Chapter 1: Laplace / Poisson equation

$$-\Delta u = 0 \quad \text{or} \quad -\Delta u = f$$

Laplace Poisson

Def. let Ω open $\subset \mathbb{R}^d$. any $u \in C^2(\Omega)$ and $\Delta u = 0$ on Ω is a <u>harmonic</u> function on Ω

<u>Relation to complex analysis.</u>

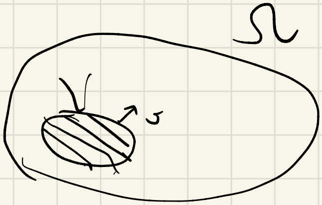
$u: \mathbb{C} \to \mathbb{C}$ analytic function

$$\rightarrow (\partial_{xx} + \partial_{yy}) \operatorname{Re} u = 0$$

or $\operatorname{Im} u$ (Tutorial)

$(\mathbb{C} \simeq \mathbb{R}^2)$

<u>Physical interpretation.</u> $\Omega \subset \mathbb{R}^d$



$u =$ density

In <u>equilibrium</u>, $\forall V \subset \Omega$

$\int_{\partial V} F \cdot \vec{\nu} \, dS = 0$ (flux of u through $V = 0$)

(Exercise) Gauss-Green $=$ $\int_V \operatorname{div}(F) \, dx$

$F = -a \nabla u$, $a > 0$

$$\Rightarrow \int_V (\Delta u)(x)\,dx = 0 \quad , \forall\, V \underset{\text{open}}{\subset} \Omega$$

$$\Rightarrow \quad \Delta u(x) = 0$$ (Fundamental theorem Calculus of variations)

(Exercise)

<u>Exercise</u> (Gauss - Green formula)

$$\int_{\partial V} F \cdot \vec{\nu}\, dS = \int_V \operatorname{div}(F)(x)\,dx$$

$$, V \text{ open} \subset \mathbb{R}^d$$

<u>Exercise</u>: If $u \in C(\Omega)$ and

$$\int_V u(x)\,dx = 0, \ \forall\, V \text{ open ball in } \Omega$$

$\Rightarrow u = 0$ almost everywhere.

<u>Remark</u>: The result holds also if $u \in L^1(\Omega)$.

Can you prove that?

Fundamental solution of Laplace equation:

$$\Delta u = 0 \quad \text{on } \mathbb{R}^d$$

Assume that u is radial; $\quad u(x) = v(r)$

with $r = |x| = \sqrt{x_1^2 + \dots + x_d^2}$. Then:

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$\partial_{x_i} u = v'(r)\, \frac{x_i}{r}$$

$$\partial_{x_i}^2 u = v''(r)\, \frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

$$\Rightarrow \quad \Delta u = v''(r) + \frac{d-1}{r} v'(r) = 0$$

$$\Rightarrow \quad \log(v')' = \frac{v''}{v'} = \frac{d-1}{r}$$

$$\Rightarrow \quad v' = \frac{\text{const}}{r^{d-1}}$$

$$\Rightarrow \quad v = \begin{cases} \text{const} \log r + \text{const} & (d=2) \\ \text{const} \dfrac{1}{r^{d-2}} + \text{const} & (d=3) \end{cases}$$

Def. Fundamental solution of Laplace equation

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & d=2 \\ \dfrac{1}{4\pi|x|} & (d=3) \\ \dfrac{1}{d(d-2)\,|B_1|\,|x|^{d-2}} & (d\geq 3) \end{cases}$$

Remark: The fundamental solution Φ is radial and satisfies

$$\Delta\Phi(x) = 0 \ , \ \forall x \in \mathbb{R}^d, \ x \neq 0$$

$$|\nabla\Phi(x)| \leq \frac{C}{|x|^{d-1}} \ , \quad |D^2\Phi(x)| \leq \frac{C}{|x|^d} \ , \forall x \neq 0$$

Remark: We do not have $\Delta\Phi = 0$ on the whole $\mathbb{R}^d$. Indeed, we will see that

$$-\Delta\Phi(x) = \delta_0(x) \rightsquigarrow \text{Dirac-delta function}$$

formally $\delta_0(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$ & $\int_{\mathbb{R}^d} \delta_0 = 1$

$\rightsquigarrow$ this will make sense in distribution theory (we discuss later)

Poisson equation:

$$-\Delta u(x) = f(x) \text{ on } \mathbb{R}^d$$

Solution: Let Φ be the fundamental solution of Laplace equation. The solution of $-\Delta u = f$ is

$$u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) \, dy$$

Theorem: If $f \in C_c^2(\mathbb{R}^d)$, then $u \in C^2(\mathbb{R}^d)$ and $\Delta u = f$ in $\mathbb{R}^d$.

