beta-\gamma|-|\gamma|})$ as $\varepsilon \to 0$ and by our constraints: $K - |\beta - \gamma| - |\gamma| \geq 0$ □

Corollary 5.28. Let $T \in S'$ be s.t. $\Delta T = 0$, i.e. $\sum_{i=1}^{d} \partial_{x_i}^2 T = 0$. Then T is a polynomial.

Proof. $0 =< \Delta T, \hat{\phi} >=< T, \Delta\hat{\phi} >= - < T, \widehat{4\pi^2|x|^2\phi} >$, i.e. $4\pi^2\xi^2\hat{T} = 0 \Rightarrow \hat{T}$ has support at the origin.

$$\Rightarrow \hat{T} = \sum_{|\gamma|\leq K} c_\gamma \partial^\gamma \delta_0$$

and $T = \sum_{|\gamma|\leq K} c_\gamma \underbrace{\mathcal{F}^{-1}(\partial^\gamma\delta_0)}_{\text{is polynomial from examples}}$ □

5.3 Remarks on Distributions vs. Tempered distributions

Definition 5.29. Let $\Omega \subset \mathbb{R}^d$ open. Define the space of distributions $D'(\Omega)$ as the dual space of $D(\Omega)$.
Here $D(\Omega) = C_c^\infty(\Omega)$, with the topology

$$\varphi_n \to \varphi \text{ } in \text{ } D(\Omega) \Leftrightarrow \begin{cases} \|D^\alpha\varphi_n - D^\alpha\varphi\|_{L^\infty} \to 0 \ \forall\alpha \\ supp\ (\phi_n) \subset K \ compact \subset \Omega \end{cases}$$

Heuristic idea:

$$D(\Omega) \subset L^2 \Rightarrow (D(\Omega))^* = D'(\Omega) \supset (L^2)^* = L^2$$

since "smaller space ⇒ larger dual"

Example 5.30. Any function $f \in L^1_{loc}(\Omega)$ is a distribution $T_f(\varphi) = \int_\Omega f\phi \ \ \forall\phi \in D(\Omega)$

Lemma 5.31 (Fundamental Lemma of Calculus of Variations)**.** If $f \in L^1_{loc}(\Omega)$ and $\int_\Omega f\varphi = 0, \ \ \forall\phi \in C_c^\infty(\Omega)$, then $f = 0$ a.e. $x\Omega$

Proof. Step 1 Case $\Omega = \mathbb{R}^d$. Let $f \in L^1_{loc}$ and $\int_{\mathbb{R}^d} f\varphi = 0, \ \forall\varphi \in C_c^\infty(\Omega)$
Take $\varphi_0 \in C_c^\infty$ and define $\varphi_{n,y}(x) = n^d\varphi(\frac{y-x}{n}) \in C_c^\infty(\mathbb{R}^d, dx)$. Also choose $1 = \int_\Omega \varphi_0 = \int_\Omega \varphi_n$.

Then: $0 = \int_\Omega f(x)\varphi_{n,y}(x)dx = \int_\Omega n^d f(x)\varphi(\frac{y-x}{n})dx \xrightarrow{n\to\infty} f(y)$, i.e. $f = 0$ a.e.
Step 2: Consider the general case $\Omega\mathbb{R}^d$, $f \in L^1_{loc}(\Omega)$. Take $\chi \in C_c^\infty(\Omega)$. Then $\int_\Omega f \underbrace{\chi\varphi}_{\in C_c^\infty}$, $\forall\varphi \in C_c^\infty(\Omega) = \int_\Omega g\varphi$ where $g = f\chi \in L^1_c(\Omega) \subset L^1_c(\mathbb{R}^d)$
By step 1, $g = f\chi = 0$ a.e. $\Rightarrow$ $f = 0$ *a.e.* □

Comparing to the tempered distributions:

Note that D' is really bigger than the space of tempered distributions!
E.g. Take $f(x) = e^x$ in $\mathbb{R}$ is a distribution but not a tempered distribution!
For proving that, consider the Schwartz function $\varphi(x) = e^{-\sqrt{1+x^2}}$ and show that $\int_\mathbb{R} \varphi e^x = \infty$ (exercise)

Derivatives: For all $T \in D$ we can define $\forall\alpha$

$$(D^\alpha T) \in D' \ by \ (D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha\varphi), \ \forall\varphi \in D$$

Q: Why is the space of distributions too big for the Fourier transform?
If we want to define the Fourier transform of a distribution $T \in D'$, then we want to do something like

$$(\mathcal{F}T)(\varphi) =< \mathcal{F}T, \varphi >=< T, \mathcal{F}^*\varphi >$$

BUT this does not work in general in D', since $\mathcal{F} : C_c^\infty(\mathbb{R}^d) \not\to C_c^\infty(\mathbb{R}^d)$
This is related to the "uncertainty principle", i.e. if f is more localized $\Rightarrow$ $\hat{f}$ is less localized.

Theorem 5.32 (Hardy Uncertainty Principle)**.** If $f : \mathbb{R}^d \to \mathbb{C}$ satisfies

$$|f(x)| \le Ce^{-\pi a|x|^2}$$

and

$$|\hat{f}(x)| \le Ce^{-\pi b|\xi|^2}$$

for parameters $a, b > 0$, then

$$ab \le 1$$

Moreover, if $ab = 1$, then $f(x) = const e^{-\pi a|x|^2}$ and $\hat{f}(x) = const e^{-\frac{\pi}{a}|\xi|^2}$

It is easy to prove, that if $f \in C_c^\infty$, then $\hat{f}$ is not compactly supported.
Let $d = 1$. Consider $\hat{f}(z) = \in_\mathbb{R} f(x)e^{-2\pi izx}dx$, $z \in \mathbb{C}$.
Since $f \in C_c^\infty(\mathbb{R})$ $\Rightarrow$ $\hat{f}(z)$ is entire. (This is sufficient, if $f \in L^1_c$)
Here, if $\hat{f}(\xi)$, $\xi \in \mathbb{R}$ is compactly supported, then $\hat{f}(\xi) = 0$ is an open integral in $\mathbb{R}$.
$\Rightarrow$ $\hat{f} = 0$ in $\mathbb{C}$ $\Rightarrow \hat{f} = 0$ in $\mathbb{R}$ $\Rightarrow$ $f = 0$ in $\mathbb{R}$

proof of Hardy Uncertainty Principle. We only consider $d = 1$. The argument can be extended to higher dimensions. We prove that if $\begin{cases} |f(x)| \leq Ce^{-\pi x^2}, \\ |\hat{f}(x)| \leq Ce^{-\pi\xi^2} \end{cases}$ for all $x, \xi \in \mathbb{R}$, then $f = ce^{-\pi x^2}$.
Define $\hat{f}(z) = \int_{\mathbb{R}} f(x)e^{-2\pi izx}dx,\ z \in \mathbb{C},\ z = \xi + i\nu,\ \xi, \nu \in \mathbb{R}$
$\Rightarrow\ \partial_z\hat{f}(z) = \int_{\mathbb{R}} f(x)(-2\pi ix)e^{-2\pi ixz}dx\ \Rightarrow\ \hat{f}(z)$ is entire.
We have the following easy upper bound:

$$\begin{aligned} |\hat{f}(z)| &= |\int_{\mathbb{R}} f(x)e^{-2\pi i\xi x + 2\pi\nu x}dx| \\ &\leq \int_{\mathbb{R}} |f(x)|e^{2\pi\nu x}dx \\ &\leq \int_{\mathbb{R}} e^{-\pi x^2}e^{2\pi\nu x}dx \\ &= \int_{\mathbb{R}} e^{-\pi|x-\nu|^2}e^{\pi\nu^2}dx \\ &= (\int_{\mathbb{R}} e^{-\pi y^2}dy)e^{\pi\nu^2} = e^{pi\nu^2} \end{aligned}$$

1.try: $F(z) = e^{\pi z^2}\hat{f}(z)$ entire. $|F(\xi)| = |e^{\pi\xi^2}\hat{f}(\xi)| \leq C$ for all $\xi \in \mathbb{R}$.
We want to somehow show $|F(z)| \leq C$ for all $z \in \mathbb{C}$.
$\Rightarrow\ F(z) = const$ in $\mathbb{C}\ \Rightarrow\ \hat{f}(z) = const e^{-\pi z^2}\ \forall z \in \mathbb{C}$
The problem here is that $|F(z)| = |F(\xi + i\nu)| = e^{\pi^2(\xi^2-\nu^2)}\hat{f}(\xi + i\nu) \leq e^{\pi(\xi^2-\nu^2)}e^{\pi\nu^2} \leq e^{\pi\xi^2}$
This is not enough to get the bound for F.

2.try: Take $\varepsilon > 0$ small, $\theta > 0$ small , $\delta > 0$ small

$$F_\varepsilon(z) = e^{(i\varepsilon e^{i\varepsilon}z^{2+\varepsilon})}e^{i\delta z^2}e^{\pi z^2}\hat{f}(z)$$

and $\Gamma_\theta = \{z = re^{i\alpha},\ 0 \leq \alpha \leq \theta\}$
Claim: $|F_\varepsilon(z)| \leq C$ as the bound of Γ_θ and $|F_\varepsilon(z)| \to 0$ as $z \in \Gamma_\theta$ and $|z| \to \infty$

$$|F_\varepsilon(z)| = |\underbrace{e^{(i\varepsilon(\cos(\varepsilon)+i\sin(\varepsilon))r^{2+\varepsilon})}}_{Ce^{-\varepsilon\underbrace{\sin(\varepsilon)}_{\geq 0} r^{2+\varepsilon}}}\underbrace{e^{i\delta z^2}e^{\pi z^2}\hat{f}(z)|}_{e^{cr^2}} \to 0\ as\ r \to \infty$$

Thus, by the maximum principle: $|F_\varepsilon(z)| \leq C$ for all $z \in \Gamma_\theta$ with C independent of ε (need to verify!)
Take $\varepsilon \to 0$ and $\delta \to 0\ \Rightarrow\ |e^{\pi z^2}\hat{f}(z)| \leq C$ for all $z \in \Gamma_\theta,\ C$ independent of θ if θ is small and fixed.
By rotations we can increase $\theta\ \Rightarrow\ |e^{\pi z^2\hat{f}(z)}| \leq C$ □

Remark. The Sobolev inequality is also a form of the uncertainty principle!

Operators defined by integral kernel:

$$(Kf)(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy$$

If we take $K = f(p)g(x)$, then this defines a kernel.

$$K\varphi(x) = f(k)(g\varphi)(x) = \mathcal{F}^{-1}(f(k)\widehat{g\varphi}(k))$$

Consequence: If $f, g \in L^2(\mathbb{R}^d)$, then $\int_{\mathbb{R}^d\times\mathbb{R}^d} |K(x,y)|^2 dxdy = \int_{\mathbb{R}^d} |\check{f}(x-y)|^2|g(y)|^2 dxdy = \|f\|_{L^2}^2 \|g\|_{L^2}^2 < \infty$

$\Rightarrow$ K is a Hilbert Schmidt operator $\Rightarrow$ K is compact, i.e. if $\varphi_n \rightharpoonup \varphi$ weakly in L^2, then $K\varphi_n \to K\varphi$ strongly in L^2.

