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Phan Thành Nam Winter Semester 2023-2024

Fourier Analysis and Nonlinear PDE

Homework Sheet 10

(Released 26.1.2024 – Discussed 2.2.2024)

E10.1 Recall the Kato space for $p > 3$ and $T \le \infty$ with the norm

$$\|u\|_{K_p(T)} = \sup_{T \ge t > 0} t^{\frac{1}{2}\left(1-\frac{3}{p}\right)} \|u(t)\|_{L^p_x(\mathbb{R}^3)}.$$

(a) Show that for $T = \infty$ this norm is invariant under the scaling $u_\lambda(x) = \lambda u(\lambda^2 t, \lambda x)$.

(b) Show that $\|e^{t\Delta} u_0\|_{K_p(\infty)}$ is equivalent to the norm of u_0 in the Besov space $\dot{B}^{\frac{3}{p}-1}_{p,\infty}(\mathbb{R}^3)$.

(c) Show that

$$\lim_{T\to 0} \|e^{t\Delta} u_0\|_{K_p(T)} = 0 \quad \forall u_0 \in L^3 \cap L^p.$$

Deduce the same result for all $u_0 \in L^3$. Hint: You can use $\|e^{t\Delta} u_0\|_{K_p(\infty)} \le C\|u_0\|_{L^p}$.

E10.2 Let $\varphi \in \mathcal{S}(\mathbb{R}^3)$ and define $\varphi_n(x) = e^{inx_1}\varphi(x)$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Show that when $n \to \infty$ we have

$$\|\varphi_n\|_{\dot{H}^{1/2}(\mathbb{R}^3)} \to \infty, \quad \|e^{t\Delta}\varphi_n\|_{K^p(\infty)} \to 0$$

for every $p > 3$. Thus $K^p(\infty)$-norm is much weaker than the energy norm $L^\infty_t \dot{H}^{1/2}_x$.

E10.3 Let B be a Banach space which is continuously embedded in $\mathcal{S}'(\mathbb{R}^3)$. Assume that B is invariant under the scaling $f_{\lambda,a}(x) = \lambda f(\lambda(x-a))$, namely

$$\|f_{\lambda,a}\|_B = \|f\|_B, \quad \lambda > 0, a \in \mathbb{R}^3.$$

Show that $B \subset \dot{B}^{-1}_{\infty,\infty}$. This means that there is no hope to solve the Navier–Stokes equation on a space larger than the homogeneous Besov space $\dot{B}^{-1}_{\infty,\infty}$.

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Homework Sheet 9

(Released 20.1.2024 – Discussed 26.1.2024)

E9.1 (a) Prove that if u is a solution to the Navier–Stokes equation

$$\partial_t u - \Delta u + \mathrm{div}(u \otimes u) = 0, \quad \mathrm{div} u = 0, \quad t > 0, x \in \mathbb{R}^d$$

then $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$, $\lambda > 0$, is also a solution.

(b) Prove that the following energy spaces are invariant under the scaling $u \mapsto u_\lambda$:

$$L^\infty_t \dot{H}_x^{\frac{d}{2}-1}, \quad L^\infty_4 \dot{H}_x^{\frac{d-1}{2}}, \quad L^t_2 \dot{H}_x^{\frac{d}{2}}.$$

E9.2 Prove that if $f \in L^2_T H^{s-1}_x(\mathbb{R}^d)$, then

$$g(t) = \int_0^t e^{(t-t')\Delta} f(s) \mathrm{d}s, \quad t \in [0,T]$$

belongs to $L^4_T \dot{H}_x^{s+\frac{1}{2}}$.

Hint: In the class we have proved that $g \in L^\infty_T \dot{H}^s_x \cap L^2_T \dot{H}^{s+1}_x$.

E9.3 Let $u_0 \in H^{1/2}(\mathbb{R}^3)$ (not only $\dot{H}^{1/2}(\mathbb{R}^3)$) with $\mathrm{div}(u_0) = 0$. Assume that the Navier–Stokes equation has a global solution $u(t)$ with the initial datum u_0.

(a) Use Leray energy condition and Sobolev embedding theorem to show that

$$\int_0^\infty \|u(t)\|^4_{\dot{H}^{1/2}(\mathbb{R}^d)} \mathrm{d}t < \infty.$$

(b) Show that $\|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^d)} \to 0$ as $t \to \infty$.

Note: These results also hold if $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, but the proof of (a) is more difficult.