Proof: We use the definition:

$$u(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) \, dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y) \, dy$$

Check: u is continuous. Take $x_n \to x$

$\rightarrow$ $u(x_n) \to u(x)$ by Dominated c.v.

[Recall measure theory, Monotone / Dominated c.v. theorem, ...]

Compute the derivatives:

$$\frac{u(x+he_i) - u(x)}{h} = \int \Phi(y) \frac{f(x+he_i-y) - f(x-y)}{h} dy$$

$$\xrightarrow[h\to 0]{} \int \Phi(y) \frac{\partial f}{\partial x_i}(x)\, dy$$

(Dominated C.V. again)

Similarly

$$D^\alpha u(x) = \int \Phi(y) (D^\alpha f)(x-y)\, dy$$

$\in$ Continuous! $\forall |\alpha| \le 2$

Thus $u \in C^2(\mathbb{R}^d)$ since $f \in C_c^2(\mathbb{R}^d)$.

Why $\Delta u = f$?

$$-\Delta_x u = \int_{\mathbb{R}^d} \Phi(y) (-\Delta_x) f(x-y)\, dy$$

$$= \int_{\mathbb{R}^d} \Phi(y)(-\Delta_y) f(x-y)\, dy = \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} + \int_{B(0,\varepsilon)}$$

The main part:

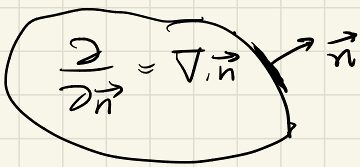
$$\int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \Phi(y)\,(-\Delta_y) f(x-y)\,dy$$

$$= \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \Phi(y)\,(-\Delta_y) f(x-y)\,dy$$

$$= \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} (\nabla_y \Phi)(y)\,(\nabla_y f)(x-y)\,dy$$

$$- \int_{\partial B(0,\varepsilon)} \Phi(y)\,\frac{\partial f}{\partial \vec{n}}(x-y)\,dS(y)$$

$$= \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \overbrace{(-\Delta \Phi)(y)}^{=0}\, f(x-y)\,dy$$



$$+ \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi}{\partial \vec{n}}(y)\, f(x-y)\,dS(y)$$

$$- \int_{\partial B(0,\varepsilon)} \Phi(y)\,\frac{\partial f}{\partial \vec{n}}(x-y)\,dS(y)$$

Direct computation:

$$\begin{cases} \nabla \Phi = -\dfrac{1}{d\,|B_1|} \cdot \dfrac{y}{|y|^d} \\ \vec{n} = -\dfrac{y}{|y|} \quad \text{on } \partial B(0,\varepsilon) \end{cases}$$

$$\Rightarrow \frac{\partial \Phi}{\partial \vec{n}} = \frac{1}{d\,|B_1|\,\varepsilon^{n-1}} \quad \text{on } \partial B(0,\varepsilon)$$

$$\Rightarrow \int_{\partial B(0,\varepsilon)} \frac{\partial \Phi(y)}{\partial \vec{n}} f(x-y)\, dS(y)$$

$$= \int_{\partial B(0,\varepsilon)} \frac{1}{d\,|B_1|\,\varepsilon^{n-1}} f(x-y)\, dS(y)$$

$$= \underbrace{⨍_{\partial B(0,\varepsilon)}}_{\text{mean value / average integral}} f(x-y)\, dS(y) \xrightarrow[\varepsilon \to 0]{} f(x)$$