Theorem 5.33. If $f, g : \mathbb{R}^d \to \mathbb{C}$ and $f, g \to 0$ at ∞, then the operator $K = f(p)g(x)$ is a compact operator in $L^2(\mathbb{R}^d)$

Remark. This implies the Sobolev embedding, i.e.

$$H^1(\mathbb{R}^d) \subset L^{2^*}(\mathbb{R}^d)$$

where $2^* = \frac{2d}{d-2}$

$$\|u\|_{L^{2^*}} \lesssim \|u\|_{H_1}$$

Sobolev compact embedding:

$$\mathbb{1}_\Omega H^1(\mathbb{R}^d) \underset{compact}{\subset} \mathbb{1}_\Omega L^p(\mathbb{R}^d) \;\; \forall \Omega \; bd. \; in \; \mathbb{R}^d \; \forall p < 2^*$$

i.e. $u_n \rightharpoonup u$ weakly in H^1, then $\mathbb{1}_\Omega u_n \to \mathbb{1}_\Omega u$ strongly in L^p for $p < 2^*$

Why? $\mathbb{1}_\Omega u_n = \underbrace{(\mathbb{1}_\Omega(x)\frac{1}{\sqrt{p^2+1}})}_{compact\ operator}\underbrace{(\sqrt{p^2+1}u_n)}_{bd.\ \rightharpoonup\sqrt{p^2+1}u} \to \mathbb{1}_\Omega u$ in L^2

Example 5.34. For $\mathbb{R}^d$, let $r \in (0, d]$, then $\frac{1}{,|x|^r} \in S'(\mathbb{R}^d)$

$$|<\frac{1}{|x|^r}, \varphi>| \leq \int_{\mathbb{R}^d} \frac{|\varphi(x)|}{|x|^r} dx \leq \int_{\mathbb{B}(0,1)} \frac{|\varphi(x)|}{|x|^r} dx + \int_{\mathbb{R}^d} |\varphi(x)| dx$$

Now fix $r = 1, d = 1$ Let $\varphi \in C_c^\infty(\mathbb{R}^d)$. We define the cauchy principal value in the sense of distributions as

$$(p.v.\left(\frac{1}{x}\right))(\varphi) := \lim_{\varepsilon\to 0} \int_{x>\varepsilon} \frac{\varphi(x)-\varphi(-x)}{x} = \int_{\mathbb{R}_+} \frac{\varphi(x)-\varphi(-x)}{x}$$

This is well defined, since

$$|\frac{1}{x}(\varphi(x)-\varphi(-x))| \leq \frac{1}{x}\int_{-x}^{x} |\varphi'(y)| dy \leq 2\|\varphi\|_{L^\infty}$$

$$\Rightarrow \ | < p.v.\frac{1}{x}, \varphi > | \le 2\int_0^1 \|\varphi'\|_{L^\infty} + \int_{\mathbb{R}} |\varphi| \ \Rightarrow \ p.v. \in S'(\mathbb{R})$$

We have

$$xp.v.\frac{1}{x} = 1$$

$$\widehat{xp.v.\frac{1}{x}} = \hat{1} = \delta_0$$

$$\partial \widehat{p.v.\frac{1}{x}} = -2\pi i \widehat{xp.v.}\frac{1}{x} = -2\pi i \delta_0$$

Further, we know that $\partial H = \delta_0$, where $H = \begin{cases} 1, \ x > 0 \\ 0, else \end{cases}$ is the Heaviside function. Thus :

$$\partial(\widehat{p.v.\frac{1}{x}} + 2\pi i H) = 0$$

Lemma 5.35. If $T \in S'(\mathbb{R})$, then

$$\partial T = 0 \ \Rightarrow \ T = const.$$

Proof. Let $\varphi \in S$ s.t. $\int_{\mathbb{R}} \varphi = 0$. Then $x \mapsto \int_{-\infty}^x \varphi(y)dy \in S(\mathbb{R})$

$$\Rightarrow \ 0 =< T, \varphi >=< T, \partial \int_{-\infty}^x \varphi(y)dy >= - < \partial, \int_{-\infty}^x \varphi(y)dy >$$

Let $\psi \in S$

$$0 =< T, \psi - \frac{e^{-x^2} \int \psi(y)dy}{\int e^{-y^2}dy} >=< T, \psi > - \frac{\int \psi(y)dy}{\int e^{-y^2}dy} < T, e^{-x^2} >$$

$$\Rightarrow \ < T, \psi >= \frac{\int < T, e^{-x^2} > \psi(y)dy}{\int e^{-y^2}dy}$$

□

$$\widehat{p.v.\frac{1}{x}} = -\partial\pi i H + C$$

Let φ be even $< \widehat{p.v.\frac{1}{x}}, \check{\varphi} >=< p.v.\frac{1}{x}, \varphi >= 0$
If ψ is even, then $\check{\psi}$ and $\hat{\psi}$ are even,

$$< \widehat{p.v.\frac{1}{x}}, \psi >=< p.v.\frac{1}{x}, \check{\psi} >= 0 = \int_{\mathbb{R}} (-2\pi i H + C)\varphi$$

and $-2\pi iH + C$ is odd, where $C = \pi i$

$$\widehat{p.v.\frac{1}{x}} = -i\pi sign(\xi)$$

Remark. This is important for defining the Hilbert transform.

$$H\varphi = \varphi \star p.v.\frac{1}{x}$$

5.4 Convolutions

We want to define convolution for distributions. Let $\nu, \varphi, \psi \in S$

$$\int (\psi \star \nu)(x)\varphi(x)dx = \int\int \psi(x-y)\nu(y)dy\varphi(x)dx = \int\int \psi^{\#}(y-x)\nu(y)dy\varphi(x)dx$$

$$= \int \nu(y)\psi^{\#} \star \varphi(y)dy$$

where $\psi^{\#}(x) = \psi(-x)$. If $\alpha \in \mathbb{N}_0^d$, $\partial^\alpha(\varphi\star\psi) = \partial^\alpha \int \varphi(x-y)\psi(y)dy = (\partial^\alpha\varphi)\star\psi = \varphi\star(\partial^\alpha\psi)$

Remark. If $\varphi \in S$, then $\psi \mapsto \varphi \star \psi$ is continuous and linear in $S \to S$

Definition 5.36. Let $T \in S'(\mathbb{R}^d)$ and $\varphi \in S(\mathbb{R}^d)$, then

$$< \varphi \star T, \psi > := < T, \varphi^{\#} \star \psi >$$

for all $\psi \in S$, where $\varphi^{\#} = \varphi(-x)$

Proposition 5.37. Let $T \in S'$, $\varphi \in S$

1. $\partial^\alpha(\varphi \star T) = \partial^\alpha(\varphi \star T) = \partial^\alpha\varphi \star T = \varphi \star \partial^\alpha T$
2. $\widehat{\varphi \star T} = \hat{\varphi}\hat{T}$

Proof. 1.

$$\begin{aligned} < \partial^\alpha(\varphi \star T), \psi > &= (-1)^{|\alpha|} < \varphi \star T, \partial^\alpha\psi > \\ &= (-1)^{|\alpha|} < T, \varphi^{\#} \star \partial^\alpha\psi > \\ &= (-1)^{|\alpha|} < T, \partial^\alpha\varphi^{\#} \star \psi > \\ &= (-1)^{|\alpha|} < \partial^\alpha\varphi \star T, \psi > \end{aligned}$$

The other equality works similarly.

2.

$$< \widehat{\varphi \star T}, \psi > = < \varphi \star T, \hat{\psi} > = < T, \varphi^{\#} \star \hat{\psi} >$$

$$\begin{aligned}
(\varphi^{\#} \star \hat{\psi})(x) &= \int \varphi(y-x)\hat{\psi}(y)dy \\
&= \int \varphi(y-x) \int \psi(z) e^{-2\pi i y z} dz dy \\
&= \int \int \varphi(y)\psi(z) e^{-2\pi i y z} e^{-2\pi i x z} dz dy \\
&= \int \hat{\varphi}(z)\psi(z) e^{-2\pi i x z} dz
\end{aligned}$$

$$< T, \mathcal{F}(\hat{\varphi}\psi) > = < \hat{T}, \hat{\varphi}\psi >$$

□

Theorem 5.38. Let $T \in S'(\mathbb{R}^d)$, $\varphi \in S(\mathbb{R}^d)$, then

$$\varphi \star T \in \mathcal{O}(\mathbb{R}^d)$$

where

$$\mathcal{O}(\mathbb{R}^d) = \{g \in C^{\infty} : \ \forall \beta \in \mathbb{N}_0^d \ C_\beta (1 + |x|)^{m_\beta}\}$$

Remark.

$$\tau_y : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$$

is clearly a homeomorphism of S, so we can use duality to find

$$< \tau_y T, \phi > = < T, \tau_y \phi >$$

for $T \in S'$

Proof.

$$\begin{aligned}
< \varphi \star T, \phi > &= < T, \varphi^{\#} \star \psi(x) > \\
&= < T, \int \varphi^{\#}(x-y)\psi(y)dy > \\
&= < T, \int \tau_y \varphi^{\#}(x)\psi(y)dy > \\
&= \int < T, \tau_y \varphi^{\#} > \psi(y)dy \\
&= \int < T, \tau_y \varphi^{\#} > \psi(y)dy
\end{aligned}$$

So we identify $(\varphi \star T)(x) =< T, \tau_x\varphi^\# >$

$\varphi \star T \in C^\infty$: Let $e_k = (0, ..., 0, \underbrace{1}_{k-th}, 0, ..., 0)$

$$\frac{\tau_{-he_k}(\varphi \star T)(x) - (\varphi \star T)(x)}{h} =< T, \frac{\tau_{-he_k}(\tau_x\varphi^\#) - (\tau_x\varphi^\#)}{h} >$$

As $h \to 0$ we get $< T, \tau_x\partial^{e_k}\varphi^\# >$. Repeat for higher order. $\Rightarrow$ $\varphi \star T \in C^\infty$

$$\begin{aligned} |\partial^\gamma(\varphi \star T)(x)| &\leq C \sum_{|\alpha|,|\beta|\leq K} \left\|y^\alpha\tau_x\partial^{\beta+\gamma}\varphi^\#\right\|_{L^\infty} \\ &= C \sum_{|\alpha|,|\beta|\leq K} \left\||x+y|^\alpha\partial^{\beta+\gamma}\varphi^\#\right\|_{L^\infty} \\ &\leq \tilde{C} \sum_{|\beta|\leq K} \left\|(1+|x|^K+|y|^K)\partial^{\beta+\gamma}\varphi^\#\right\|_{L^\infty} \end{aligned}$$

So $\varphi \star T \in \mathcal{O}(\mathbb{R}^d)$ □

Corollary 5.39. If $T \in \mathcal{E}'(\mathbb{R}^d) = \{U \in S'(\mathbb{R}^d) : \textit{ supp } U \textit{ is compact}\}$, then for any $\varphi \in S$: $\varphi \star T \in S(\mathbb{R}^d)$

Proof. Let $\varphi \in S$ be 1 in *supp* T, then $\varphi T = T$ (exercise).
Thus $\hat{T} = \widehat{\varphi T} = \hat{\varphi} \star \hat{T}\mathcal{O}(\mathbb{R}^d)$ by thm 5.38. Since for all $\psi \in S$ and $g \in \mathcal{O}(\mathbb{R}^d)$ we have $g\psi \in S$ and $\widehat{\psi \star T} = \hat{\psi}\hat{T} \in S(\mathbb{R}^d)$ □

Definition 5.40. Let $T \in S'(\mathbb{R}^d)$, $U \in \mathcal{E}'$.

$$< U \star T, \varphi >:=< T, \varphi U^\# >$$

where $< U^\#, \varphi >:=< U, \varphi^\# >$

Exercise. Let $T \in S'$, $U \in \mathcal{E}'$, then

$$\widehat{U \star T} = \hat{U}\hat{T}$$

Chapter 6

Applications of Fourier transform

6.1 Distribution Theory and PDEs

Let $p(\partial) = \sum_{|\alpha \leq K|} c_\alpha \partial^\alpha$, then we call $T \in S'$ a fundamental solution, if $p(\partial)T = \delta_0$.
If we want to solve for $f \in S$

$$p(\partial)u = f \qquad (*)$$

then

$$u = f \star T$$

solves $(*)$:

$$p(\partial)(f \star T) = f \star (p(\partial)T) = f \star \delta_0 = f$$

In particular, $m = \frac{1}{4\pi^2|x|^2} \in S'$.
Let $T = \mathcal{F}^{-1}m$

$$\begin{aligned} < \widehat{-\Delta T}, \varphi > &= < 4\pi|\xi|^2\hat{T}, \varphi > \\ &= < \hat{T}, 4\pi^2|\xi|^2\varphi(\xi) > \\ &= \int \frac{4\pi^2|\xi|^2\varphi(\xi)}{4\pi^2|\xi|^2} d\xi \\ &= \int \varphi =< 1, \varphi > \end{aligned}$$

Thus, $-\widehat{\Delta T} = 1$, so $-\Delta T = \delta_0$

6.1.1 Poisson equation

$$-\Delta u = f \quad in \ \mathbb{R}^d$$

$$\Rightarrow \ u(x) = (G \star f)(x)$$

with $G = \begin{cases} \frac{c}{|x|^{d-2}}, \ if \ d \geq 3 \\ c \log|x|, \ if \ d = 2 \\ c|x|, \ if \ d = 1 \end{cases}$

Remark. It is not trivial to see how to transfer the regularity of f to the regularity of u, i.e. "Which properties should f have, s.t. we get that u is a classical solution?"

Theorem 6.1 (Regularity of Poisson equation)**.** Assume f is compactly supported in $\mathbb{R}^d$.