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Homework Sheet 8

(Released 21.12.2023 – Discussed 19.1.2024)

E8.1 Prove that for every $s \in \mathbb{R}$, the Besov space $B^s_{2,2}(\mathbb{R}^d)$ coincides with the Sobolev space $H^s(\mathbb{R}^d)$.

E8.2 Prove the Sobolev embedding for nonhomogeneous Besov spaces:

$$B^s_{p_1,r_1}(\mathbb{R}^d) \subset B^{s'}_{p_2,r_2}(\mathbb{R}^d), \quad s' = s - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$$

for all $s \in \mathbb{R}$ and $1 \le p_1 \le p_2 \le \infty$, $1 \le r_1 \le r_2 \le \infty$.

E8.3 Prove that if $s > 0$, then for all $p, r \in [1, \infty]$ we have $B^s_{p,r} = \dot{B}^s_{p,r} \cap L^p$.
Hint: For the inclusion $B^s_{p,r} \subset L^p$, it suffices to consider $r = \infty$ and use $u = \sum_{j \ge -1} \Delta_j u$.

E8.4 (Hard) Prove that for all $d \ge 1$, $s \in \mathbb{R}$, $p \in [1, \infty]$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\|\phi u\|_{B^s_{p,\infty}(\mathbb{R}^d)} \lesssim_{d,s,p,\phi} \|u\|_{B^s_{p,\infty}(\mathbb{R}^d)}.$$

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Homework Sheet 7

(Released 2.12.2023 – Discussed 15.12.2023)

E7.1 Consider the inhomogeneous heat equation

$$\partial_t u(t,x) = \Delta u(t,x) + f(t,x), \quad x \in \mathbb{R}^d, t > 0,$$

with initial datum $u_0 = 0$ and the function f satisfies $\operatorname{supp} \hat{f} \subset \{\lambda \leq |k| \leq 2\lambda\}$. Prove that for all parameters $1 \leq a \leq b \leq \infty$, $1 \leq p \leq q \leq \infty$, we have

$$\|u\|_{L^q_t L^b_x} \leq C\lambda^\alpha \|f\|_{L^p_t L^a_x}$$

where

$$\alpha = 2\left(\frac{1}{p} - \frac{1}{q} - 1\right) + d\left(\frac{1}{a} - \frac{1}{b}\right).$$

E7.2 Let $u \in L^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that $\operatorname{supp}\hat{u} \subset \{\lambda \leq |k| \leq 2\lambda\}$. Prove that

$$\|u^N\|_{L^2} \leq C\lambda^{-1}\|\nabla(u^N)\|_{L^2}.$$

Note that the Fourier transform of u^N is not necessarily supported in an annulus.

E7.3 Let $u \in \mathcal{S}'_h(\mathbb{R}^d)$ and $u_N(x) = u(2^N x)$. Prove that for all $s \in \mathbb{R}$ and $p, r \in [1,\infty]$ we have

$$\|u_N\|_{\dot{B}^s_{p,r}} = 2^{N\left(s - \frac{d}{p}\right)}\|u\|_{\dot{B}^s_{p,r}}.$$

Hint: In the lecture we proved the L^2 case by Plancherel's theorem. Now you may perform the scaling with the convolution in x-space.

E7.4 Explain why we do not expect any inclusion between $\dot{B}^{s_1}_{p,r}$ and $\dot{B}^{s_2}_{p,r}$ if $s_1 \neq s_2$.

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Homework Sheet 6

(Released 24.11.2023 – Discussed 1.12.2023)

E6.1 Prove that $(L^q_t L^r_x)' = L^{q'}_t L^{r'}_x$ for all $q, r \in (1,\infty)$, where

$$\|f\|_{L^q_t L^r_x} = \|\|f(t,x)\|_{L^r_x(\mathbb{R}^d)}\|_{L^q_t(\mathbb{R})}.$$

E6.2 Prove that for all $d \ge 1$ and $p \in (2,\infty)$,

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{R}^d)} \le t^{-d(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L^{p'}(\mathbb{R}^d)}, \quad \forall u_0 \in L^{p'}(\mathbb{R}^d).$$

Hint: You can use Riesz–Thorin interpolation theorem.

E6.3 Let $d \ge 1$. Explain why

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

is a necessary condition to have the Strichartz estimate

$$\|e^{it\Delta}u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)}.$$

Hint: You can consider $u_\lambda(x) = \lambda^{d/2}u_0(\lambda x)$.

