Mathematisches Institut der Universität München
Phan Thành Nam, Eman Hamza Winter Semester 2021-2022
4.2.2022

Partial Differential Equations

Homework Sheet 13

(Discussed on 9.2.2022)

E13.1 (d'Alembert formula for wave equation in 1D)
Let $g \in C^2(\mathbb{R}), h \in C^1(\mathbb{R})$ and define

$$u(x,t) = \frac{1}{2}\Big(g(x+t) + g(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y, \quad \forall x \in \mathbb{R}, t > 0.$$

Prove that $u \in C^2(\mathbb{R} \times (0,\infty))$ and

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, \quad \forall x \in \mathbb{R}, t > 0, \\ \lim_{t\to 0^+} u(x,t) = g(x), \quad \lim_{t\to 0^+} \partial_t u(x,t) = h(x), \quad \forall x \in \mathbb{R}. \end{cases}$$

E13.2 (Poisson's formula for wave equation in 2D)
Let $g \in C^3(\mathbb{R}^2), h \in C^2(\mathbb{R}^2)$ and define

$$u(x,t) = \frac{t}{2} \fint_{B(x,t)} \frac{g(y) + \nabla g(y)\cdot(y-x) + th(y)}{\sqrt{t^2 - |x-y|^2}}\mathrm{d}y, \quad \forall x \in \mathbb{R}^2, t > 0.$$

Prove that $u \in C^2(\mathbb{R}^2 \times (0,\infty))$ and

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, \quad \forall x \in \mathbb{R}^2, t > 0, \\ \lim_{t\to 0^+} u(x,t) = g(x), \quad \lim_{t\to 0^+} \partial_t u(x,t) = h(x), \quad \forall x \in \mathbb{R}^2. \end{cases}$$

E13.3 Let $g \in C_c^3(\mathbb{R}^3), h \in C_c^2(\mathbb{R}^3)$. Assume that $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ satisfies the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0, \quad \forall x \in \mathbb{R}^3, t > 0, \\ u(x,0) = g(x), \quad \partial_t u(x,0) = h(x), \quad \forall x \in \mathbb{R}^3. \end{cases}$$

Prove that there exists a constant $C > 0$ such that

$$|u(x,t)| \le \frac{C}{t}, \quad \forall x \in \mathbb{R}^3, t > 0.$$

E13.4 Let $g \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Let $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ be the solution of the Schrödinger equation with the initial data g, namely

$$u(x,t) = (e^{it\Delta}g)(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{i|x-y|}{4t}} g(y)\mathrm{d}y.$$

Prove that for all $2 < p \le \infty$ we have

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} = 0.$$

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Partial Differential Equations

Homework Sheet 12

(Discussed on 2.2.2022)

E12.1 Let $\Omega \subset \mathbb{R}^d$ be open and $u \in C^2(\Omega)$. Assume that $x_0 \in \Omega$ is a local maximizer of u, namely there exists some $r > 0$ such that

$$u(x_0) \geq u(x), \quad \forall x \in B_r(x_0) \subset \Omega.$$

(a) Prove that the Hessian matrix $H = (D^\alpha u(x_0))_{|\alpha|=2}$ is negative semi-definite, namely

$$y \cdot Hy \leq 0, \quad \forall y \in \mathbb{R}^d.$$

(b) Prove that $\Delta u(x_0) \leq 0$.

Recall that we used (b) for the maximum principle by Hopf's method.

E12.2 We will prove the maximum principle for a general elliptic operator

$$Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}\partial_{x_j}u(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}u(x), \quad x = (x_i)_{i=1}^d.$$

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $a_{ij}, b_i \in C(\overline{\Omega})$ such that for all $x \in \Omega$,

$$A(x) = (a_{ij}(x))_{i,j=1}^d \geq \mathbb{1} \quad \text{(as matrices)}.$$

Prove that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu(x) \geq 0$ for all $x \in \Omega$, then

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

Hint: By Hopf's method you should first consider the case $Lu > 0$.

