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Winter Term 2021–2022
15.12.2021

Partial Differential Equations
Midterm exam

Family name: . . . . Matriculation no.: . . . .

First name: . . . . Semester: . . . .

Study course: . . . .

Signature: . . . .

You have **3 hours** of official working time and **one additional hour** to prepare and finalize your upload. Solutions must be uploaded until the deadline at 14:00 o'clock on 15.12.2021 via Uni2Work in PDF-format. Make sure to follow these rules:

- Solutions must be **handwritten** (pen on paper & scanned, or digital pen tablet). Do **not** use the colours **red** or **green**.
  If you do not use the official exam preprint (this file), you must follow the official formatting instructions for "plain-paper submissions" given in uni2work.
- Solve each problem on the respective sheet. If you need more space, you can use the extra sheets; in this case please state your name and the problem you refer to.
- All answers and solutions must provide sufficiently detailed arguments. You may refer to all results from the lectures, homeworks and tutorials.
- Solutions must be prepared by yourself. You are not allowed to share information about any of the problems or their solutions of this exam with others before the hand-in deadline.
- With your signature you agree to the rules of the exam.

Before uploading please check whether your pdf-scan is readable and contains all your solutions (in total there are **four problems**). Do not forget to write your name on each sheet. Good luck!

Problem 1	Problem 2	Problem 3	Problem 4	$\sum$
(max 2)	(max 2)	(max 2)	(max 4)	

Problem Overview (you do not have to include this page in your submission).

Problem 1 (2 points). Let Ω be an open, bounded subset of $\mathbb{R}^d$ ($d \geq 1$) such that $0 \in \Omega$. Let $\{u_n\}_{n=1}^{\infty} \subset C(\overline{\Omega})$ satisfy that for every $n \geq 1$, u_n is a real-valued, harmonic function in Ω and

$$u_n(x) = \frac{1}{ne^{-n|x|} + 1}, \quad \forall x \in \partial\Omega.$$

Prove that $u_n(0) \to 1$ as $n \to \infty$.

Problem 2 (1+1 points).

(a) Prove that the function $g(x) = |x|^{-1}$ satisfies that $\Delta g(x) \leq 0$ on $\mathbb{R}^4 \backslash \{0\}$.

(b) Prove that for every non-negative, radial function $f \in C_c(\mathbb{R}^4)$, we have

$$\int_{\mathbb{R}^4} \frac{f(y)}{|x-y|} \mathrm{d}y \leq \frac{1}{|x|} \int_{\mathbb{R}^4} f(y) \mathrm{d}y, \quad \forall x \in \mathbb{R}^4 \backslash \{0\}.$$

Problem 3 (2 points). Let $B = B(0,1)$ be the unit open ball in $\mathbb{R}^2$. Let $u \in C(\overline{B})$ satisfy that u is a positive, harmonic function in B and $u(0) = 1$. Prove that

$$\frac{1-|x|}{1+|x|} \leq u(x) \leq \frac{1+|x|}{1-|x|}, \quad \forall x \in B.$$

Hint: You can use the mean-value theorem and Poisson's formula

$$u(x) = \fint_{\partial B} \frac{1-|x|^2}{|y-x|^2} u(y) \mathrm{d}S(y), \quad \forall x \in B.$$

Problem 4 (1+1+2 points). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be compactly supported and $f \in L^p(\mathbb{R}^3)$ for some $p > 3/2$. Let $u \in L^2(\mathbb{R}^3)$ satisfy that

$$-\Delta u = f * f \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

(a) Prove that $u \in C^1(\mathbb{R}^3)$.

(b) Prove that the Fourier transforms of u and f satisfy

$$|2\pi k|^2 \widehat{u}(k) = \widehat{f}(k)^2, \quad \text{for a.e. } k \in \mathbb{R}^3.$$

Hint: You may integrate the equation against test functions and use Plancherel theorem.

(c) Prove that if $p = 2$, then $u \in C^2(\mathbb{R}^3)$.

Hint: You can prove that if $\hat{g} \in L^1(\mathbb{R}^3)$, then $g \in C(\mathbb{R}^3)$ (by Dominated Convergence).