E6.4 Assume that the cubic NLS

$$i\partial_t u = -\Delta u + |u|^2 u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3, \quad u_0 \in H^1(\mathbb{R}^3),$$

has a global solution $u(t,x)$ which satisfies

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^3)} = \|u\|_{S^0} + \|\nabla u\|_{S^0} < \infty, \quad \|u\|_{S^0} = \|u\|_{L^\infty_t L^3_x} + \|u\|_{L^4_t L^3_x}.$$

(a) Prove that we have the (uniform in time) bound

$$\left\|\int_0^t e^{i(t-s)\Delta}(|u|^2 u)(s)\mathrm{d}s\right\|_{L^2(\mathbb{R}^d)} \lesssim \|u\|^3_{S^1}.$$

(b) Prove that there exist limits $u_\pm \in L^2(\mathbb{R}^3)$ such that

$$\|e^{-it\Delta}u(t,x) - u_\pm(x)\|_{L^2(\mathbb{R}^3)} \to 0, \quad t \to \pm\infty.$$

(c) Prove that

$$\|e^{-it\Delta}u(t,x) - u_\pm(x)\|_{H^1(\mathbb{R}^3)} \to 0, \quad t \to \pm\infty.$$

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Homework Sheet 5

(Released 18.11.2023 – Discussed 24.11.2023)

E5.1 Let $d \geq 3$ and $u \in \dot{H}^1(\mathbb{R}^d)$ is radially symmetric decreasing, namely $u(x) = f(|x|)$ where $t \mapsto |f(t)|$ is decreasing for $t \in (0, \infty)$. Use Hardy's inequality to prove Sobolev's inequality

$$\|u\|_{L^{2^*}(\mathbb{R}^d)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^d)}, \quad 2^* = \frac{2d}{d-2}.$$

E5.2 Let $d \geq 1$ and $u_0 \in H^1(\mathbb{R}^d)$. Prove that for all $t > 0$,

$$\|e^{t\Delta}u_0 - u_0\|_{L^2} \leq C\sqrt{t}\|u_0\|_{H^1}, \quad \|e^{it\Delta}u_0 - u_0\|_{L^2} \leq C\sqrt{t}\|u_0\|_{H^1}.$$

E5.3 Consider the nonlinear Schrödinger equation in 1D in Duhamel's form

$$u(t,.) = e^{it\Delta}u_0(.) + \int_0^t e^{i(t-s)\Delta}\Big(|u(s,.)|^2 u(s,.)\Big)ds, \quad u_0 \in H^1(\mathbb{R}^d).$$

(a) Prove that for every $u_0 \in H^1(\mathbb{R})$, there exist $-T_* < 0 < T^*$ and a unique local solution $u(t,.) \in H^1(\mathbb{R})$ for $t \in (-T_*, T^*)$. Hint: The local Lipschitz condition can be deduced using

$$\||f|^2 f - |g|^2 g\|_{H^1(\mathbb{R})} \leq C\|f - g\|_{H^1(\mathbb{R})}(\|f\|^2_{H^1(\mathbb{R})} + \|g\|^2_{H^1(\mathbb{R})}).$$

(b) Prove the conservation laws for $t \in (-T_*, T^*)$

$$\|u(t,.)\|_{L^2} = \|u_0\|_{L^2}, \quad \mathcal{E}(u(t,.)) := \int_{\mathbb{R}} \Big(|\nabla u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4 \mathrm{d}x\Big) = \mathcal{E}(u_0).$$

Deduce that $\|u(t,.)\|_{H^1(\mathbb{R})}$ is uniformly bounded in t.

(c) Prove that there exists a unique global solution $u(t,.) \in H^1(\mathbb{R})$ for $t \in (-\infty, \infty)$.

E5.4 Let $d \geq 1$ and let $f \in S(\mathbb{R}^d)$ such that $\mathrm{supp}\, \hat{f} \subset \{1 \leq |k| \leq 2\}$.

(a) Prove that for every $n \in \mathbb{N}$, we can write

$$f = \sum_{|\alpha|=n} g_\alpha * D^\alpha f$$

where $\hat{g}_\alpha(k) = c_\alpha k^\alpha |k|^{-2n}\chi(k)$ with $c_\alpha \in \mathbb{C}$, $\chi \in C_c^\infty(\mathbb{R}^d \backslash \{0\})$, $\chi = 1$ on $\{1 \leq |k| \leq 2\}$.

(b) Deduce that

$$\sup_{|\alpha|=n} \|D^\alpha f\|_{L^p(\mathbb{R}^d)} \geq C\|f\|_{L^p(\mathbb{R}^d)}$$

for all $p \geq 1$, where the constant C is independent of f and p.