1. If $f \in L^p(\mathbb{R}^d)$ with $p > \frac{d}{2}$, then $u \in C^{0,\alpha}(\mathbb{R}^d)$, i.e. u is Hölder continuous:
$$|u(x) - u(y)| \leq C|x-y|^\alpha, \ \forall x, y \in \mathbb{R}^d$$
2. If $f \in L^p$, $p > d$, then $u \in C^{1,\alpha}$ with $0 < \alpha < 1 - \frac{d}{p}$.
3. If $f \in C^{k,\alpha}$ for $k \geq 0$ and $0 < \alpha < 1$, then $u \in C^{k+2,\alpha}$

Remark. $f \in C^{k,\alpha}$, iff $\left\|D^\beta f\right\|_\infty < \infty$ for all $|\beta| \leq k$ and $\left\|D^\beta f\right\|_{C^{0,\alpha}} < \infty$ for all $|\beta| = k$

Proof. 1) $f \in L^p$, $p > \frac{d}{2}$, $u = G \star f$, $d \geq 3$, $G \sim \frac{1}{|x|^{d-2}}$

$$|u(x) - u(y)| = |\int_{\mathbb{R}^d} (G(x-y) - G(z-y))f(y)dy| \lesssim \int_{\mathbb{R}^d} (\frac{1}{|x-y|^{d-2}} - \frac{1}{|z-y|^{d-2}})|f(y)|dy$$

By the triangle inequality:

$$|\frac{1}{|x-y|^{d-2}} - \frac{1}{|z-y|^{d-2}}| \leq C\frac{||x-y|^{d-2} - |z-y|^{d-2}|}{|x-y|^{d-2}|z-y|^{d-2}}$$

$$\leq C\frac{||x-y| - |z-y||}{|x-y|^{d-2}|z-y|^{d-2}}(|x-y|^{d-3} - |z-y|^{d-3}) \leq C\frac{|x-z|}{|x-y|^{d-2}|z-y|^{d-2}}(|x-y|^{d-3} - |z-y|^{d-3})$$

First try:

$$|u(x-u(z))| \leq C|x-z| \int_{\mathbb{R}^d} (\frac{1}{|x-y||z-y|^{d-2}} + \frac{1}{|x-y|^{d-2}|z-y|})|f(y)|dy$$

$$\leq C|x-z| \int_{\mathbb{R}^d} (\frac{1}{|x-y|^{d-1}} + \frac{1}{|z-y|^{d-1}})|f(y)|dy$$

By Hölder:

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}|f(y)|dy} \leq (\int_{\mathbb{R}^d} |f(y)|^p dy)^{\frac{1}{p}} (\int_{supp \ f} \frac{1}{|x-y|^{(d-1)p'}} dy)^{\frac{1}{p'}} < \infty$$

if $(d-1)p' < d$, i.e. $1 - \frac{1}{d} < \frac{1}{p'} = 1 - \frac{1}{p} \Leftarrow p > d$ with $\frac{1}{p} + \frac{1}{p'} = 1$. But this is not enough!

Second try:

$$||x-y| - |z-y|| \leq \min(|x-z|, |x-y| + |z-y|) \leq |x-z|^\alpha (|x-y|^{1-\alpha} + |z-y|^{1-\alpha})$$

Then:

$$|u(x)-u(y)| \leq C|x-z|^{\alpha} \int_{\mathbb{R}^d} \underbrace{(\frac{|x-y|^{d-2-\alpha}+|z-y|^{d-2-\alpha}}{|x-y|^{d-2}|z-y|^{d-2}})}_{\leq C(\frac{1}{|x-y|^{d-2+\alpha}}+\frac{1}{|z-y|^{d-2+\alpha}})}|f(y)|dy$$

By Hölder:

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2+\alpha}}|f(y)|dy \leq (\int |f|^p)^{\frac{1}{p}} (\int_{supp\ f} \frac{1}{|x-y|^{(d-2+\alpha)p'}}dy)^{\frac{1}{p'}}$$

where $\frac{1}{p}+\frac{1}{p'}=1$
The last integral is finite if $(d-2+\alpha)p' < d$

$$\Leftarrow\ 1-\frac{2-\alpha}{d}=\frac{d-2+\alpha}{d}<\frac{1}{p'}=1-\frac{1}{p} \Leftarrow \frac{2-\alpha}{d}>\frac{1}{p}$$

Remark: By the same analysis, we can obtain the Sobolev embedding $\underbrace{H^s(\mathbb{R}^d)}_{(1-\Delta)^{\frac{s}{2}}f\in L^2} \subset C^{0,\alpha}(\mathbb{R}^d)$ if $s>\frac{d}{2}$

2)If $f\in L^p$ with $p>d$ using $u=G\star f$

$$\Rightarrow\ \partial_i u(x)=(\partial_i G)\star f(x)=\int_{\mathbb{R}^d} \underbrace{\partial_i G(x-y)}_{\frac{(x-y)_i}{|x-y|^d}} f(y)dy$$

$$\Rightarrow\ |\partial_i u(x)-\partial_i u(z)| \leq \int_{\mathbb{R}^d} |\partial_i G(x-y)-\partial_i G(z-y)||f(y)|dy$$

By the triangle inequality:

$$|\partial_i G(x-y)-\partial_i G(z-y)| \leq |\frac{(x-y)_i}{|x-y|^d}-\frac{(z-y)_i}{|z-y|^d}|$$

$$\leq |\frac{x-y}{|x-y|^d}-\frac{z-y}{|z-y|^d}| \leq c|x-z|^{\alpha}(\frac{1}{|x-y|^{d-1+\alpha}}-\frac{1}{|z-y|^{d-1+\alpha}})$$

$$\Rightarrow\ |\partial_i u(x)-\partial_i u(z)| \leq c|x-z|^{\alpha}\int_{\mathbb{R}^d}(\frac{1}{|x-y|^{d-1+\alpha}}-\frac{1}{|z-y|^{d-1+\alpha}})|f(y)|dy$$

Hölder:

$$\int_{\mathbb{R}^d}\frac{1}{|x-y|^{d-1+\alpha}}|f(y)|dy \leq \|f\|_{L^p}(\int_{supp\ f}\frac{1}{|x-y|^{(d-1+\alpha)p'}})^{\frac{1}{p'}}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. We need $(d-1+\alpha)p'<d \Leftrightarrow\ \alpha<1-\frac{d}{p}$

3) Let $f \in C^{0,\alpha}$, then $u = G \star f \in C^{1,\alpha}$ (this only needs $f \in L^p$, $p > d$)
We will compute $\partial^2_{i,j} u$. Formally $\partial^2_{i,j} G \star f$ is ill defined:

$$\partial_i G(x) = \partial(\frac{1}{|x|^{d-2}}) = -(d-2)\frac{1}{|x|^{d-1}} \underbrace{\partial|x|}_{\frac{x_i}{|x|}} = \frac{-(d-2)x_i}{|x|^d}$$

$$\partial^2_{i,j} G(x) \sim \partial_j(\frac{x_i}{|x|^d}) = \delta_{i,j}\frac{1}{|x|^d} + x_i\partial_j(\frac{1}{|x|^d}) = \delta_{i,j}\frac{1}{|x|^d} - \frac{x_i x_j}{|x|^d}) = \frac{1}{|x|^d}(\delta_{i,j} - d\frac{x_i}{|x|}\frac{x_j}{|x|})$$

$$\Rightarrow \ |\partial^2_{i,j} G(x)| \le \frac{c}{|x|^d} \notin L^1_{loc}(\mathbb{R}^d)$$

Take $\varphi \in C^\infty_c(\mathbb{R}^d)$. Then:

$$(\partial_{i,j} u)(\varphi) = -\int_{\mathbb{R}^d} (\partial_i u)(x)\partial_j\varphi(x)dx \sim \int_{\mathbb{R}^d\times\mathbb{R}^d} (\partial_i G)(x-y)f(y)\partial_j\varphi(x)dxdy$$

$$= \int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d} \partial_j\varphi(x)\partial_i G(x-y)dx = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} f(y) \underbrace{\int_{|x-y|\ge\varepsilon} \partial_j\varphi(x)\partial_i G(x-y)dx}_{=\underbrace{\int_{|x-y|=\varepsilon}(\varphi\partial_i G)\nu_j dx}_{(I)} - \underbrace{\int_{|x-y|\ge\varepsilon}(\varphi\partial^2_{i_j} G)dx}_{(II)}}$$

$$(I) = c\int_{|x-y|=\varepsilon} \varphi\frac{(x-y)_i}{|x-y|^{d-i}}\frac{(x-y)_j}{|x-y|} = c\int_{|x-y|=\varepsilon} \varphi(x)(x-y)_i(x-y)_i(x-y)_j\varepsilon^{-(d+1)}dx$$

$$\to c\delta_{i,j}\varphi(y) \ as \ \varepsilon \to 0$$

$$(II) = \int_{|x-y|\ge 1} \varphi\partial^2_{i_j} G dx + \int_{1>|x-y|\ge\varepsilon} (\varphi\partial^2_{i_j} G)dx$$

We don't need to worry about the first integral, since $\partial_{i,j} G(x-y)$ is smooth in the $|x-y| \ge 1$ domain.

$$\int_{1>|x-y|\ge\varepsilon} \varphi\partial^2_{i,j} G dx = \int_{1>|x-y|\ge\varepsilon} \underbrace{(\varphi(x)-\varphi(y))}_{c|x-y|} \underbrace{\partial^2_{i,j} G(x-y)}_{\frac{c}{|x-y|^d}} dx \to good\ limit$$

Thus:

$$\begin{aligned} &(\partial_{i,j} u)(\varphi) = -\int_{\mathbb{R}^d} (\partial_i u)(\partial_j\varphi(x)dx \\ = \ & c\delta_{i,j}\int_{\mathbb{R}^d} f(y)\varphi(y)dy + \int_{|x-y|\ge 1} f(y)\varphi(x)\partial^2_{i,j} G(x-y)dxdy \\ &+ \int_{|x-y|<1} \underbrace{f(y)(\varphi(x)-\varphi(y))}_{(f(x))-f(y))\varphi(x)} \partial^2_{i,j} G(x-y)dxdy \end{aligned}$$

$$\Rightarrow \ (\partial_{i,j}u)(x) = c\delta_{i,j}f(x) + \int_{|x-y|\geq 1} f(y)\partial^2_{i,j}G(x-y)dy + \int_{|x-y|<1} (f(x)) - f(y))\partial^2_{i,j}G(x-y)dy$$

Now we want to show, that the RHS is $C^{0,\alpha}(dx)$. It is already clear that $f \in C^{0,\alpha}$ and $f \star (\partial^2_{i,j}G\mathbb{1}_{|x|\geq 1}) \in C^{0,\alpha}$. The main difficulty is in the last term.
To simplify the notation $z = 0$:

$$\rightsquigarrow \int_{|y|<1} \underbrace{\partial_{i,j}G(y)}_{|\cdot|\leq \frac{c}{|y|^d}} \underbrace{(f(x+y) - f(y))}_{|\cdot|\leq C|x|^\alpha} dy$$

$$= \int_{4|x|<|y|<1} \partial_{i,j}G(y)(f(x+y) - f(y))dy + \int_{|y|<4|x|} \partial_{i,j}G(y)(f(x+y) - f(y))dy$$

Second term:

$$|\int_{|y|<4|x|} \partial_{i,j}G(y)(f(x+y) - f(y))dy| = |\int_{|y|<4|x|} \partial_{i,j}G(y)(f(x+y) + f(x) - f(y) + f(0))dy|$$

$$\leq C\int_{|y|<4|x|} \frac{|y|^\alpha}{|y|^d}dy = C|x|^\alpha$$

First term:

$$\begin{aligned}
&\int_{4|x|<|y|<1} \partial_{i,j}G(y)(f(x+y) - f(y))dy \\
= \ &\int_{\underbrace{4|x| < |y-x| < 1}_{A}} \partial_{i,j}G(y-x) \underbrace{f(y)}_{f(y)-f(0)} dy - \int_{\underbrace{4|x| < |y-x| < 1}_{B}} \partial_{i,j}G(y) \underbrace{f(y)}_{f(y)-f(0)} dy \\
= \ &\int_{A\cap B} (\partial_{i,j}G(y-x) - \partial_{i,j}G(y))(f(y) - f(0))dy \\
&+ \int_{A\setminus B} \partial_{i,j}G(y)(f(y) - f(0))dy - \int_{B\setminus A} \partial_{i,j}G(y)(f(y) - f(0))dy
\end{aligned}$$

For these three integrals we have:

$$|\int_{A\cap B} \partial_{i,j}G(y)(f(y) - f(0))dy| \leq \int_{A\cap B} \underbrace{|\partial_{i,j}G(y)|}_{c|x|(\frac{1}{|y-x|^{d+1}} + \frac{1}{|y|^{d+1}})} \underbrace{|f(y) - f(0)|}_{\leq c|y|^\alpha} dy$$

$$\leq c|x| \int_{1>|y|>4|x|} \frac{|y|^\alpha}{|y|^{d+1}}dy \leq c|x|(|x|^{\alpha-1}) = c|x|^\alpha$$