On the other hand:

$$\left| \int_{\partial B(0,\varepsilon)} \Phi \frac{\partial f}{\partial \nu}(x-y)\, dS(y) \right|$$

$$\leq C\|\nabla f\|_\infty \int_{\partial B(0,\varepsilon)} |\Phi| \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

Similarly:

$$\left| \Delta_x \int_{B(0,\varepsilon)} \Phi(y) f(x-y) \right|$$

$$= \left| \int_{B(0,\varepsilon)} \Phi \, \Delta_x f(x-y) \right|$$

$$\leq \|\Delta f\|_{L^\infty} \int_{B(0,\varepsilon)} |\Phi| \rightarrow 0$$

$C\varepsilon$ if $d \geq 3$

$C\varepsilon|\ln \varepsilon|$ $d=2$

Thus we conclude that

$$\Delta u(x) = f(x), \quad \forall x \in \mathbb{R}^d$$

if $u = \Phi * f$ and $f \in C_c^2(\mathbb{R}^d)$.

Harmonic functions in a domain $\Omega \subset \mathbb{R}^d$

Let Ω open $\subset \mathbb{R}^d$. Let $u \in C^2(\Omega)$ and

$$\Delta u = 0 \quad \text{in } \Omega.$$

Theorem: (Mean-value formula for harmonic function)

If $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω, then:

$$u(x) = \fint_B u = \fint_{\partial B} u \quad , \forall \text{ ball } B \subset \Omega.$$

Proof: In 1D, $\Delta u = 0 \to u$ is linear $\to$ obvious.

In general case: consider

$$f(r) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{\partial B(0,1)} u(x+rz)\, dS(z)$$

$$\to f'(r) = \fint_{\partial B(0,1)} \nabla u(x+rz) \cdot z \, dS(z)$$

$$= \fint_{\partial B(x,r)} \nabla u(y) \frac{y-x}{r} \, dS(y)$$

$$\dots = \fint_{\partial B(x,r)} \frac{\partial u}{\partial \vec{n}} \, dS(y)$$

Green's formula

$$= \frac{r}{d} \fint_{B(x,r)} \Delta u(y) \, dy = 0$$

$$\Rightarrow f(r) = \text{const} \Rightarrow f(r) = \lim_{t \to 0} f(t) = u(x).$$

Consequently, by polar coordinates:

$$\int_{B(x,r)} u \, dy = \int_{B(0,r)} u(x+y) \, dy$$

$$= \int_0^r \left(\int_{\partial B(0,s)} u \right) ds$$

$$= \int_0^r |\partial B(0,s)| \, u(x) \, ds$$

$$= |B(0,r)| \, u(x) = |B(x,r)| \, u(x). \square$$

Reverse $\rightsquigarrow$ exercise!!

Theorem (Maximum principle) Ω open $\subset \mathbb{R}^d$.
Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then:

a) $$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

b) Assume Ω is connected. If $\exists\, x_0 \in \Omega$ s.t.
$$u(x_0) = \max_{\overline{\Omega}} u$$
Then $u \equiv$ const in Ω.

Proof: b) Assume $\exists\, x_0 \in \Omega$ s.t.
$$u(x_0) = \max_{\overline{\Omega}} u.$$
Then $\forall$ ball B : $x_0 \in B \subset \Omega$ we have
$$u(x_0) = \fint_B u(x)\,dx \leq \max_{\overline{\Omega}} u = u(x_0)$$
$$\Rightarrow \quad u(x) = u(x_0), \ \forall x \in B$$
The set $\{x : u(x) = u(x_0)\}$ is both open & closed within Ω $\rightarrow$ it is Ω as Ω is connected.

(b) $\Rightarrow$ (a) ✓

Theorem (Uniqueness) Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$. Then $\exists$ at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof. By linearity $\rightsquigarrow f = g = 0$.

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \Longrightarrow u = 0 \text{ in } \Omega.$$

Exercise: Assume Ω open, connected $\subset \mathbb{R}^d$.
Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Prove that 1) if $g \geq 0$, then $u \geq 0$ in Ω

2) if $g \geq 0$ and $g \not\equiv 0$, then $u > 0$ in Ω.