E12.3 Consider the inhomogeneous heat equation

$$\begin{cases} (\partial_t - \Delta_x)u(x,t) = f(x,t), & (x,t) \in \mathbb{R}^d \times (0,T), \\ u(x,0) = g(x), & x \in \mathbb{R}^d \end{cases}$$

with $f \in C_1^2(\mathbb{R}^d \times (0,T))$ compactly supported, and $g \in C(\mathbb{R}^d \times [0,T]) \cap L^\infty(\mathbb{R}^d \times [0,T])$. Assume that there exists a solution $u \in C_1^2(\mathbb{R}^d \times (0,T)) \cap C(\mathbb{R}^d \times [0,T])$ satisfying

$$u(x,t) \leq Me^{M|x|^2}, \qquad (x,t) \in \mathbb{R}^d \times [0,T].$$

Prove that

$$\max_{(x,t) \in \mathbb{R}^d \times [0,T]} |u(x,t)| \leq \|g\|_{L^\infty} + T\|f\|_{L^\infty}.$$

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Partial Differential Equations

Homework Sheet 11

(Discussed on 26.1.2022)

E11.1 Consider the fundamental solution of the heat equation with initial data $g \in L^2(\mathbb{R}^d)$:

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y) \mathrm{d}y, \quad \Phi(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

(a) Prove that $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$. Hint: In the lecture we already proved that for every $t > 0$, $u(\cdot,t) \in C^\infty(\mathbb{R}^d)$ by Sobolev embedding theorem.

(b) Prove that

$$\|u(\cdot,t) - g\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } t \to 0^+$$

and

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } t \to \infty.$$

(c) Prove that if we assume further $g \in H^1(\mathbb{R}^d)$, then

$$\|u(\cdot,t) - g\|_{L^2(\mathbb{R}^d)} \le C\sqrt{t}, \quad \text{as } t \to 0^+.$$

E11.2 Consider the heat equation in a bounded domain

$$\begin{cases} \partial_t u(x,t) = \Delta_x u(x,t), & \forall x \in \Omega, t > 0, \\ u(x,t) = 0, & \forall x \in \partial\Omega, t > 0, \\ u(x,0) = g(x), & \forall x \in \Omega. \end{cases}$$

Let us focus on the simplest case $\Omega = (0,1)$. Prove that for every $g \in C_c^1(0,1)$, the function

$$u(x,t) = \sum_{n=1}^{\infty} g_n e^{-t\pi^2 n^2} \sin(n\pi x), \qquad \text{where } g_n = 2\int_0^1 g(y)\sin(n\pi y) \mathrm{d}y$$

is a classical solution to the above heat equation.

E11.3 Let $g(t) = e^{-1/t^2}$ and denote $g^{(n)}(t)$ the n-th derivative of g. Define

$$u(x,t) = \sum_{n=0}^{n} \frac{g^{(n)}(t)}{(2n)!} x^{2n}, \quad \forall x \in \mathbb{R}, t > 0.$$

Prove that u is a classical solution to the heat equation

$$\begin{cases} \partial_t u(x,t) = \Delta_x u(x,t), & \forall x \in \mathbb{R}, t > 0 \\ \lim_{t\to 0} u(x,t) = 0, & \forall x \in \mathbb{R}. \end{cases}$$

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Partial Differential Equations

Homework Sheet 10

(Discussed on 19.1.2022)

Let us discuss the boundary problems in one-dimension. Here we always take $\Omega = (a,b) \subset \mathbb{R}$ be an open, bounded interval. For every $u \in H^1(\Omega)$, the values $u(a)$ and $u(b)$ are determined uniquely by trace theory, or by Sobolev's embedding theorem.

E10.1 (Sobolev inequalities) (a) Prove that $H^1(\mathbb{R}) \subset \Big(C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \Big)$.

Hint: You can use the Fourier transform.

(b) Prove that $H^1(\Omega) \subset C(\overline{\Omega})$.

E10.2 (Poincare inequality) Prove that there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C \|u'\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$ such that $u(a) = 0$.