The second:

$$|\int_{A\setminus B} \partial_{i,j}G(y)(f(y) - f(0))dy| \leq \int_{4|x|<|y-x|<1, \ |y|\leq 4|x|} |\partial_{i,j}G(y-x)||f(y) - f(0)|dy$$

$$\leq \int_{|y|\leq 4|x|} \frac{|y|^\alpha}{|x|^d}dy \leq C|x|^\alpha$$

The third:

$$
\begin{aligned}
&|\int_{B\setminus A}|... \le \int_{4|x|<|y|<1,\ |x-y|\le 4|x|} ... + \int_{4|x|<|y|<1,\ |x-y|\le 4|x|} ... \\
\le\ & \int_{4|x|<|y|<5|x|} |\partial_{i,j}G(y)||f(y)-f(0)|dy + \int_{1-|x|<|y|<1} |\partial_{i,j}G(y)||f(y)-f(0)|dy \\
\le\ & \int_{4|x|<|y|<5|x|} \frac{|y|^\alpha}{|y|^\alpha} dy + c\int_{[1-|x|,1]} \frac{r^\alpha}{r^d} r^{d-1} dr \\
\le\ & c|x|^\alpha + c'|x|^\alpha \sim |x|^\alpha
\end{aligned}
$$

For general k :, e.g $f \in C^{1,\alpha}$ we have

$$
-\Delta u = f \Rightarrow \ -\Delta(\partial_i u) = \underbrace{(\partial_i f)}_{C^{0,\alpha}}
$$

$$
\overset{k=0}{\Rightarrow}\ \partial_i u \in C^{2,\alpha}\ \forall i \Rightarrow u \in C^{3,\alpha}
$$

□

Remark. We need $D^\alpha D^\beta T = D^\beta D^\alpha T$ for all distributions T. This can be proved by 1) seeing that this is true for $\varphi \in C_c^\infty$ instead of a distribution T (Schwarz's theorem) 2) $(D^\alpha D^\beta T)(\varphi) = T(D^\alpha D^\beta \varphi)(-1)^{|\alpha|+|\beta|} = (D^\beta D^\alpha T)(\varphi)$

6.1.2 Heat equation

$$
\partial_t u(t,x) = \Delta_x u(t,x), \quad t \ge 0, x \in \mathbb{R}^d
$$

By Fourier transform in x:

$$
\partial_t \hat{u}(t,x) = -|2\pi k|^2 \hat{u}(t,k) \Rightarrow\ \hat{u}(x,t) = e^{-t|2\pi k|^2}\hat{u}(0,k) = e^{-t|2\pi k|^2}\hat{f}(k)
$$

$$
\begin{aligned}
u(t,x) &= \int_{\mathbb{R}^d} e^{-t|2\pi k|^2}\hat{f}(k)e^{2\pi ikx}dx = (G_t \star f)(x) \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4t}} f(y)dy
\end{aligned}
$$

where $G_t(x) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac{d}{2}}}$ is the heat kernel $\hat{G}_t(k) = e^{-t|2\pi k|^2}$

Theorem 6.2. If $f \in L^2(\mathbb{R}^d)$, then the solution of the heat equation $u(t,x) = (e^{t\Delta}f)(x) = (G_t \star f)(x)$ satisfies

1. $\lim\limits_{t\to 0} u(t,x) = f(x)$ in $L^2(\mathbb{R}^d)$
2. $\lim\limits_{t\infty} u(t,x) = 0$ in $L^2(\mathbb{R}^d)$

6.1.3 Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = \Delta_x u(t,x), \ t \geq 0, \ x \in \mathbb{R}^d \\ u(0,x) = f(x), \ x \in \mathbb{R}^d \end{cases}$$

By following the heat equation formally, $t \mapsto it$ (imaginary time)

$$u(t,x) = \frac{1}{(4\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4it}} f(y)dy \quad (*)$$

Theorem 6.3.
1. If $f \in L^1 \cap L^2$, then $u(t,x) = (e^{it\Delta} f)(x)$ given by $(*)$
2. $\forall f \in L^2$, we can define $u(t,x)$ by a limiting argument. We have: $\hat{u}(t,k) = e^{-it|2\pi k|^2} \hat{f}(k)$

Proof. First take $f \in L^1 \cap L^2$, then use $(*)$

$$\Rightarrow \ \hat{u}(t,k) = e^{-t|2\pi k|^2} \hat{f}(k)$$

$$\Rightarrow \ \|u(t,\cdot)\|_{L^2} = \|\hat{u}(t,\cdot)\|_{L^2} = \left\|\hat{f}\right\|_{L^2} = \|f\|_{L^2}$$

For any $f \in L^2$, find $f_1 \in L_1 \cap L^2$, $f_n \to f$, then $u_n = e^{-it\Delta} f_n$ well-defined and a cauchy sequence in L^2

$$\|u_m(t) - u_n(t)\|_{L^2(dx)} = \left\|e^{-it\Delta}(f_n - f)\right\|_{L^2(dx)} = \|f_n - f_m\|_{L^2} \to 0$$

So $\exists u(t,x) = \lim_{n\to\infty} u_n(t,x)$ in L^2 □

Theorem 6.4. Take $f \in L^2(\mathbb{R}^d)$ and $u(t,x) = e^{-it\Delta} f(x)$. Then:

1. $\lim_{t\to 0} u(t,x) = f(x)$ in L^2
2. $\lim_{t\to\infty} \int_B |u(t,x)|^2 dx = 0$, for all $B \subset \mathbb{R}^d$ bounded (scattering theory)

Proof. Consider the case $f \in L^1 \cap L^2$. Then:

$$|u(t,x)| = \frac{1}{(4\pi t)^{\frac{d}{2}}} \|f\|_{L^1} \to 0 \ as \ t \to \infty$$

If B bounded, then:

$$\int_B |u(t,x)|^2 dx \leq |B| \, \|u(t)\|_{L^\infty} \leq \frac{|B| \, \|f\|_{L^1}}{t^{\frac{d}{2}}} \to 0 \ as \ t \to \infty$$

General case $f \in L^2$: Take $f_n \to f$ in L^2, $f_n \in L^1 \cap L^2$.
Define $u_n = e^{-it\Delta} f_n$, then $u_n(t,x) \to u(t,x)$ in L^2.

$$\int_B |u(t,x)|^2 dx \leq 2\int_B |u_n(t,x)|^2 dx + 2\int_B |(u-u_n)(t,x)|^2 dx \leq \frac{|B| \, \|f\|_{L^1}}{t^{\frac{d}{2}}} + C\|f_n - f\|_{L^2}$$

Then take $t \to \infty$ and $n \to \infty$ □

6.1.4 Well posedness of PDEs

Stability: small errors of data $\rightsquigarrow$ a small error of equation

Schrödinger equation:

$$\begin{cases} i\partial_t u = \Delta_x u \\ u(0,x) = f(x) \end{cases}$$

$$\Rightarrow \left\| \underbrace{e^{-it\Delta} f}_{u} - e^{-it\Delta} f_\varepsilon \right\|_{L^2} = \|f - f_\varepsilon\|$$

which is exactly what we want.

Heat equation:

$$\begin{cases} \partial_t u = \Delta_x u \\ u(0,x) = f(x) \end{cases}$$

$$\Rightarrow \left\| e^{t\Delta} f - e^{t\Delta} f_\varepsilon \right\|_{L^2} = \|f - f_\varepsilon\| \quad (exercise)$$

Heat equation:

$$\begin{cases} \partial_t u = \Delta_x u \\ u(T,x) = f(x) \end{cases}$$

$$\Rightarrow \left\| \underbrace{u(0,x)}_{e^{-T\Delta} f} - e^{t\Delta} f_\varepsilon \right\|_{L^2} \xrightarrow{may} \infty$$

even if $\|f_\varepsilon - f\| \to 0$. This is because

$$\left\| \underbrace{u(0,x)}_{e^{-T\Delta} f} - e^{t\Delta} f_\varepsilon \right\|_{L^2} = \int_{\mathbb{R}^d} e^{2T|2\pi k|^2} |\hat{f}(k) - \hat{f}_\varepsilon(k)|^2 >> \int |\hat{f}(k) - \hat{f}_\varepsilon(k)|^2 = \|f_\varepsilon - f\|_{L^2}^2$$

Theorem 6.5. Let $f \in L^2$ s.t. $e^{-T\Delta f} \in L^2$, i.e. $e^{T|2\pi k|^2} \hat{f}(k) \in L^2$ for $T > 0$. Let u be the solution of the backward heat equation. Given $f_\varepsilon \in L^2$, $\|f_\varepsilon - f\|_{L^2} \leq \varepsilon$. Define $\hat{u}_\varepsilon(0,k) = e^{T|2\pi k|^2} \hat{f}_\varepsilon(k) \mathbb{1}_{|k| \leq \delta_\varepsilon}$ with $\delta_\varepsilon := |\log \varepsilon|^{\frac{1}{4}}$. Then:

$$\|u_\varepsilon(0,x) - u(0,x)\|_{L^2} \xrightarrow{\varepsilon \to 0} 0$$

6.2 Oscillatory Integrals

Definition 6.6.

$$I(t) = \int_{\mathbb{R}^d} e^{it\varphi(x)} f(x) dx$$

is called an oscillatory integral, where $f \in L^1(\mathbb{R}^d)$ complex valued and φ real valued.

Q: How does $I(t)$ decay as $t \to \infty$?
Fourier $\rightsquigarrow$ $t\varphi(x) = 2\pi k x$, $k \in \mathbb{R}^d$. But here we consider a general phase.

Theorem 6.7 (φ has no critical point)**.** Assume $f \in C_c^\infty(\mathbb{R}^d)$, $\varphi \in C^\infty$, $\nabla\varphi(x) \neq 0$ $\forall x \in$ *supp* f, then

$$|I(t)| = |\int_{\mathbb{R}^d} e^{it\varphi(x)} f(x) dx| \leq \frac{C_N}{|t|^N} \quad as\ |t| \to \infty$$

for all $N \geq 1$

Proof. Step 1: In 1D: $\varphi'(x) \neq 0$ for all $x \in$ *supp* f $\Rightarrow$ φ is monotone in *supp* f. We can change the variable in $u = \varphi(x)$.

$$\Rightarrow dx = d(\varphi^{-1}(\varphi(x))) = (\varphi^{-1})'(u) du$$

$$I(t) = \int_{\mathbb{R}^d} e^{it\varphi(x)} f(x) dx = \int_{\mathbb{R}^d} \underbrace{f(\varphi^{-1}(u))(\varphi^{-1})'(u)}_{\in C_c^\infty} du$$

$\rightsquigarrow$ usual Fourier transform.

$$\Rightarrow |I(t)| \leq \frac{C_N}{|t|^N} \ \forall N \geq 1$$

as $I(t)$ is Schwartz.

Step 2: $(d \geq 1)$ Assume $\partial_{x_i}\varphi(x) \neq 0$ for all $x \in$ *supp* f, then

$$I(t) = \int_{\mathbb{R}^d} e^{it\varphi(x)} f(x) dx = \int_{\mathbb{R}^{d-1}} \underbrace{(\int_{\mathbb{R}} e^{it\varphi(x)} f(x) dx_1)}_{|\cdot| \leq \frac{C_N(x_2,...,x_d)}{|t|^N}} dx_2 ... dx_d$$

since $C_N(x_2, ..., x_d)$ is nice enough:

$$\Rightarrow \ |I(t)| \leq \frac{\tilde{C}_N}{|t|^N}$$

In general, we only assume $\nabla\varphi(x) \neq 0 \Leftrightarrow (\partial_{x_1}\varphi, ..., \partial_{x_d}\varphi) \neq 0$. We can find a partition of unity $1 = \sum_{k=1}^{d} \chi_k$, $\chi_k \in C^\infty, 0 \leq \chi_k \leq 1$ s.t. $\partial_{x_k}\varphi \neq 0$ on $supp\ f\chi_k$

$$\Rightarrow\ |I(t)| \leq \sum_k = 1^d | \int e^{it\varphi(x)} f(x)\chi_k(x)dx| \leq \frac{C_N}{|t|^N} \quad \forall N \geq 1$$

□

Remark (geometric method)**.** Assume $\Omega_1, ..., \Omega_l \subset \mathbb{R}^d$ open, $\bigcup \Omega_j = \mathbb{R}^d$, then for all Ω_j we can find a function

$$a_i : \mathbb{R}^d \to [0,1],\ s.t.\ a_i \simeq \mathbb{1}_{\Omega_i} \begin{cases} a_i(x) = 1\ if\ x \in \Omega_i \\ a_i(x) = 0\ if\ dist(x, \Omega_i) \geq \varepsilon \end{cases} \qquad (Urgushen\ lemma)$$

$$\rightsquigarrow \chi_i = \frac{a_i}{\sum_j a_j}$$

$\Rightarrow\ \sum \chi_i = 1,\ supp\ \chi_i = supp\ a_i,\ \sum_j a_j > 0$ since $\bigcup_i \Omega_i = \mathbb{R}^d$

Problem: We will see the oscillatory integral with φ having critical points!