<u>Def:</u> Let Ω open $\subset \mathbb{R}^d$ and $u \in C^2(\Omega)$.

- u is <u>subharmonic</u> if $\Delta u \geq 0$ in Ω
- u is superharmonic if $\Delta u \leq 0$ in Ω.

<u>Remark:</u> In 1D, subharmonic = convex

<u>Exercise:</u> let Ω open $\subset \mathbb{R}^d$ and $u \in C^2(\Omega)$, $\Delta u \geq 0$ (i.e. u is subharmonic).

(a) Prove the mean-value inequality

$$\fint_{\partial B(x,r)} u(y)\,dS(y) \geq \fint_{B(x,r)} u(y)\,dy \geq u(x)$$

for all $x \in B(x,r) \subset \Omega$.

(b) Assume that Ω is connected and $u \in C(\bar{\Omega})$.
Prove the strong maximum principle, i.e. either

- $u =$ const in Ω, or
- $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.

<u>Theorem:</u> (Regularity) If $u \in C(\Omega)$ and

$$u(x) = \fint_{\partial B} u \,, \quad \forall \text{ ball } B \subset \Omega .$$

Then $u \in C^2(\Omega)$ and $\Delta u = 0$, i.e. u is harmonic.

Moreover, $u \in C^\infty(\Omega)$ and u is analytic in Ω.

<u>Proof:</u> Let $\eta \in C_c^\infty(\mathbb{R}^d)$, radial, $\int \eta = 1$.

$$\eta_\varepsilon = \varepsilon^{-d} \eta(\varepsilon^{-1} x) \qquad \eta = 0 \text{ if } |x| \geq 1$$

$$u_\varepsilon = \eta_\varepsilon * u \in C^\infty(\Omega_\varepsilon)$$

where $\Omega_\varepsilon = \{x \in \Omega ; \ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Then:
$$u_\varepsilon(x) = \int_\Omega \eta_\varepsilon(x-y)\, u(y)\, dy$$

$$= \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\, dy$$

$$= \frac{1}{\varepsilon^d} \int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \underbrace{\left(\int_{\partial B(x,r)} u \, dS\right)}_{u(x)\,|\partial B(x,r)|} dr$$

$$= u(x) \int_{B(0,\varepsilon)} \eta_\varepsilon \, dy = u(x).$$

Since $u_\varepsilon \in C^\infty(\Omega_\varepsilon) \Rightarrow u \in C^\infty(\Omega_\varepsilon), \forall \varepsilon.$

To prove that u is analytic in Ω, we need to show that $\forall x_0 \in \Omega$, $\forall r > 0$ s.t. $\forall x \in B(x_0, r)$

$$u(x) = u(x_0) + \sum_{\alpha \neq 0} c_\alpha (x - x_0)^\alpha$$

where $y^\alpha = y_1^{\alpha_1} \cdots y_d^{\alpha_d}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and series converge absolutely, i.e

$$\sum_\alpha c_\alpha r^{|\alpha|} < \infty.$$

We know that $u \in C^\infty(\Omega)$. Hence, by Taylor's expansion

$$u(x) = u(x_0) + \sum_{0 < |\alpha| < N} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha + R_N(x)$$

where $\alpha! = \alpha_1! \dots \alpha_d!$, $\alpha = \alpha_1 + \dots + \alpha_d$, and

$$R_N(x) = \sum_{|\alpha| = N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}$$

We need to prove that

$$|R_N(x)| \to 0 \quad \text{uniformly in } x \in B(x_0, r).$$

Lemma: (Estimates on derivatives) If u is harmonic in $\Omega \subset \mathbb{R}^d$ and $B(x_0, r) \subset \Omega$, then $\forall |\alpha| = N$:

$$|D^\alpha u(x_0)| \leq \frac{(C_d N)^N}{r^{d+N}} \int_{B(x_0,r)} |u|$$

where C_d depends only on d.

We will prove the lemma later. Now we complete the proof of the analyticity.

For $x_0 \in \Omega$, let $0 < r < \frac{1}{L+1} \operatorname{dist}(x_0, \partial\Omega)$.