E10.3. (Neumann boundary condition) Let $u \in H^2(\Omega)$ and $f \in L^2(\Omega)$. Prove that the following statements are equivalent:

(1) u solves the equation

$$\begin{cases} -u'' = f \quad \text{in } D'(\Omega), \\ u'(a) = u'(b) = 0. \end{cases}$$

(2) u solves the variational problem

$$\int_\Omega u'\varphi' = \int_\Omega f\varphi, \quad \forall \varphi \in H^1(\Omega).$$

E10.4. (Robin boundary condition) Let $f \in L^2(\Omega)$.

(a) Prove that there exists a unique $u \in M := \{\varphi \in H^1(\Omega), u(a) = 0\}$ such that

$$\int_\Omega u'\varphi' = \int_\Omega f\varphi, \quad \forall \varphi \in M.$$

(b) Prove that the above function u is the unique solution to the equation

$$\begin{cases} -u'' = f \quad \text{in } D'(\Omega), \\ u(a) = 0, \quad u'(b) = 0. \end{cases}$$

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Partial Differential Equations

Homework Sheet 9

(Discussed on 12.1.2022)

We only consider real-valued functions.

E9.1. Let Ω be an open, bounded domain in $\mathbb{R}^d$ $(d \geq 1)$ with C^1-boundary. Let $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$. Prove that the following statements are equivalent:

(1) $-\Delta u = f$ in $\mathcal{D}'(\Omega)$.

(2) For all $\varphi \in H_0^1(\Omega)$ we have

$$\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) \mathrm{d}x = \int_\Omega f(x)\varphi(x)\mathrm{d}x.$$

(3) u is a minimizer for the variational problem

$$E = \inf_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla v(x)|^2 \mathrm{d}x - \int_\Omega f(x)v(x)\mathrm{d}x \right).$$

Hint: You may adapt the proof of Dirichlet's principle to weak solutions.

E9.2. Recall that $Q = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x'| < 1, |x_d| < 1\}$ and

$$Q_+ = Q \cap \{x_d > 0\}, \quad Q_- = Q \cap \{x_d < 0\}, \quad Q_0 = Q \cap \{x_d = 0\}.$$

For every function $u \in H^1(Q_+)$, define the extension $Bu : Q \to \mathbb{R}$ as

$$Bu(x) = \begin{cases} u(x), & \forall x \in Q_+, \\ u(x', -x_d), & \forall x = (x', x_d) \in Q_-. \end{cases}$$

(a) Prove that for every $i \in \{1, 2, ..., d-1\}$ we have

$$\partial_i Bu(x) = \begin{cases} (\partial_i u)(x), & \forall x \in Q_+, \\ (\partial_i u)(x', -x_d), & \forall x = (x', x_d) \in Q_-. \end{cases}$$

(From this and the computation of $\partial_d(Bu)$ in the lecture we obtain $Bu \in H^1(Q)$.)

(b) Find an example where $u \in H^2(Q_+)$ but $Bu \notin H^2(Q)$.

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Partial Differential Equations

Homework Sheet 8

(Discussed on 14.12.2021)

E8.1. Let $u \in W^{1,1}_{\rm loc}(\mathbb{R}^d)$. Let

$$B = u^{-1}(\{0\}) = \{x \in \mathbb{R}^d : u(x) = 0\}.$$

Prove that $\nabla u(x) = 0$ for a.e. $x \in B$.

Remark: This result holds if $\mathbb{R}^d$ is replaced by an open set Ω. Moreover, if u is real-valued, we may replace $u^{-1}(\{0\})$ by $u^{-1}(A)$ for any Borel set $A \subset \mathbb{R}$ of zero measure.

E8.2. Let Ω and U be two open, bounded subsets of $\mathbb{R}^d$ ($d \geq 1$) such that $U \cap \Omega$ is non-empty. Let $\chi \in C_c^\infty(U)$ and $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < \infty$. Prove that $\chi u \in W_0^{1,p}(U \cap \Omega)$. Recall the definition $W_0^{1,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$.