Remark. 1D: If we bound $|\int_a^b e^{it\varphi(x)}dx| \lesssim \frac{c}{|t|^\alpha}$, then

$$|\int_a^b \underbrace{e^{it\varphi(x)}}_{F'} f(x)| \overset{Ibp}{=} |[Ff]_a^b - \int_a^b F(x)f'(x)dx| \leq \|F\|_{L^\infty} (\|f\|_{L^\infty} + \int_a^b |f'|)$$

Theorem 6.8. (1D, Van der Corput)
(a) Take $a < b$ real $\varphi \in C^k$ for some $k \geq 2$, $|\varphi^{(k)}(t)| \geq 1$. Then

$$|\int_a^b e^{it\varphi(x)}dx| \leq \frac{Ck}{|t|^{\frac{1}{k}}}$$

(b) If $\varphi \in C^1$, φ' monotone and $|\varphi'| \geq 1$, then

$$|\int_a^b e^{it\varphi(x)}dx| \leq \frac{C}{|t|}$$

Example 6.9 (Bessel function)**.**

$$\hat{\mathbb{1}}_{[-1,1]}(t) \rightsquigarrow |\int_a^b e^{it\cos(x)}\cos(x)dx| \leq \frac{c}{|t|^{\frac{1}{2}}} \; as \; t \to \infty$$

Here $\varphi(x) = \cos(x)$. $\varphi'(x) = -\sin(x)$, $\varphi''(x) = -\cos(x)$. We can divide $[0,\pi]$ to sub-intervals where either $|\varphi'| \geq \frac{1}{2}$ and φ' monotone or $|\varphi''| \geq \frac{1}{2}$
In our proof of the Gauss circle problem bound we used this decay of the Bessel function.

Proof. (b) Assume $\varphi' \geq 1$ in $[a,b]$ and φ' monotone.

$$|\int_a^b e^{it\varphi(x)}dx| = |\int_a^b \frac{(e^{it\varphi(x)})'}{it\varphi'(x)}dx| = \frac{1}{t}|-\int_a^b (e^{it\varphi}(\frac{1}{\varphi'})' + \left[\frac{e^{it\varphi}}{\varphi'}\right]_a^b)|$$

$$\leq \frac{1}{t}|\frac{1}{\varphi'(a)} + \frac{1}{\varphi'(b)}| + \underbrace{\int_a^b |(\frac{1}{\varphi'})'|}_{\overset{(*)}{=}\int_a^b(\frac{1}{\varphi'})' = |\frac{1}{\varphi'(b)} - \frac{1}{\varphi'(a)}|} \leq \frac{4}{t}$$

In $(*)$ we used that φ' monotone $\Rightarrow$ $\frac{1}{\varphi'}$ monotone $\Rightarrow$ $(\frac{1}{\varphi'})'$ either ≥ 0 on $[a,b]$ or ≤ 0 on $[a,b]$

For (a) we need some new tools! We will prove (a) later. □

Remark. Without the assumption φ' monotone, this fails!

Lemma 6.10. $\forall k \geq 1$, $\forall \{a_m\}_{m=0}^k \subset [a,b]$ distinct, $\forall f \in C^k([a,b])$ real valued. Then $\exists y \in [a,b]$ s.t. $f^{(k)}(y) = \sum\limits_{m=0}^{k} c_m f(a_m)$ where $c_m = (-1)^k k! \prod\limits_{l\in\{0,\dots,k\}\setminus\{m\}} \frac{1}{a_l - a_m}$

Proof. Induction:
<u>$k=1$:</u> $f \in C^1$, $a_0 < a_1$, $\exists y \in (a_0, a_1)$ s.t.

$$f'(y) = \frac{f(a_1) - f(a_0)}{a_1 - a_0}, \quad c_0 = \frac{-1}{a_1 - a_0}, \; c_1 = \frac{1}{a_1 - a_0}$$

$\rightsquigarrow$ Rolle's theorem
<u>$k \geq 1$</u>: We will try to apply Rolle's theorem many times. More precisely, we find a polynomial $p(x) = \sum\limits_{m=0}^{k} b_m x^m$ s.t. the function $F(x) = f(x) - p(x)$ has zeros $\{a_m\}_{m=0}^k$, i.e. $F(a_0) = F(a_1) = \dots = F(a_k) = 0$.
$\exists a_0 < b_0 < a_1 < b_1 < \dots < b_{k-1} a_k$ s.t. $F'(b_0) = F'(b_1) = \dots = F'(b_{k-1}) = 0$

$\Rightarrow \exists y \in [a,b]$ s.t. $F^{(k)}(y) = 0$
$\Rightarrow f^{(k)}(y) - k!b_k = 0$

The claim follows from the fact that $k!b_k = \sum_{m=0}^{k} c_m f(a_m)$. How can we find $p(x) = \sum_{m=0}^{k} b_m x^m$? We want $\underbrace{p(a_l)}_{\sum_{m=0}^{k} b_m a_l^m} = f(a_l)$ for all $l = 0, ..., k$.

$$\Leftrightarrow \begin{pmatrix} 1 & a_0^1 & \dots & a_0^k \\ \vdots & & \ddots & \vdots \\ 1 & a_k^1 & \dots & a_k^k \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_k \end{pmatrix} = \begin{pmatrix} f(a_0) \\ \vdots \\ f(a_k) \end{pmatrix}$$

determine the $\{b_l\}$ uniquely.

$\det(\cdot) = Van\ der\ monde\ determinant = \prod_{l=0}^{k-1} \prod_{j=l+1}^{k} (a_l - a_j) \neq 0$

$\Rightarrow \ b_m = \sum_{m=0}^{k} (-1)^m f(a_m) \prod_{l \in \{0,...,k\} \setminus \{m\} \frac{1}{a_l - a_m} (-1)^{k-m}}$ □

Lemma 6.11. Let $E \subset \mathbb{R}$ measurable, $0 < |E| < \infty$, then $\forall k \geq 1, \ \exists \{a_m\}_{m=0}^{k} \subset E$ s.t.

$$\prod_{l \neq j} |a_l - a_j| \geq \left(\frac{|E|}{2e}\right)^k, \quad e = 2.7...$$

Proof. (exercise) □

Lemma 6.12 (Controlling of sub-level set)**.** $\forall k \geq 1, \ \forall \varphi \in C^k$ real-valued, $|\varphi^{(k)}(t)| \geq 1$. Then

$$|\{t : \ |\varphi(t)| \leq \alpha\}| \leq Ck\alpha^{\frac{1}{k}}$$

Proof. By the previous lemma $E = \{|\varphi| \leq \alpha\}$, $\exists \{a_m\}_{m=0}^{k}$ s.t.

$$\frac{|E|^k}{(2l)^k} \leq \prod_{l \neq j} |a_l - a_j|$$

By lemma 6.10, $\exists y : \varphi^{(k)}(y) = (-1)^k k! \sum_{m=0}^{k} \varphi(a_m) \prod_{l \neq j} |a_l - a_j|$

$$\Rightarrow \ 1 \leq |\varphi^{(k)}| \leq k! \alpha \prod_{l \neq j} |a_l - a_j| \leq k! \alpha \frac{(2e)^k}{|E|^k}$$

$$\Rightarrow \ |E| \leq (k!\alpha(2e)^k)^{\frac{1}{k}} \leq Ck\alpha^{\frac{1}{k}}$$

□

proof of theorem 6.8 (a). $|\varphi^{(k)}(t)| \geq 1$. We consider
$R_1 = \{t : |\varphi'(t)| \geq \alpha\}$
$R_2 = \{t : |\varphi'(t)| \geq \alpha\}$
For $R_2 :$ $|R_2| \leq Ck\alpha^{\frac{1}{k-1}}$ by previous sub-level set lemma.
For $R_1 :$ $\varphi^{(k)} \neq 0$ on $(a,b) \Rightarrow$ $\varphi^{(2)}$ has at most $2k$ zeros.
$\Rightarrow R_1 =$ union of at most k intervals where $\varphi^{(2)} \neq 0$ in each interval $\Rightarrow$ φ' is monotone in each interval.

Consider 1 interval (c,d), $|\varphi'| \geq 0$ on (c,d) and φ' is monotone in (c,d). (b) $\Rightarrow$ $|\int_c^d e^{it\varphi(x)}dx| \leq \frac{c}{|t|\alpha}$

$\overset{triangle\ inequality}{\Rightarrow}$ $|\int_{R_1} e^{it\varphi(x)}dx| \leq \frac{Ck}{|t|\alpha}$.
Conclusion:

$$|\int_a^b e^{it\varphi(x)}dx| \leq |R_2| + |\int_{R_1} e^{it\varphi(x)}dx|$$

$$Ck\alpha^{\frac{1}{k-1}} + \frac{Ck}{|t|\alpha} = Ck(\alpha^{\frac{1}{k-1}} + \frac{1}{|t|\alpha}) \ \ \forall\alpha$$

$$\overset{optimize}{\Rightarrow} \quad \min_{\alpha>0}(\alpha^{\frac{1}{k-1}} + \frac{1}{|t|\alpha}) \lesssim \frac{1}{|t|^{\frac{1}{k}}}$$

□

6.3 Number Theory

Recall that for the Gauss circle problem we used:

Lemma 6.13 (Poisson summation formula)**.** Let $f \in S(\mathbb{R}^d)$, then

$$\sum_{x\in\mathbb{Z}^d} f(x) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)$$

Lemma 6.14 (Bessel)**.**

$$|\int_0^\pi e^{it\cos(\theta)}\cos(\theta)d\theta| \lesssim \frac{1}{|t|^{\frac{1}{2}}} \ as\ t \to \infty$$

Question: What is the distribution of the first digit of 2^m?
$2^0 = \underline{1}, \quad 2^1 = \underline{2}, \quad 2^2 = \underline{4}, \quad 2^3 = \underline{8}, \quad 2^4 = \underline{1}6, \quad 2^5 = \underline{3}2, \ \ldots$
One would expect that all digits are equally distributed. In reality, e.g. 1 appears more than any other digit!

Theorem 6.15. Define $\{1, \ldots, 9\} \ni \alpha(n) = \mathit{first\ digit\ of\ n}$, then

$$\frac{\#\{1 \leq k \leq N : \ a(2^k) = m\}}{N} \to \log_{10}(1 + \frac{1}{m})$$

for all $m \in \{1, \ldots, 9\}$

Theorem 6.16. If $\alpha \in \mathbb{R}$ is irrational, then $\{k\alpha - \lfloor k\alpha \rfloor\}_{k=1}^{\infty}$ is equidistributed in $[0, 1)$, i.e.

$$\frac{1}{N}\#\{1 \leq k \leq N : k\alpha - \lfloor k\alpha \rfloor \in Q\} \to |Q|$$

for all $Q \subset [0, 1)$ measurable.

Theorem 6.15 is a consequence of this:

Proof. $\alpha(2^k) = m \Leftrightarrow m10^s \leq 2^k < (m + 1)10^s$ for some $s \in \mathbb{N}$.
$2^k = \underbrace{m...}_{s\ digits} \ \Leftrightarrow \ s + \log_{10}(m) \leq k \log_{10} 2 < s + \log_{10}(m + 1)$
$\Rightarrow \log_{10}(m) \leq k \log_{10}(2) - \lfloor k \log_{10}(2) \rfloor$

$\forall m \in \{1, ..., 9\}$, we have that $\alpha = \log_{10} 2$ is irrational and

$$\frac{1}{N}\#\{1 \leq k \leq N : \ k\alpha - \lfloor k\alpha \rfloor \in [log_{10}(m), log_{10}(m + 1)]\} \to log_{10}(1 + \frac{1}{m})$$

□

Theorem 6.17. Take $\{a_k\}_{k=1}^{\infty} \subset \underbrace{\mathbb{T}^d}_{torus}$, d fixed. Then TFAE:

(a)Take $\{a_k\}$ is equidistributed, namely

$$\frac{1}{N}\#\{k \leq N : a_k \in Q\} \to |Q|, \quad \forall Q \subset \mathbb{T}^d \ measurable$$

(b) $\forall f$ smooth $\frac{1}{N} \sum\limits_{k=1}^{N} f(a_k) \to \int_{\mathbb{T}^d} f(x) dx$

(c) $\forall m \in \mathbb{Z}^d \setminus \{0\},\ \frac{1}{N}\sum\limits_{k=1}^{N} e^{2\pi i m a_k} \to 0$

Remark. To us the torus is simply $\mathbb{T}^d = [0,1]^d$

Proof. (a) $\Rightarrow$ (b): We can conclude $(a) \Rightarrow (b)$ for smooth f by approximating it with step functions and then using a density argument.