Then $\forall x \in B(x_0, r)$ we have

$$B(x, Lr) \subset B(x_0, (L+1)r) \subset \Omega$$

$$\overset{\text{Lemma}}{\Longrightarrow} \quad |D^\alpha u(x)| \leq \frac{(C_d N)^N}{(Lr)^{N+d}} \int_{B(x, Lr)} |u|$$

$$\leq \left(\frac{C_d N}{Lr}\right)^N \underbrace{\frac{1}{(Lr)^d} \int_{B(x_0,(L+1)r)} |u|}_{\leq M}$$

$$\Rightarrow \quad \|D^\alpha u\|_{L^\infty(B(x_0,r))} \leq M \left(\frac{C_d N}{Lr}\right)^N, \quad \forall |\alpha| = N.$$

Multinomial Theorem:

$$d^N = (1+1+\dots+1)^N = \sum_{|\alpha|=N} \frac{|\alpha|!}{\alpha!} = N! \sum_{|\alpha|=N} \frac{1}{\alpha!}$$

Stirling's formula: (exercise)

$$N! \sim \sqrt{2\pi N}\left(\frac{N}{e}\right)^N,$$

namely

$$\frac{N!}{\sqrt{2\pi N}\left(\frac{N}{e}\right)^N} \longrightarrow 1 \quad \text{as } N \to \infty.$$

Thus:

$$\sum_{|\alpha|=N} \frac{\|D^\alpha u\|_{L^\infty(B(x_0,r))} r^N}{\alpha!} \leq M \left(\frac{d C_d N}{L}\right)^N \cdot \frac{1}{N!}$$

$$\leq M \left(\frac{d C_d e}{L}\right)^N \longrightarrow 0 \quad \text{as } N \to \infty$$

if we take $L = L_d > d C_d e$.

In conclusion, we get the series expansion:

$$u(x) = u(x_0) + \sum_{\alpha \neq 0} \frac{D^\alpha u(x_0)}{\alpha!} (x-x_0)^\alpha$$

for all $x \in B(x_0, r)$.

Proof of the derivative bound:

For $|\alpha| = 0$, by the mean-value theorem

$$u(x_0) = \fint_{B(x_0,r)} u(y)\,dy$$

$$\Rightarrow |u(x_0)| \leq \frac{1}{|B_1|r^d} \int_{B(x_0,r)} |u|$$

For $|\alpha| = 1$, note that $\partial_{x_i} u$ is also harmonic

$$\Rightarrow \quad |\partial_{x_i} u(x_0)| = \left| \fint_{B(x_0,r/2)} \partial_{x_i} u(y)\,dy \right|$$

$$= \left| \frac{1}{|B_1|\left(\frac{r}{2}\right)^d} \int_{B(x_0,r/2)} \partial_{x_i} u(y)\,dy \right|$$

$$= \left| \frac{1}{|B_1|\left(\frac{r}{2}\right)^d} \int_{\partial B(x_0,r/2)} u\, n_i\, dS \right|$$

$$\leq \frac{|S_1|}{|B_1|\left(\frac{r}{2}\right)} \|u\|_{L^\infty(\partial B(x_0,r/2))}$$

and from the case $\alpha = 0$

$$\|u\|_{L^\infty(\partial B(x_0,r/2))} \leq \frac{1}{|B_1|(r/2)^d} \int_{B(x_0,r)} |u|$$

More generally: if $|\alpha| = N$, then

$$D^\alpha u = \partial_{x_i}(D^\beta u) \quad \text{with} \quad |\beta| = N-1$$

$$\Rightarrow |D^\alpha u(x_0)| = \left| ⨍_{B(x_0, r/N)} \partial_{x_i}(D^\beta u) \right|$$

$$= \left| \frac{1}{|B_1|\left(\frac{r}{N}\right)^d} \left(\int_{\partial B(x_0, r/N)} D^\beta u \cdot n_i \, dS \right) \right|$$

$$\leq \frac{|S_1|}{|B_1|\left(\frac{r}{N}\right)} \| D^\beta u \|_{L^\infty(B(x_0, r/N))}$$