E8.3. Let Ω and U be two open, bounded subsets of $\mathbb{R}^d$ ($d \geq 1$) such that there is a C^1-diffeomorphism $h : \overline{U} \to \overline{\Omega}$. Prove that if $y \mapsto u(y) \in W_0^{1,p}(\Omega)$ for some $1 \leq p < \infty$, then $x \mapsto u(h(x)) \in W_0^{1,p}(U)$.

E8.4. (Partition of unity) Let Γ be a compact subset of $\mathbb{R}^d$ ($d \geq 1$) and let $\{U_i\}_{i=1}^N$ be open subsets of $\mathbb{R}^d$ such that

$$\Gamma \subset \bigcup_{i=1}^N U_i.$$

Prove that there exist functions $\{\chi_i\}_{i=0}^N \subset C^\infty(\mathbb{R}^d)$ such that

(1) $\chi_i \geq 0$ for all i and $\sum_{i=0}^N \chi_i = 1$;
(2) $\operatorname{supp} \chi_i \subset U_i$ for all $i = 1, 2, ..., N$;
(3) $\operatorname{supp} \chi_0 \subset \mathbb{R}^d \backslash \Gamma$.

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Partial Differential Equations

Homework Sheet 7

(Discussed on 8.12.2021)

E7.1. Let Ω be an open, bounded domain in $\mathbb{R}^d$ with C^1-boundary. Assume that for every $x \in \Omega$, there exists a solution $\phi_x \in C^2(\overline{\Omega})$ to

$$\begin{cases} \Delta_y \phi_x(y) = 0, \quad \forall y \in \Omega, \\ \phi_x(y) = G(y-x), \quad \forall y \in \partial\Omega \end{cases}$$

where G is the fundamental solution of Laplace's equation in $\mathbb{R}^d$. Prove that

$$\phi_x(y) = \phi_y(x), \quad \forall x, y \in \Omega.$$

E7.2. Recall $\mathbb{R}^d_+ = \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > 0\}$ and Poisson's kernel

$$K(x,y) = \frac{2}{d|B_1||x-y|^d}, \quad \forall x \in \mathbb{R}^d_+, \quad \forall y \in \partial\mathbb{R}^d_+.$$

Prove that for every $x \in \mathbb{R}^d_+$ we have

$$\int_{\partial\mathbb{R}^d_+} K(x,y)\mathrm{d}y = 1.$$

(You may try the case $d = 2$ first.)

E7.3. Let $g \in C(\partial\mathbb{R}^d_+) \cap L^\infty(\partial\mathbb{R}^d_+)$ satisfy $g(x) = |x|$ if $x \in \partial\mathbb{R}^d_+ \cap B(0,1)$. Let

$$u(x) = \int_{\partial\mathbb{R}^d_+} K(x,y)g(y)\mathrm{d}y, \quad \forall x \in \mathbb{R}^d_+$$

with the above Poisson's kernel $K(x,y)$. Prove that $|\nabla u(x)|$ is unbounded in $\mathbb{R}^d_+ \cap B(0,r)$ for every $r > 0$. Here $B(0,r)$ is the open ball in $\mathbb{R}^d$.

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Partial Differential Equations

Homework Sheet 6

(Discussed on 1.12.2021)

E6.1. Let $\chi \in C^\infty(\mathbb{R}^d)$ and $f \in W^{1,p}(\mathbb{R}^d)$. Prove that $\chi f \in W^{1,p}(\mathbb{R}^d)$ and

$$\partial_i(\chi f)(x) = (\partial_i \chi)(x) f(x) + \chi(x)\partial_i f(x) \quad \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

for all $i = 1, 2, ..., d$.

E6.2. Let $f \in L^p(\mathbb{R}^2)$ be compactly supported. Let $G(x) = -\frac{1}{2\pi}\ln|x|$ be the fundamental solution of Laplace's equation. Prove that:

(a) If $p = 1$, then $G * f \in L^q_{\text{loc}}(\mathbb{R}^2)$ for all $q < \infty$.