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Homework Sheet 4

(Released 11.11.2023 – Discussed 17.11.2023)

E4.1 (Hölder continuity) Let $f \in H^s(\mathbb{R}^d)$ with $s > d/2$. Prove that f is Hölder continuous, namely there exist constants $\alpha > 0$ and $C > 0$ such that

$$|f(x) - f(y)| \le C|x-y|^\alpha, \quad \forall x, y \in \mathbb{R}^d.$$

E4.2 (Perron-Frobenius principle). Let Ω be an open subset of $\mathbb{R}^d$. Let $V \in L^1(\Omega)$. Assume that the Schrödinger equation

$$(-\Delta - V(x))\psi(x) = 0, \quad x \in \Omega,$$

has a positive solution $0 < \psi \in C^2(\Omega)$. Prove that

$$\int_\Omega |\nabla \varphi(x)|^2 \mathrm{d}x - \int_\Omega V(x)|\varphi(x)|^2 \mathrm{d}x \ge 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Hint: You may consider the function $g = \varphi/\psi$.

E4.3 (Hardy inequality) Let $d \ge 3$ and $\Omega = \mathbb{R}^d \backslash \{0\}$.

(a) Find a positive solution $0 < \psi \in C^2(\Omega)$ for the Schrödinger equation

$$\left(-\Delta - \frac{(d-2)^2}{4|x|^2}\right)\psi(x) = 0.$$

(b) Use the Perron-Frobenius principle and a density argument to conclude that

$$\int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 \mathrm{d}x \ge \frac{(d-2)^2}{4} \int_\Omega \frac{|\varphi(x)|^2}{|x|^2} \mathrm{d}x, \quad \forall \varphi \in \dot{H}^1(\mathbb{R}^d).$$

E4.4. For every open set $\Omega \subset \mathbb{R}^d$, we denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega) \subset C_c^1(\mathbb{R}^d)$ under the $H^1(\mathbb{R}^d)$-norm. Let $B : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace operator.

(a) Prove that $Bu = 0$ on $\partial\Omega$ for every $u \in H_0^1(\Omega)$.

(b) Let $u \in H^1(\mathbb{R})$. Prove that $u_{|\Omega} \in H_0^1(\Omega)$ with $\Omega = (0,1)$ iff $u(0) = u(1) = 0$.

E4.5. Let $g(t) = \exp(-t^{-2})$, $t > 0$. Prove that $\partial_t^n g(t) = P_n(t^{-1})g(t)$ where P_n is a polynomial of degree 3n satisfying

$$P_{n+1}(t^{-1}) = (P_n(t^{-1}))' + 2t^{-3}P_n(t^{-1})$$

Deduce that

$$|P_n(s)| \le \max_{0 \le k \le n} 2^{n+k}(3n)^{n-k} s^{n+2k}, \quad \forall s > 0.$$

Hint: $P_n(t^{-1})$ can be obtained from $P_0 = 1$ by applying (n times) either ∂_t or $2t^{-3}$.

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Homework Sheet 3

(Released 4.11.2023 – Discussed 10.11.2023)

E3.1. Let $s>0$. Prove that the following two statements are equivalent:

(a) $f_n \rightharpoonup f$ weakly in $H^s(\mathbb{R}^d)$ when $n\to\infty$;

(b) $f_n \rightharpoonup f$ weakly in $L^2(\mathbb{R}^d)$ and $\{f_n\}_{n=1}^\infty$ is bounded in $H^s(\mathbb{R}^d)$.

E3.2. Prove that $H^s(\mathbb{R}^d)\subset L^\infty(\mathbb{R}^d)$ with continuous embedding if $s>d/2$. Prove that it fails to hold if $s=d/2$.

E3.3. Let $s=d/2$. Prove that there exist constant $\varepsilon>0, C>0$ depending only on d such that the Moser-Trudinger inequality holds

$$\int_{\mathbb{R}^d} (e^{\varepsilon |f(x)|^2}-1)\mathrm{d}x \le C, \qquad \forall \|f\|_{H^s}\le 1.$$

Hint: You can mimic the Chemin-Xu's proof of Sobolev's inequality.

E3.4. Let $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Prove that if $\{f_n\}_{n=1}^\infty$ is bounded in $H^s(\mathbb{R}^d)$ for some $s\in\mathbb{R}$, then

$$\sup_{|k|\le R}\sup_{n\ge 1} |\widehat{\varphi f_n}(k)|<\infty.$$

(We used this bound in the proof of Sobolev's compact embedding.)