and by the induction hypothesis

$$\| D^\beta u \|_{L^\infty B(x_0, \frac{r}{N})} \leq \frac{[C_d(N-1)]^{N-1}}{r^{d+N-1}} \int_{B(x_0,r)} |u|$$

$$\Rightarrow |D^\alpha u(x_0)| \leq \frac{C_d N}{r} \cdot \frac{[C_d(N-1)]^{N-1}}{r^{d+N-1}} \int_{B(x_0,r)} |u|$$

$$\leq \frac{(C_d N)^N}{r^{N+d}} \int_{B(x_0,r)} |u| . \qquad \square$$

Note that if $x \in B(x_0, \frac{r}{N})$, then

$$B\left(x, r\frac{(N-1)}{N}\right) \subset B(x_0, r).$$

Hence, by the induction hypothesis

$$\|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))} \leq \sup_{x \in B(x_0, \frac{r}{N})} |D^\beta u(x)|$$

$$\leq \frac{[C_d(N-1)]^{N-1}}{\left[r\frac{(N-1)}{N}\right]^{d+N-1}} \int_{B(x_0,r)} |u|$$

$$= \frac{C_d^{N-1}}{\left[r \cdot \frac{N-1}{N}\right]^d} \cdot \frac{1}{(r/N)^{N-1}} \cdot \int_{B(x_0,r)} |u|$$

$$\leq \frac{2^d C_d^{N-1}}{r^d} \cdot \frac{1}{(r/N)^{N-1}} \int_{B(x_0,r)} |u|$$

$$\Rightarrow |D^\alpha u(x_0)| \leq \frac{|S_1|}{|B_1|(r/N)} \cdot \|D^\beta u\|_{L^\infty(B(x_0, \frac{r}{N}))}$$

$$\leq \frac{2^d |S_1|}{|B_1|} \cdot C_d^{N-1} \cdot \frac{1}{r^d} \cdot \frac{1}{(r/N)^N} \int_{B(x_0,r)} |u|$$

$$\leq (C_d N)^N / r^{d+N} \int_{B(x_0,r)} |u|.$$

Theorem (Liouville's theorem) If $u \in C^2(\mathbb{R}^d)$ is harmonic and bounded, then $u = \text{const}$.

Proof:

$$|Du(x_0)| \le \frac{C_d}{r^{d+1}} \int_{B(x_0,r)} |u|$$

$$\le \frac{C_d}{r} \longrightarrow 0 \quad \text{as } r \to \infty$$

Theorem (Uniqueness of Poisson eq. in $\mathbb{R}^d$)

Let $f \in C^2_c(\mathbb{R}^d)$, $d \geq 3$. Then any bounded, $C^2(\mathbb{R}^d)$ solution of Poisson eq. $-\Delta u = f$ in $\mathbb{R}^d$ is of the form

$$u(x) = \Phi * f + C = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy + C$$

Here C is a constant and Φ the fundamental solution of Laplace equation in $\mathbb{R}^d$.

Proof: $\Phi(x) \to 0$ as ∞ $\Rightarrow$ u is bounded

$\Rightarrow$ uniqueness. (In 2D, $\Phi \to \infty$ as ∞)

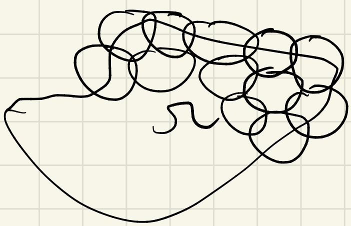
Exercise: (Harnack's inequality)

Let $u \in C^2(\mathbb{R}^d)$ be harmonic and non-negative. Prove that for every open, bounded, connected set $\Omega \subset \mathbb{R}^d$ we have:

$$\sup_{x \in \Omega} u(x) \leq C_\Omega \inf_{x \in \Omega} u(x)$$

for a finite constant C_Ω depending only on Ω.

Hint: Consider first the case $\Omega =$ a ball. In the general case, Ω can be covered by a finite collection of balls & one of them is contained completely inside Ω. $\square$



- So far we did not construct a solution of Poisson equation $-\Delta u = f$ in Ω. This can be done using Green function. But before doing so, let us discuss some basic facts of the convolution & Fourier transform.