(b) If $p > 2$, then $G * f \in C^{1,\alpha}(\mathbb{R}^2)$ for all $0 < \alpha < 1 - 2/p$.

Hint: This low regularity result has been discussed in the lecture for $d \geq 3$. Here you need to adapt the proof to $d = 2$.

E6.3. Let Ω be an open subset of $\mathbb{R}^d$. Let $f \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for some $0 < \alpha < 1$.

(a) Prove that for every open ball $B \subset \overline{B} \subset \Omega$, there exists a function $f_B \in C^{0,\alpha}(\mathbb{R}^d)$ such that f_B is compactly supported and $f_B(x) = f(x)$ for all $x \in B$.

(b) Prove that if $u \in \mathcal{D}'(\Omega)$ satisfies

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\Omega),$$

then $u \in C^{2,\alpha}_{\text{loc}}(\Omega)$.

Hint: For (b) you can use the result that $G * f_B \in C^{2,\alpha}(\mathbb{R}^d)$.

E6.4. Assume that $u, f \in L^2(\mathbb{R}^d)$ and

$$-\Delta u = f \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Prove that $u \in W^{2,2}(\mathbb{R}^d)$ and

$$\|u\|_{W^{2,2}(\mathbb{R}^d)} \leq C(\|u\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)}).$$

Here the constant $C = C_d$ is independent of u and f.

Hint: You can use the Fourier transform.

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Partial Differential Equations

Homework Sheet 5

(Discussed on 24.11.2021)

E5.1. Prove that if f is a harmonic function in $\mathbb{R}^d$ and $g \in C_c(\mathbb{R}^d)$ is radial, then

$$\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x = f(0)\int_{\mathbb{R}^d} g(x)\mathrm{d}x.$$

E5.2. Let $1 \le p < \infty$. Let $\Omega \subset \mathbb{R}^d$ be open. Consider the Sobolev space

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) : \partial_{x_i} f \in L^p(\Omega), \forall i = 1, 2, ..., d\}$$

with the norm

$$\|f\|_{W^{1,p}} = \|f\|_{L^p} + \sum_{i=1}^d \|\partial_{x_i} f\|_{L^p(\Omega)}.$$

Prove that $W^{1,p}(\Omega)$ is a Banach space. Here $x = (x_i)_{i=1}^d \in \mathbb{R}^d$.
Hint: You can use the fact that $L^p(\Omega)$ is a Banach space.

E5.3. Let f be a real-valued function in $W^{1,p}(\mathbb{R}^d)$ for some $1 \le p < \infty$. Prove that $|f| \in W^{1,p}(\mathbb{R}^d)$ and

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x), & \text{if } f(x) > 0, \\ -\nabla f(x), & \text{if } f(x) < 0, \\ 0, & \text{if } f(x) = 0. \end{cases}$$

Hint: You can use the chain rule for $G_\varepsilon(f)$ with $G_\varepsilon(t) = \sqrt{\varepsilon^2 + t^2} - \varepsilon \to |t|$ as $\varepsilon \to 0$.

E5.4. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $f \in L^1(\Omega)$. Let G be the fundamental solution of Laplace equation in $\mathbb{R}^d$. Define

$$u(x) = \int_\Omega G(x-y)f(y)\mathrm{d}y, \quad \forall x \in \mathbb{R}^d.$$

Prove that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $-\Delta u = f$ in $D'(\Omega)$.
Hint: In the lecture we already discussed the case $\Omega = \mathbb{R}^d$. Here you need to consider a general bounded domain.

E5.5. Let $B = B(0, 1/2) \subset \mathbb{R}^3$. Consider $u : B \to \mathbb{R}$ defined by

$$u(x) = \ln(|\ln|x||).$$

Prove that the distributional derivative $f = -\Delta u$ is a function in $L^{3/2}(B)$.