(b) $\Rightarrow$ (a): We will conclude this by using the idea from the Gauss circle problem. We can find 2 smooth functions g, h s.t. $0 \le g(x) \le \mathbb{1}_Q(x) \le 1$ of every x. Then:

$$\underbrace{\frac{1}{N}\sum_{k=1}^{N} g(a_k)}_{\to \int_{\mathbb{T}^d} g(x)dx} \le \frac{1}{N}\sum_{k=1}^{N} \mathbb{1}_Q(a_k) \le \underbrace{\frac{1}{N}\sum_{k=1}^{N} h(a_k)}_{\to \int_{\mathbb{T}^d} h(x)dx}$$

(b) $\Rightarrow$ (c): Use (b) for $x \mapsto e^{2\pi i m x} = f(x)$

$$\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i m a_k} = \frac{1}{N}\sum_{k=1}^{N} f(a_k) \to \int f(x)dx = \int_{\mathbb{T}^d} e^{2\pi i m x} dx = \begin{cases} 1, & if\ m = 0 \\ 0, & if\ m \in \mathbb{Z}^d \setminus \{0\} \end{cases}$$

(c) $\Rightarrow$ (b):

$$\frac{1}{N}\sum_{k=1}^{N} f(a_k) = \frac{1}{N}\sum_{k=1}^{N}\Big(\sum_{l \in \mathbb{Z}^d} \hat{f}(l) e^{2\pi i l a_k}\Big) = \frac{1}{N}\sum_{l}\sum_{k} = \hat{f}(0) + \sum_{l \neq 0} \underbrace{\Big(\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i l a_k}\Big)}_{\to 0,\ as\ N \to \infty} \hat{f}(l)$$

$$\to \hat{f}(0) = \int f(x)dx$$

□

Exercise. Use this thm to prove $k\alpha - \lfloor k\alpha \rfloor$ is equidistributed.

Riemann Zeta function:

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \forall s > 1$$

Basel problem: $\zeta(s)$ is transcendental $\forall s \in \mathbb{N},\ s > 1$, i.e. $\zeta(s)$ is not a root of any polynomial of rational coefficients.

Theorem 6.18 (Euler)**.** The Basel problem is true if $s \in \mathbb{N}$ is even.

Proof. Consider B-polynomial $\{B_m\}_{m=0}^{\infty}$, $B_0 = 1$, $B'_m(x) = mB_{m-1}(x)$, $\int_0^1 B_m(x)dx = 0$.
Actually: $B_m(x) = -m! \sum\limits_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{2\pi ikx}}{(2\pi ik)^m}$, $\forall m \geq 2$ the series converges absolutely.

$$B'_m(x) = -m! \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{e^{2\pi ikx}}{(2\pi ik)^{m-1}} = mB_{m-1}(x)$$

(for $m = 1$ the series converges conditionally)

Application: If m is even, $x = 0$, then:

$$B_m(0) = \left(\underbrace{c_m}_{\in \mathbb{Q}} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx)}{(2\pi k)^m} \right) \Bigg|_{x=0} = c_m \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^m} = \frac{c_m}{(2\pi)^m} \zeta(m)$$

Conclusion follows from $B_m(0)$ being rational! Since $B'_m(x) = mB_{m-1}(x)$, $\int B_m = 0$
$\Rightarrow$ $B_m(x)$ has rational coefficients $\forall m$ $\Rightarrow$ $B_m \in \mathbb{Q}$ ☐

Chapter 7

Decomposition Methods

7.1 Caldéron-Zugmund decomposition

Idea: For a constant $\alpha > 0$ decompose the domain of $0 \geq f \in L^1$ into cubes s.t. restricted to one of the cubes either $f \leq \alpha$ or $f \sim \alpha$.

Theorem 7.1. Let $0 \geq f \in L^1(\mathbb{R}^d)$. Then $\forall \alpha > 0$ $\exists$ countable family of disjoint cubes $\{Q_k\}_{k=1}^{\infty}$ s.t.

1. If $x \notin \Omega = \bigcup_k Q_k$, then $f(x) \leq \alpha$ *a.e.*
2. $\alpha < \fint_{Q_k} f \leq 2^d \alpha, \ \forall k$

Proof. The proof uses a stopping time argument. Assume f is compactly supported, then we cover *supp* f by a big cube Q_0. We can assume that $\fint_{Q_0} f = \frac{1}{|Q_0|} \int_{\mathbb{R}^d} f < \alpha$. Now we divide Q_0 int 2^d sub-cubes of half-length side. For a sub cube Q of Q_0, if

$$\fint_Q f > \alpha \to \textit{bad cube} \ \to \ \textit{stop} \ \to \ \textit{add to collection}$$

Otherwise, if $\fint_Q f \leq \alpha \to$ good cube $\to$ divide Q again in 2^d sub-cubes and repeat.
By induction, we obtain a countable collection $\{Q_k\}_{k=1}^{\infty}$ of sub-cubes s.t. all Q_k are bad cubes.
Clearly, $\{Q_k\}$ are disjoint. Now we check the properties 1) and 2).

1) If $x \notin \Omega = \bigcup_k Q_k \Rightarrow \ \exists$ a sequence of good cubes $\{g_n\}_{n=1}^{\infty}$ s.t. $x \in g_n, \ \forall n$, and $|g_n| \to 0$ as $n \to \infty$. Then:

$$\fint_{g_n} f \leq \alpha, \ \forall n \Rightarrow \ f(x) = \lim_{n \to \infty} \fint_{g_n} f \leq \alpha.$$

for a.e. x by Lebesgue differentiation theorem.

2) $\forall$ bad cubes Q_k, then by the definition $⨍_{Q_k} f > \alpha$. From the construction, Q_k is a sub-cube of a good cube $\tilde{Q}$ s.t. $|Q_k| = 2^{-d}|\tilde{Q}|$ and $⨍_{\tilde{Q}} f \leq \alpha$
Consequently:

$$⨍_{Q_k} f = \frac{1}{|Q_k|}\int_{Q_k} f \leq \frac{1}{2^{-d}|\tilde{Q}|}\int_{\tilde{Q}} f = 2^d ⨍_{\tilde{Q}} f \leq 2^d\alpha$$

□

Remark. From the above construction, we have:

$$|\Omega| = \sum_k |Q_k| < \sum_k \frac{1}{\alpha}\int_{Q_k} f \leq \frac{1}{\alpha}\int_{\mathbb{R}^d} f$$

Note that if $x \in \Omega^c \Rightarrow f(x) \leq \alpha$. This means $|\Omega| \rightsquigarrow |\{f > \alpha\}|$ related.
$\rightarrow$ recall that $|\{f > \alpha\}| \lesssim \frac{1}{\alpha}\int_{\mathbb{R}^d} |f|$

Lemma 7.2 (Reverse maximal inequality)**.** Let $0 \leq f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, then

$$|\{Mf > c\alpha\}| \geq \frac{1}{2^d\alpha}\int_{\{f>\alpha\}} f$$

for some constant $c = c_d > 0$.

Proof. Let $\{Q_k\}_{k=1}^{\infty}$ be the collection of cubes obtained by the Caldéron-Zugmund decomposition. Then for $\Omega = \bigcup_k Q_k$:

$$\int_{\{f>\alpha\}} f \overset{1)}{\leq} \int_\Omega f = \sum_k \int_{Q_k} f \overset{2)}{\leq} \sum_k 2^d\alpha|Q_k| = 2^d\alpha|\Omega|$$

$$\forall k: \quad ⨍_{Q_k} f > \alpha \;\Rightarrow\; \{Mf > c_d\alpha\} \supset \Omega$$

We conclude that

$$|\{Mf > c_d\alpha\}| \geq |\Omega| \geq \frac{1}{2^d\alpha}\int_{\{f>\alpha\}} f$$

□

Definition 7.3.

$$\|f\|_{L^1 locL^1(B)} = \|f\|_{L^1} + \int_B |f(y)| \log(\frac{|f(y)|}{\|f\|_{L^1}})dy$$

We can apply the lemma from above to obtain:

Theorem 7.4. Let $f \in L^1(B)$ for $B \subset \mathbb{R}^d$ ball. Then:

$$Mf \in L^1(B) \Leftrightarrow \; f \in L^1 log L^1(B)$$

Remark. Recall that $f \in L^p \Leftrightarrow Mf \in L^q, \; \forall p > 1$.

Proof. Step 1: (Assume B unit ball and $f \geq 0$ for simplicity) We prove that

$$\int_B |Mf| \lesssim \|f\|_{L^1 log L^1(B)}$$

By layer-cake:

$$\begin{aligned} \int_B |Mf| &= \int_0^\infty |\{x \in B : \; Mf(x) > \lambda\}| d\lambda \\ &\leq \int_0^\delta ... + \int_\delta^\infty ... \\ &\leq \int_0^\delta |B| d\lambda + \int_\delta^\infty \frac{1}{\lambda} \int_{\{f > \lambda\}} f d\lambda \\ &= |B|\delta + \int_{\mathbb{R}^d f(y) (\int_\delta^{f(y)} \frac{1}{\lambda} d\lambda) dy} \\ &= |B|\delta + \int_{\mathbb{R}^d} f(y) \log(\frac{f(y)}{\delta}) dy \overset{\delta = \|f\|_{L^1}}{\leq} \|f\|_{L^1 log L^1(B)} \end{aligned}$$

We used the weak L^1 inequality: $|\{Mf > \lambda\}| \leq \frac{1}{\lambda} \|f\|_{L^1} \rightsquigarrow \underbrace{|\{Mf > \lambda\}|}_{\leq |\{Mf \mathbb{1}_{|f| \geq \frac{\lambda}{2}} > \frac{\lambda}{2}\}|} \lesssim \frac{1}{\lambda} \left\| f \mathbb{1}_{|f| > \frac{\lambda}{2}} \right\|_{L^1}$

Step 2: We prove

$$\int_B |Mf| \gtrsim \|f\|_{L^1 log L^1(B)}$$

Recall lemma $|\{Mf > \lambda\}| \gtrsim \frac{1}{\lambda} \int_{\{f > \lambda\}} f$ (We proved this by CZ decomposition)
By layer cake:

$$\int_B |Mf| = \int_0^\infty |\{x \in B : \; Mf > \lambda\}| dy \gtrsim \int_0^\infty (\frac{1}{\lambda} \int_{\{f > \lambda\}} f) d\lambda = \int_{\mathbb{R}^d} f(y) (\int_{\lambda_0}^{f(y)} \frac{1}{\lambda} d\lambda) dy \sim \|f\|_{L^1 log L^1(B)}$$

Here $\lambda_0 \geq c_d \|f\|_{L^1}$

$$\Rightarrow \; |\{x \in B : \; Mf > \lambda\}| \gtrsim \frac{1}{\lambda} \int_{\{f > \lambda\}} f$$

□

Theorem 7.5 (A variant of CZ decomposition)**.** Let $0 \geq f \in L^1(\mathbb{R}^d)$, assume $\int_{\mathbb{R}^d} f \geq c_d$. Then $\forall 0 < \Lambda \leq c_d$, $\forall \alpha_d > 0$, we can find a cover $\mathbb{R}^d$ of disjoint cubes $\{Q_k\}_{k=1}^N$ s.t.

1. $\int_{Q_k} f \leq \Lambda, \ \forall k$
2. $\sum_Q \frac{1}{|Q_k|^\alpha}(\int_{Q_k} f - \alpha_d \Lambda) \geq 0$

Proof. We use again the stopping time argument. Assume *supp* $f \subset Q_0$ big cube. We divide Q_0 into 2^d sub-cubes. For all Q sub-cubes of Q_0 one of the following two holds:

- $\int_Q f \leq \Lambda$, then stop and add Q to collection
- $\int_Q f > \Lambda$, then divide again in 2^d sub-cubes and repeat!

After finitely many steps, we stop and get a collection $\{Q_k\}$. Now we have to check 1) and 2). 1) is obvious.
2) By writing the cubes as a tree, we can divide $\{Q\} \to \bigcup_F \{Q\}_{Q\in F}$ where Q are the nodes of the tree and F are the branches of the tree. So we get a union of disjoint collections $\{\{Q\}_{Q\in F}\}_F$ s.t. for all F we have:
$\forall m, \ \exists$ at most 2^d cubes $Q \subset F : \ |Q| = m$. Now we prove that $\forall F : \ \sum_{Q\in F} \frac{1}{|Q|^\alpha}(\int_Q f - c_d\Lambda) \geq 0$.
Take $m =$ smallest volume of cubes in F. Then $\forall Q \in F \Rightarrow |Q| = m2^{dn}$, then

$$\sum_{Q\in F} \frac{1}{|Q|^\alpha} \leq \sum_{n=0}^{\infty} \frac{2^d}{(m2^{dn})^\alpha} \lesssim c_d \frac{1}{m^\alpha}$$

On the other hand:

$$\sum_{Q\in F} \frac{1}{|Q|^\alpha}(\int_Q f) \geq \sum_{Q\in F, \ |Q|=m} \frac{1}{m^\alpha}\int_Q f = \frac{1}{m^\alpha}\int_{\tilde{Q}} f \geq \frac{\Lambda}{m^\alpha}$$

where $\int_{\tilde{Q}} f > 1$

$$\Rightarrow \sum_{Q\in F} \frac{1}{|Q_k|^\alpha}(\int_{Q_k} f - \alpha_d\Lambda) \geq \frac{\Lambda}{m^\alpha} - \underbrace{c_d\tilde{c}_d}_{<1}\frac{\Lambda}{m^\alpha} \geq 0$$

□

Theorem 7.6. $\forall d \geq 1$, if $\{u_n\}_{n=1}^N$ ONF, in $L^2(\mathbb{R}^d)$, then

$$\sum_{n=1}^N \int_{\mathbb{R}^d} |\nabla u_n|^2 \geq K_d \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}}$$

where $\rho(x) = \sum_{n=1}^N |u_n(x)|^2$ density of the system. Here K_d is indipendent of N. In particular, we can take $N \to \infty$.