E3.5. Use Sobolev's compact embedding to deduce the following statement: If $f_n \rightharpoonup f$ weakly in $H^s(\mathbb{R}^d)$ for some $s>0$, then up to a subsequence, we have

- $\mathbb{1}_{B_R} f_n \to \mathbb{1}_{B_R} f$ strongly in $L^2(\mathbb{R}^d)$ for all $R>0$; and
- $f_n(x)\to f(x)$ for a.e. $x\in\mathbb{R}^d$.

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Fourier Analysis and Nonlinear PDE

Homework Sheet 2

(Released 27.10.2023 – Discussed 3.11.2023)

E2.1. Let $d \geq 1$ and $s \in (0,1)$. Prove that
$$\int_{\mathbb{R}^d} \frac{|e^{ik\cdot y} - 1|^2}{|y|^{d+2s}} = C_{d,s}|k|^{2s}$$
for a constant $C_{d,s} > 0$ independent of $k \in \mathbb{R}^d$.

E2.2. Let $-\infty < r < s < \infty$.

(a) Prove that $\dot{H}^s(\mathbb{R}^d)$ and $\dot{H}^r(\mathbb{R}^d)$ cannot be compared for the inclusion.

(b) Prove that $\Big(\dot{H}^s(\mathbb{R}^d) \cap \dot{H}^r(\mathbb{R}^d)\Big) \subset \dot{H}^p(\mathbb{R}^d)$ for all $p \in (r,s)$.

E2.3. We say that a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is compactly supported if there exists $R > 0$ such that
$$f(\varphi) = 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) \text{ such that } \operatorname{supp}(\varphi) \subset B_R^c = \mathbb{R}^d \backslash B_R.$$
Prove that the set of compactly supported tempered distributions is dense in $\mathcal{S}'(\mathbb{R}^d)$.

E2.4. Let $f \in \dot{H}^s(\mathbb{R}^d)$ with $s < d/2$. Prove that there exists a sequence $\{f_n\}_{n=1}^\infty \subset \dot{H}^s(\mathbb{R}^d)$ such that $f_n \to f$ in $\dot{H}^s(\mathbb{R}^d)$ as $n \to \infty$, and for every n we have
$$\hat{f}_n(k) = 0, \quad \forall |k| \leq n^{-1}.$$
Hint: You can use the fact that $\dot{H}^s(\mathbb{R}^d)$ is a Hilbert space.

E2.5. Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then we know that $\partial_{x_1} f \in \mathcal{D}'(\mathbb{R}^d)$ is well-defined. Prove or disprove the following: if $\partial_{x_1} f \in \mathcal{S}'(\mathbb{R}^d)$, then $f \in \mathcal{S}'(\mathbb{R}^d)$.

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Fourier Analysis and Nonlinear PDE

Homework Sheet 1

(Released 20.10.2023 – Discussed 27.10.2023)

E1.1. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g = 1$. Define $g_n(x) = n^d g(nx)$.

(a) Prove that $g_n * f \to f$ in $\mathcal{S}(\mathbb{R}^d)$ as $n \to \infty$.

(b) Prove that the condition $g \in \mathcal{S}(\mathbb{R}^d)$ can be replaced by the weaker condition that $g \in L^1(\mathbb{R}^d)$ and $g(x)$ decays faster than any polynomial at infinity.

E1.2. Prove that $C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$.

E1.3. Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ be a linear mapping. Prove that $T \in \mathcal{S}'(\mathbb{R}^d)$ if and only if there exist $k \in \mathbb{N}$ and $C \in (0, \infty)$ such that

$$|T(\varphi)| \le C \|\varphi\|_{k,\mathcal{S}}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

E1.4. (a) Let $f(x) = \sum_{|\alpha| \le N} c_\alpha x^\alpha$ be a polynomial with variable $x \in \mathbb{R}^d$ and coefficients $c_\alpha \in \mathbb{C}$. Let $g \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$. Prove that $fg \in \mathcal{S}'(\mathbb{R}^d)$.

(b) Prove that the function $f(x) = e^{|x|}$, $x \in \mathbb{R}^d$, is not an element of $\mathcal{S}'(\mathbb{R}^d)$.

E1.5. Let $x_0 \in \mathbb{R}^d$ and denote the Dirac delta function δ_{x_0} as

$$\delta_{x_0}(\varphi) = \varphi(x_0), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(a) Prove that $\delta_{x_0} \in \mathcal{S}'(\mathbb{R}^d)$.

(b) Compute the Fourier transform of δ_{x_0}.

E1.6. Prove that the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is bijective.