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Partial Differential Equations

Homework Sheet 4

(Discussed on 17.11.2021)

E4.1. Prove that if $T_n \to T$ in $\mathcal{D}'(\mathbb{R}^d)$, then $D^\alpha T_n \to D^\alpha T$ in $\mathcal{D}'(\mathbb{R}^d)$ for all $\alpha = (\alpha_j)_{j=1}^d$.

E4.2. Let δ_x be the Dirac delta function in $\mathcal{D}'(\mathbb{R}^d)$. Prove that

$$(D^\alpha \delta_x)(\varphi) = (-1)^{|\alpha|}(D^\alpha \varphi)(x), \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d), \quad \forall \alpha = (\alpha_j)_{j=1}^d.$$

E4.3. Let $f \in L^1(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} f = 1$. For every $\varepsilon > 0$, denote $f_\varepsilon(x) = \varepsilon^{-d} f(\varepsilon^{-1}x)$. Prove that $f_\varepsilon \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^d)$ when $\varepsilon \to 0$.

E4.4. Let $\{f_n\} \subset L^1(\mathbb{R}^d)$ such that $\mathrm{supp} f_n \subset B(0,1)$ for all $n \geq 1$ and

$$f_n \to f \quad \text{in } L^1(\mathbb{R}^d)$$

as $n \to \infty$. Prove that for every $g \in C_c^\infty(\mathbb{R}^d)$

$$f_n * g \to f * g \quad \text{in } \mathcal{D}(\mathbb{R}^d).$$

E4.5. Compute the distributional derivatives $f', f'' \in \mathcal{D}'(\mathbb{R})$ of the function

$$f(x) = x|x-1|, \quad x \in \mathbb{R}.$$

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Partial Differential Equations

Homework Sheet 3

(Discussed on 10.11.2021)

E3.1. (Lebesgue Differentiation Theorem) Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. Prove that for almost every $x \in \mathbb{R}^d$ we have

$$\lim_{r \to 0} \fint_{B(x,r)} |f(x) - f(y)| \mathrm{d}y = 0.$$

E3.2. Let $1 \le p, q, r \le 2$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Recall that if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$ by Young's inequality, and its Fourier transform is well-defined by the Hausdorff-Young inequality. Prove that

$$\widehat{f * g}(k) = \widehat{f}(k)\widehat{g}(k) \quad \text{for a.e. } k \in \mathbb{R}^d.$$

Hint: In the lecture we already discussed the case $f, g \in C_c(\mathbb{R}^d)$.

E3.3. Prove that if $f \in C_c^\infty(\mathbb{R}^d)$, then for all $N \ge 1$ we have

$$|\widehat{f}(k)| \le \frac{C_N}{(1 + |k|)^N}, \quad \forall k \in \mathbb{R}^d$$

where the constant $C_N > 0$ is independent of k.

E3.4. Prove that the Fourier transform of a Gaussian in $\mathbb{R}^d$ is

$$\mathcal{F}(e^{-\pi|x|^2}) = e^{-\pi|k|^2},$$

and more generally

$$\mathcal{F}(e^{-\pi\lambda^2|x|^2}) = \lambda^{-d} e^{-\pi|k|^2/\lambda^2}, \quad \forall \lambda > 0.$$

Hint: For $d = 1$, you can show that the function

$$\int_{\mathbb{R}} e^{-\pi(x+ik)^2} \mathrm{d}x$$

is independent of $k \in \mathbb{R}$.

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Phan Thành Nam, Eman Hamza 29.10.2021

Partial Differential Equations

Homework Sheet 2

(Discussed on 03.11.2021)

E2.1. Let $u \in C^2(\mathbb{R}^d)$ and let $H(x) = (D^\alpha u(x))_{|\alpha|=2}$ be the Hessian matrix of u, namely

$$H_{ij}(x) = \partial_{x_i}\partial_{x_j} u(x), \quad \forall i,j \in \{1,2,...,d\}, \quad \forall x = (x_1,...,x_d) \in \mathbb{R}^d.$$

Assume that u is convex, namely

$$u(tx + (1-t)y) \le tu(x) + (1-t)u(y), \quad \forall t \in [0,1], \quad \forall x, y \in \mathbb{R}^d.$$

(a) Prove that for every $x \in \mathbb{R}^d$, the Hessian matrix $H(x)$ is positive semidefinite.