Remark. (d= 3) We have the Sobolev inequality:

$$\int_{\mathbb{R}^3} |\nabla u|^2 \geq C(\int_{\mathbb{R}^3} |u|^6)^{\frac{1}{2}} \overset{\text{Hölder}}{\geq} C\frac{\int_{\mathbb{R}^3} |u|^{\frac{10}{3}}}{(\int_{\mathbb{R}^3} |u|^2)^{\frac{2}{3}}} = c\int_{\mathbb{R}^3} \rho^{\frac{5}{3}}$$

$\rho = |u|^2$, $\int_{\mathbb{R}^3} |u|^2 = 1$. This is the LT inequality for $N = 1$.

Lemma 7.7. For any normalized functions $\{u_n\}_{n=1}^N \subset L^2(\mathbb{R}^d)$ we have:

$$(*) \quad \sum_{n=1}^{N} \int_{\mathbb{R}^d} |\nabla u_n|^2 \geq \frac{c_d}{N^{\frac{2}{d}}} \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}} \quad (c_d > 0 \textit{ indipendent of } N)$$

Proof. If $u_n = u \Rightarrow (*)$becomes the LT inequality for $N = 1$, namely

$$LHS\ of\ (*) = N\int_{\mathbb{R}^d} |\nabla u|^2 = \int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2 \leq \frac{c\int \rho^{1+\frac{2}{d}}}{(\int \rho)^{\frac{2}{d}}} = \frac{c}{N^{\frac{2}{d}}}\int \rho^{1+\frac{2}{d}}$$

$\rho = N|u|^2$

For general $\{u_n\}_{n=1}^N$ normalized in L^2, then for $\rho(x) = \sum_{n=1}^{N} |u_n(x)|^2$

$$|\nabla\sqrt{\rho}|^2|\nabla\sqrt{\sum_n |u_n|^2}|^2 \leq \left|\frac{\sum_n |u_n||\nabla u_n|}{\sqrt{\sum_n |u_n|^2}}\right| \leq \sum_n |\nabla u_n|^2$$

(conevexity of the gradient. Brezis/Hoffman-Ostenhof) $\Rightarrow LHS(*) = \int_{\mathbb{R}^d}\sum_n |\nabla u_n|^2 \geq \int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2 \geq \frac{c_d}{N^{\frac{2}{d}}}\int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}}$ by the previous estimate. □

History of LT:

- 1969 Dyson-Lenard: proof of Stability of matter. Key observation: For kinetic energy of fermions in a cube φ: $\infty > c > \frac{1}{|\varphi|^{\frac{2}{d}}}(N-1)$ (Pauli-exclusion priciple)
- 1975 Lieb-Thirring: Nice proof of stability of matter using their inequality.
  LT = uncertainty + exclusion principle
  In 1975, LT proved their inequality in the dual form

$$|\sum_{\lambda<0} \lambda(-\Delta + V(x))| \leq c_d \int_{\mathbb{R}^d} |v|^{1+\frac{d}{2}}$$

Lemma 7.8 (Dyson-Lenard)**.**

$$\sum_{n=1}^{N} \int_Q |\nabla u_n|^2 =: T_Q \geq \frac{(2\pi)^2}{|Q|^{\frac{2}{d}}}(\int_Q \rho - 1),\ \rho(x) = \sum_{n=1}^{N} |u_n(x)|^2$$

Proof.

$$\begin{aligned} T_Q &= \sum_{n=1}^{N} \\ &\overset{(*)}{\gtrsim} \frac{1}{|Q|^{\frac{2}{d}}} \sum_{n=1}^{N} \int_Q |u_n - \overline{u_n}^Q|^2 \\ &\overset{(**)}{\gtrsim} \frac{1}{|Q|^{\frac{2}{d}}} \sum_{n=1}^{N} \int_Q (|u_n|^2 - 2|\overline{u_n}^Q|^2) \\ &= \frac{1}{|Q|^{\frac{2}{d}}} \int_Q \rho - 2|Q| \sum_{n=1}^{N} |\fint_Q u_n|^2 \end{aligned}$$

Here : $\sum_{n=1}^{N} |\int_Q u_n|^2 = \sum_{n=1}^{N} | < 1, u_n >_{L^2(Q)} |^2 \overset{ONF}{\leq} \|1\|_{L^2}^2 = |Q|$

Conclusion: $T_Q \gtrsim \frac{1}{|Q|^{\frac{2}{d}}}(\int_Q \rho - 2)$ The improved one $\frac{(2\pi)^2}{|Q|^{\frac{2}{d}}}(\int_Q \rho - 1)$ needs a bit more careful analysis.

$(**)$: We used $|a-b|^2 \geq \frac{1}{2}|a|^2 - 2|b|^2$

$(*)$: In d=3: Sobolev in $L^2(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} |\nabla u|^2 \gtrsim (\int_{\mathbb{R}^3} |u|^6)^{\frac{1}{3}}$$

Poincaré: $\underbrace{\int_Q |\nabla u|^2}_{=\nabla(u-c)} \gtrsim (\int_Q |u - \overline{u_n}^Q|^6)^{\frac{1}{6}} \gtrsim \frac{1}{|Q|^{\frac{2}{3}}} |u - \overline{u_n}^Q|^2$, $c = \overline{u_n}^Q = \fint_Q u$ □

proof of LT. Denote $T = \sum_{n=1}^{N} \int_{\mathbb{R}^d} |\nabla u_n|^2$ and $T_Q = \sum_{n=1}^{N} \int_Q |\nabla u_n|^2$. Take $\{Q\}$ a collection of disjoint cubes in $\mathbb{R}^d$. Then: $T \geq \sum_Q T_Q$

For $d \geq 3$:

- We proved $T_Q \gtrsim \frac{1}{|Q|^{\frac{2}{d}}}(\int_Q \rho - 1)$, $\rho = \sum_n |u_n|^2$
- By the convexity of gradient/H-O inequality:

$$T_Q \geq \int_Q |\nabla \sqrt{\rho}|^2 \overset{\text{Poincaré}}{\gtrsim} (\int_Q |\rho - \overline{\rho}^Q|^p)^{\frac{2}{p}}, \quad p = 2^* = \frac{2d}{d-2}$$

$$\geq \frac{\int_Q |\sqrt{\rho} - \overline{\sqrt{\rho}}^Q|^{2(1+\frac{2}{d})}}{(\int_Q |\sqrt{\rho} - \overline{\sqrt{\rho}}^Q|^2)^{\frac{2}{d}}} \gtrsim \frac{\int_Q \rho^{1+\frac{2}{d}}}{(\int_Q \rho)^{\frac{2}{d}}} - \frac{c_d}{|Q|^{\frac{2}{d}}} \int_Q \rho$$

$(\overline{\sqrt{\rho}}^Q = \frac{1}{|Q|}\int_Q |\sqrt{\rho}| \leq \frac{1}{|Q|^{\frac{1}{2}}}(\int_Q \rho)^{\frac{1}{2}})$
We can combine the two bounds:

$$
\begin{aligned}
(1+\varepsilon)T_Q &\gtrsim \sum_Q \left(\frac{\varepsilon \int_Q \rho^{1+\frac{2}{d}}}{(\int_Q \rho)^{\frac{2}{d}}} - \frac{\varepsilon c_d}{|Q|^{\frac{2}{d}}} \int_Q \rho + \frac{1}{|Q|^{\frac{2}{d}}} (\int_Q \rho - 1) \right) \\
&\geq \sum_Q \frac{\varepsilon \int_Q \rho^{1+\frac{2}{d}}}{(\int_Q \rho)^{\frac{2}{d}}} + \sum_Q \frac{1}{|Q|^{\frac{2}{d}}} ((1-\varepsilon c_d) \int_Q \rho - 1) \\
&\stackrel{?}{\geq} \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}}
\end{aligned}
$$

To get the conclusion, we need to choose $\{Q\}$ nicely. By our variant of CZ decomposition, we can choose $\{Q\}$ disjoint s.t. $\bigcup_Q Q = \mathbb{R}^d$ and

1. $\int_Q \rho \leq \Lambda$
2. $\sum_Q \frac{1}{|Q|^\alpha}(\int_Q \rho - \alpha_d \Lambda) \geq 0$, $\alpha = \frac{2}{d}$

Then:

$$
(1+\varepsilon)T \geq \sum_Q \frac{\varepsilon \int_Q \rho^{1+\frac{2}{d}}}{\Lambda^{\frac{2}{d}}} + \underbrace{(1-\varepsilon c_c)}_{>0} \sum_Q \frac{1}{|Q|^{\frac{2}{d}}} (\int_Q \rho - \frac{1}{1-\varepsilon c_d}) = \varepsilon \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}}
$$

Here we can take $\varepsilon > 0$ small s.t. $(1 - \varepsilon c_d) > 0$. Then we choose $\Lambda > 0$ large s.t. $\alpha_d \Lambda > \frac{1}{1-\varepsilon c_d}$

$$
\Rightarrow \; T \geq (\frac{\varepsilon}{1+\varepsilon}\frac{1}{\Lambda^{\frac{2}{d}}}) \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}}
$$

(good for $\int_{\mathbb{R}^d} \rho \geq \Lambda$)
Actually if $\int_{\mathbb{R}^d} \rho = N \leq \Lambda$, then we can use the simple lemma:

$$
T \geq \frac{c_d}{N^{\frac{2}{d}}} \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}} \geq \frac{c_d}{\Lambda^{\frac{2}{d}}} \int_{\mathbb{R}^d} \rho^{1+\frac{2}{d}}
$$

□

7.2 Littlewood-Paley decomposition

We write $f = \sum_{n\in\mathbb{Z}} f_n$, $\hat{f}_n(k)\mathbb{1}_{\{2^{n-1}\leq|k|\leq 2^n\}}$. (Similar to dyadic decomposition)

Theorem 7.9 (Bernstein inequality/Reverse Poincaré)**.** *supp* $\hat{f}_n \subset \underbrace{\{2^{n-1} \leq |k| \leq 2^n\}}_{\Omega}$, $\forall d \geq 1, \; \forall p \in [1,\infty]$:

$$
\tilde{c}_{d.p} \|f\|_{L^p(\mathbb{R}^d)} \leq 2^{-n} \|\nabla f_n\|_{L^p(\mathbb{R}^d)} \leq c_{d,p} \|f\|_{L^p(\mathbb{R}^d)}
$$

Proof. (d =2, Sketch)

$$\|\nabla f_n\|_{L^2}^2 = \int_{\mathbb{R}^d} |k|^2 \mathbb{1}_\Omega |\hat{f}_n(k)|^2 dk \leq \int_{\mathbb{R}^d} |k|^2 \chi(k) |\hat{f}_n(k)|^2 dk = \|\nabla(G \star f_n)\|_{L^2}^2$$

$$= \|(\nabla G) \star f_n\|_{L^2}^2 \overset{Young}{\leq} \|\nabla G\|_{L^1}^2 \, \|f_n\|_{L^2}^2$$

for $\hat{G}(k) = \chi(k) \simeq \mathbb{1}_{2^{n-1} \leq |k| \leq 2^n}$, $\chi, G \in C^\infty$ □

Smooth dyadic decomposition:

Take $\psi \in C_c^\infty(\mathbb{R}^d)$, $\begin{cases} \psi(k) = 1, \ if \ |k| \leq 1 \\ \psi(k) = 0, \ if \ |k| \geq 2 \end{cases}$