(b) Prove that u is sub-harmonic in $\mathbb{R}^d$.

E2.2. (Newton's theorem) Let $d \ge 3$.

(a) Prove that for all $r > 0$ and $x \in \mathbb{R}^d$, we have

$$\fint_{\partial B(x,r)} \frac{\mathrm{d}S(y)}{|y|^{d-2}} = \frac{1}{\max(|x|,r)^{d-2}}$$

where $\mathrm{d}S(y)$ is the surface measure on the sphere $\partial B(x,r) \subset \mathbb{R}^d$.

(b) Let $0 \le f_1, f_2 \in L^1(\mathbb{R}^d)$ be radial functions with $\int_{\mathbb{R}^d} f_i = M_i$. Prove that for all $z_1, z_2 \in \mathbb{R}^d$ we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f_1(x-z_1) f_2(y-z_2)}{|x-y|^{d-2}} \mathrm{d}x\mathrm{d}y \le \frac{M_1 M_2}{|z_1 - z_2|^{d-2}}$$

Moreover, prove that we have the equality if f_1, f_2 are compactly supported and $|z_1 - z_2|$ is sufficiently large.

Hint: For (a) you may use the mean-value theorem (the function $1/|x|^{d-2}$ is harmonic in Ω if $0 \notin \Omega$). For (b) you may use (a) and polar coordinates.

E2.3. Prove Young's inequality

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)}$$

for all $d \ge 1$ and for all $1 \le p, q, r \le \infty$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

Mathematisches Institut der Universität München — Winter Semester 2021-2022
Phan Thành Nam, Eman Hamza — 21.10.2021

Partial Differential Equations

Homework Sheet 1

(Discussed on 27.10.2021)

E1.1. (Gauss–Green formula) Let $f = (f_i)_{i=1}^d \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Prove that for every open ball $B \subset \mathbb{R}^d$ we have

$$\int_{\partial B} f(y) \cdot \overrightarrow{n}_y \mathrm{d}S(y) = \int_B \mathrm{div}(f)(x) \mathrm{d}x.$$

Here $\overrightarrow{n}$ is the outward unit normal vector and $\mathrm{d}S$ is the surface measure on the sphere.

E1.2. Assume that $u \in C(\mathbb{R}^d)$ and $\int_B u = 0$ for every open ball $B \subset \mathbb{R}^d$. Prove that $u(x) = 0$ for all $x \in \mathbb{R}^d$.

E1.3. Let $f \in C_c^1(\mathbb{R}^d)$ with $d \geq 2$. Let

$$u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y) \mathrm{d}y$$

where Φ is the fundamental solution of Laplace equation in $\mathbb{R}^d$. Prove that $u \in C^2(\mathbb{R}^d)$ and $-\Delta u(x) = f(x)$ for all $x \in \mathbb{R}^d$. (The proof for $f \in C_c^2(\mathbb{R}^d)$ was already discussed in the lecture. Here you need to verify the extension to $f \in C_c^1(\mathbb{R}^d)$.)

E1.4. Let Ω be an open subset of $\mathbb{R}^d$. Let $u \in C^2(\Omega)$ and $\Delta u \geq 0$ (namely u is a *subharmonic* function).

(a) Prove that u satisfies the mean-value inequality

$$\fint_{\partial B(x,r)} u(y) \mathrm{d}S(y) \geq \fint_{B(x,r)} u(y) \mathrm{d}y \geq u(x)$$

for all $x \in B(x,r) \subset \Omega$.

(b) Assume further that Ω is connected and $u \in C(\overline{\Omega})$. Prove that u satisfies the strong maximum principle, namely either

- u is a constant in Ω, or
- $\sup_{y \in \partial\Omega} u(y) > u(x)$ for all $x \in \Omega$.