$\varphi(k) = \psi(k) - \psi(2k)$, $\varphi_n(x) = \psi(2^{-n}k) - \psi(2^{-n+1}k)$, $\forall n \in \mathbb{N}$, $\varphi_0 = \psi(x)$

$$\Rightarrow \sum_{n=0}^{N} \varphi_n(k) = \psi(2^{-N}k) \to 1, \ as \ N \to \infty$$

$$\Rightarrow 1 = \sum_{n=0}^{\infty} \varphi_n$$

Definition 7.10. $f = \sum_{n=0}^{\infty} f_n, \ \hat{f}_n(k) = \varphi_n(k)\hat{f}(k)$

Remark. If $f \in S(\mathbb{R}^d) \ \Rightarrow f_n \in S(\mathbb{R}^d), \ \forall n.$
$\|D^\alpha f_n\|_{L^p(\mathbb{R}^d)} \sim 2^{n|\alpha|} \, \|f_n\|_{L^p}$

Lemma 7.11. Let $0 \leq f \in L^1(\mathbb{R}^d)$, $\alpha > 0$, then we can find disjoint cubes $\{Q\} \subset \mathbb{R}^d$ s.t. $f = g_\alpha + b_\alpha = g_\alpha + \sum_Q (b_\alpha)|_Q$, where

1. $supp(g_\alpha) \subset \{\bigcup_Q Q\}^c$, $|g_\alpha| \lesssim \alpha$
2. $\forall Q: \ \int_Q b_\alpha = 0, \ ⨍_Q |b_\alpha| \lesssim \alpha$
3. $\sum_Q |Q| \lesssim \frac{c}{\alpha} \int_{\mathbb{R}^d} |f|$

Proof. (exercise) □

Lemma 7.12 (Hörmander)**.** Let φ and φ_n be as above. Define $K_n(x) := \check{\varphi}_n(x) = 2^{nd}\check{\varphi}(2^n x)$, then

$$\int_{|x|>2|y|} \underbrace{\|K_n(x-y) - K_n(x)\|_{l^2(n)}}_{=\sqrt{\sum_n |K_n(x-y)-K_n(x)|^2}} dx$$

Proof. We will use Minkowski's inequality: $\forall 1 \leq p \leq \infty$:

$$(\int_{\Omega_2} |\int_{\Omega_1} F(x,y)d\mu_1(x)|^p d\mu_2)^{\frac{1}{p}} \leq \int_{\Omega_1} (\int_{\Omega_2} |F(x,y)|^p d\mu_2(x))^{\frac{1}{p}} d\mu_1$$

with μ_1, μ_2 sigma finite.
Now consider

$$|K_n(x-y) - K_n(x)| \leq \int_0^{|y|} |\nabla K_n(x-te)|dt$$

where $e = \frac{y}{|y|}$

$$\begin{aligned}
\Rightarrow \qquad & \int_{|x|>2|y|} \sqrt{\sum_n |K_n(x-y) - K_n(x)|^2} dx \leq \int_{|x|>2|y|} \sqrt{\sum_n \int_0^{|y|} |\nabla K_n(x-te)|dt|^2} dx \\
\overset{Minkowski}{\leq} \quad & \int_{|x|>2|y|} \int_0^{|y|} \sqrt{\sum_n |\nabla K_n(\underbrace{x-te}_{z})|^2} dt dx \qquad ((d\mu_2, \Omega_2) \rightsquigarrow \text{ counting on } \mathbb{N}) \\
\leq \quad & \int_{|z|>|y|} \int_0^{|y|} \sqrt{\sum_n |\nabla K_n(z)|^2} dt dz \\
= \quad & |y| \int_{|z|>|y|} \sqrt{\sum_n |\nabla K_n(z)|^2} dz
\end{aligned}$$

Recall: $K_n(z) = 2^{nd}\check{\varphi}(2^n z), \varphi \in C_c^\infty \subset S(\mathbb{R}^d) \Rightarrow \check{\varphi} \in S(\mathbb{R}^d)$

$\Rightarrow |\nabla K_n(z)| \leq 2^{n(d+1)} |\nabla \check{\varphi}(2^n z)| \leq 2^{n(d+1)} \min(1, |2^n z|^{-2(d+2)})$

$\Rightarrow \sum_n |\nabla K_n(z)|^2 \leq \sum_n \min(2^{n(d+1)}, 2^{-2n}|z|^{-2(d+2)})$

$\leq \sum_{|n|\leq L} 2^{2n(d+1)} + \sum_{n \geq L} 2^{-2n} |z|^{-2(d+2)}$

$\lesssim 2^{2L(d+1)} + 2^{-2L}|z|^{-2L}|z|^{-2(d+2)} \overset{\text{opt over } L}{\rightsquigarrow} |z|^{-2(d+1)}$

$$\Rightarrow (LHS \ lemma) \leq |y| \int_{|z|\geq|y|} |z|^{-(d+1)} dz \lesssim C$$

□

Theorem 7.13 (L^p theory of Littlewood-Paley decomposition)**.** Let $f = \sum_{n=0}^{\infty} f_n$, then

$$C^{-1} \|f\|_{L^p(\mathbb{R}^d)} \leq \left\| \sqrt{\sum_{n=0}^{\infty} |f_n|^2} \right\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

Proof. $p = 2$:

$$\left\| \sqrt{\sum_n |f_n|^2} \right\|_{L^2}^2 = \int \sum_n |f_n|^2 = \int \sum_n |\hat{f}_n|^2 = \int \sum_n |\varphi_n(k)|^2 |\hat{f}(k)|^2 dk \sim \int |\hat{f}|^2 = \|f\|_{L^2}$$

The main part of the proof is to extend the bound to all $p \in (1, \infty)$.
Define $S(f)(x) = \sqrt{\sum_{n=0}^{\infty} |f_n(x)|^2}$, $f = \sum_n f_n$. We need to prove that $S : L^p \to L^p$ is bounded. We will prove this by real-interpolation:

- $|S(f+g)(x)| \leq S(f)(x) + S(g)(x)$ (Sub-additivity) ✓
- $S : L^2 \to L^2$ bounded ✓
- $S : L^1 \to L^{1,\infty}$ bounded (difficult)

From those, we find that $S : L^p \to L^p$ bounded $\forall p \in (1, 2]$. To get the result for $\infty > p > 2$, we use a duality argument. $\infty > p > 2 \Leftrightarrow 1 < p' < 2$

$$\begin{aligned} \|S(f)\|_{L^p} &\sim \sup |\sum_n \int_{\mathbb{R}^d} S(f)(x) h_n(x) dx| \\ &= \sup_{\sum_n \|h_n\|_{L^{p'}}^2 \leq 1} |\int_{\mathbb{R}^d} f(x) S^*(h_n) dx| \\ &\leq \sup_{\sum_n \|S^* h_n\|_{L^{p'}}^2 \leq 1} \sim \|f\|_{L^p} \end{aligned}$$

We used $S^* : L^{p'} \to L^{p'}$ bounded.
Let us prove the weak $(1,1)$ property.

$$|\{x : \ S(f)(x) > \alpha\}| \leq \frac{c}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}, \quad c = c_d \text{ (universal)}$$

We use the variant of Calderon-Zugmund from above. Using the sub-additivity $S(f) = S(g_\alpha + b_\alpha) \leq S(g_\alpha) + S(b_\alpha)$

$$\Rightarrow |\{x : \ S(f)(x) > \alpha\}| \leq |\{x : \ S(g_\alpha) > \frac{\alpha}{2}\}| + |\{x : \ S(b_\alpha) > \frac{\alpha}{2}\}|$$

good one: We use $S : L^2 \to L^2$ bounded and $|g_\alpha| \lesssim \alpha$

$$|\{x : \ S(g_\alpha)(x) > \frac{\alpha}{2}\}| \lesssim \int_{\mathbb{R}^d} \frac{|S(g_\alpha)(x)|^2}{\alpha^2} dx$$

$$\lesssim \int_{\mathbb{R}^d} \frac{|g_\alpha(x)|^2}{\alpha^2} dx \lesssim \int_{\mathbb{R}^d} \frac{|g_\alpha(x)|}{\alpha} dx \leq \frac{1}{\alpha} \int_{\mathbb{R}^d} |f|$$

bad one:

$b_\alpha = \sum_Q (b_\alpha)|_Q$, c_Q = center of Q. Define $\tilde{Q}$ = const. Q s.t. $c_{\tilde{Q}} = c_Q$ and $|x - c_Q| \geq 2|y - c_Q|$, $\forall x \in \tilde{Q},\ y \in Q$. Then:

$$|\bigcup_Q \tilde{Q}| \leq \sum_Q |\tilde{Q}| \lesssim \sum_Q |Q| \lesssim \frac{1}{\alpha}\int_{\mathbb{R}^d} |f|$$

Here, it suffices to bound:

$$|\{x:\ S(b_\alpha)(x) > \frac{\alpha}{2}\} \cap (\bigcup_Q \tilde{Q})^c| \lesssim \frac{1}{\alpha}\int_{(\bigcup_Q \tilde{Q})^c} S(b_\alpha)(x)dx$$

We need to prove $\int_{(\bigcup_Q \tilde{Q})^c} S(b_\alpha)(x)dx \lesssim \|f\|_{L^1}$.
We conclude the bound with Hörmander's lemma from above. So let $K_n(x) = \check{\varphi}_n(x) = 2^{nd}\check{\varphi}(2^n x)$ as in the lemma. Then: $\hat{f}_n(k) = \varphi_n(k)\hat{f}(k) \Rightarrow f_n(x) = (\check{\varphi}_n \star f)(x) = (K_n \star f)(x)$.
We prove $\int_{(\bigcup \tilde{Q})^c} S(b_\alpha)(x)dx \lesssim \|f\|_{L^1}$.
With Hörmander's lemma and Minkowski we get:

$$\int_{(\bigcup \tilde{Q})^c} S(b_\alpha)(x)dx = \int_{(\bigcup \tilde{Q})^c} \sqrt{\sum_n |K_n \star b_\alpha|^2} dx$$

$$= \int_{(\bigcup \tilde{Q})^c} \sqrt{\sum_n |\sum_Q (K_n \star b_{\alpha,Q})(x)|^2} dx, \qquad b_\alpha = \sum_Q b_{\alpha,Q}$$

Here note that $\int_Q b_{\alpha,Q} = 0$, hence:

$$(K_n \star b_{\alpha,Q})(x) = \int_{y\in Q} K_n(x-y)b_\alpha(y)dy = \int_{y\in Q} (K_n(x-c_Q-(y-c_Q)) - K_n(x-c_Q))b_\alpha(y)dy$$

Thus:

$$\int_{(\bigcup \tilde{Q})^c} S(b_\alpha)dx \overset{\text{C.S.ineq, Minkowski}}{\leq} \sum_Q \int_{(\bigcup \tilde{Q})^c} dx \int_Q |b_\alpha(y)| \sqrt{\sum_n |K_n(x-y) - K(x-c_Q)|^2} dy$$

$$\overset{\text{Fubini}}{\leq} \sum_Q \int_Q |b_{\alpha,Q}| \underbrace{\int_{|x-c_Q|>2|y-c_Q|} dx \sqrt{\sum_n |K_n(x-y) - K(x-c_Q)|^2}}_{\leq C<\infty \text{ (H lemma)}} dy \lesssim \sum_Q \int_Q |b_\alpha(y)|dy \lesssim \|f\|_{L^1}$$

Thus:

$$|\{x:\ S(b_\alpha) > \frac{\alpha}{2}\}| \leq |\{x:\ S(b_\alpha) > \frac{\alpha}{2}\} \cap (\bigcup \tilde{Q})| + |(\bigcup \tilde{Q})| \lesssim \frac{1}{\alpha}\|f\|_{L^1}$$

$$\Rightarrow \|S(f)\|_{L^{1,\infty}} \lesssim \|f\|_{L^1}$$

By interpolation: $\|S(f)\|_{L^p} \lesssim \|f\|_{L^p},\ \forall p \in (1,2]$
Duality:

$$\|(f_n)_n\|_{L^p,l^2(n)} = \left\|\sqrt{\sum_n |f_n(x)|^2}\right\|_{L^p(x)} = \sup_{\|h_n\|_{L^{p'},l^2(n)}\leq 1} |\sum_n \int (\bar{f}_n)(h_n)|$$

If $p > 2$:

$$\|S(f)\|_{L^p} = \|(f_n)\|_{L^p,l^2(n)} = \sup_{\|h_n\|_{L^{p'},l^2(n)} \leq 1} |\sum_n \int (\bar{f}_n)(h_n)| = \sup_{\|h_n\|_{L^{p'},l^2(n)} \leq 1} |\int f \sum_n (\varphi_n \hat{h}_n)^v|$$

$$\overset{\text{Hölder}}{\leq} \|f\|_{L^p} \sup \left\| \sum_n (\varphi_n \hat{h}_n)^v \right\|_{L^{p'}} \lesssim f_{L^p}$$

□

Remark. L^p theory for a function can be extended to orthonormal family to prove the Lieb Thirring inequality (Julien Sabin 2015)