Chapter 5: Heat equation

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^d \times (0,\infty) \\ u_{|t=0} = g & \text{(initial data)} \end{cases}$$

Fundamental solution:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^d, t > 0$$

$$\begin{cases} \partial_t \Phi = \Delta \Phi & \text{in } \mathbb{R}^d \times (0,\infty) \\ \int_{\mathbb{R}^d} \Phi(x,t)\,dx = 1, & \forall t > 0 \\ \lim_{t \to 0} \Phi(x,t) = \delta_0(x) & \text{in } D'(\mathbb{R}^d). \end{cases}$$

Theorem: If $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t)\, g(y)\,dy$$

satisfies (i) $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$

(ii) $\partial_t u = \Delta u, \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty)$

(iii) $\lim_{t \to 0} u(x,t) = g(x), \quad \forall x \in \mathbb{R}^d.$

Theorem: (Nonhomogeneous problem)

Let $f \in C_1^2(\mathbb{R}^d, [0,\infty))$, compact support.

Define
$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(x-y, t-s) f(y,s)\, dy\, ds$$

Then: (i) $u \in C_1^2(\mathbb{R}^d \times (0,\infty))$

(ii) $\partial_t u = \Delta u + f \quad , \forall x \in \mathbb{R}^d, \; t > 0$

(iii) $\lim_{t \to 0} u(x,t) = 0 \quad , \forall x \in \mathbb{R}^d.$

Proof: We write

$$u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s) f(x-y, t-s)\, dy\, ds$$

$$\to \quad \partial_t u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\, \partial_t f(x-y, t-s)\, dy\, ds$$

$$+ \int_{\mathbb{R}^d} \Phi(y,s) f(x-y, 0)\, dy$$

and $$\partial_{ij} u(x,t) = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\, \partial_{ij} f(x-y, t-s)\, dy$$

$\rightsquigarrow$ $\partial_t u$, $\partial_{ij} u$ (and u, $\partial_i u$) are in $C(\mathbb{R}^d \times (0,\infty))$.

Next we calculate:

$$\partial_t u - \Delta u = \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\,(\partial_t - \Delta_x) f(x-y,t-s)\,dy\,ds$$

$$+ \int_{\mathbb{R}^d} \Phi(y,s)\, f(x-y,0)\,dy$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^d} \cdots + \int_0^\varepsilon \int_{\mathbb{R}^d} \cdots + \int_{\mathbb{R}^d} \Phi(y,s)\, f(x-y,0)\,dy$$

$$= I_\varepsilon + J_\varepsilon + K.$$

Here $|J_\varepsilon| \leq \|(\partial_t - \Delta_x) f\|_{L^\infty} \int_0^\varepsilon \int_{\mathbb{R}^d} \Phi(y,s)\,dy\,ds$

$$\leq C\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^d} \Phi(y,s)\,(-\partial_s - \Delta_y) f(x-y,t-s)\,dy\,ds$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^d} \underbrace{(\partial_s - \Delta_y)\,\Phi(y,s)}_{=0}\, f(x-y,t-s)\,dy\,ds$$

$$- \int_{\mathbb{R}^d} \Phi(y,s)\, f(x-y,t-s) \Big]_{s=\varepsilon}^{s=t}$$

$$= \int_{\mathbb{R}^d} \Phi(y,\varepsilon)\, f(x-y,t-\varepsilon)\,dy \;- K$$

$$\Rightarrow \quad I_\varepsilon + K = \int_{\mathbb{R}^d} \Phi(y,\varepsilon) f(x-y, t-\varepsilon)\, dy \xrightarrow[\varepsilon \to 0]{} f(x,t)$$

Thus

$$\partial_t u - \Delta u = f(x,t), \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty).$$

Finally:

$$\|u(\cdot,t)\|_{L^\infty} \leq \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^d} \Phi(y,s)\, dy\, ds$$

$$= \|f\|_{L^\infty} t \to 0 \quad \text{as } t \to \infty.$$

<u>Remark</u>: As f & g given as above,

$$u(x,t) = \int_{\mathbb{R}^d} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^d} \Phi(x-y,t-s) f(y,s)\, ds$$

solves
$$\begin{cases} \partial_t - \Delta u = f \\ u(t=0) = g. \end{cases}$$

(Exercise)

<u>Remark</u>: (Duhamel formula)

Consider the ODE: $\partial_t w(t) = A w(t)$, $A \in \mathbb{R}$

$$\Rightarrow \quad w(t) = e^{tA} w(0)$$

More generally:

If $\quad \partial_t w(t) = A w(t) + f(t)$

$$\Rightarrow \quad \partial_t \left(e^{-tA} w(t) \right) = e^{-tA} \left(\partial_t w(t) - A w(t) \right)$$
$$= e^{-tA} f(t)$$

$$\Rightarrow \quad e^{-tA} w(t) = w(0) + \int_0^t e^{-sA} f(s)\, ds$$

$$\Rightarrow \quad w(t) = e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s)\, ds$$

More generally, if A is an operator (independent of time) then:

$$\partial_t w(t) = A w(t) + f(t)$$

$$\Rightarrow \quad w(t) = e^{tA} w(0) + \int_0^t e^{(t-s)A} f(s)\, ds.$$

<u>Application</u>: If $A = \Delta$, then the operator $e^{t\Delta}$ has kernel

$$e^{t\Delta}(x,y) = \Phi(x-y,t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}$$

$\rightsquigarrow$ called the heat kernel.

Theorem (L^2-data) For every $g \in L^2(\mathbb{R}^d)$, define
$$u(t,x) = \int_{\mathbb{R}^d} \Phi(x-y,t)\, g(y)\, dy.$$
Then $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and it solve the heat eq.
$$\begin{cases} \partial_t u = \Delta_x u & \text{in } \mathbb{R}^d \times (0,\infty) \\ \lim_{t\to 0} u(\cdot,t) = g & \text{in } L^2(\mathbb{R}^d). \end{cases}$$

Proof: Recall the heuristic computation from the heat equation using the Fourier transform
$$\partial_t u(x,t) = \Delta_x u(x,t)$$
$$\Leftrightarrow \quad \partial_t \hat{u}(k,t) = -|2\pi k|^2 \hat{u}(k,t)$$
$$\Leftrightarrow \quad \partial_t \left(e^{+t|2\pi k|^2} \hat{u}(k,t) \right) = 0$$
$$\Leftrightarrow \quad e^{t|2\pi k|^2} \hat{u}(k,t) = \hat{u}(k,0) = \hat{g}(k)$$
$$\Leftrightarrow \quad \hat{u}(k,t) = e^{-t|2\pi k|^2} \hat{g}(k) = \hat{\Phi}(k,t)\, \hat{g}(k) = \widehat{\Phi * g}$$
$$\Leftrightarrow \quad u(x,t) = \Phi * g = \int_{\mathbb{R}^d} \Phi(x-y,t)\, g(y)\, dy$$

Here we only need the direction "$\Leftarrow$" which is rigorous if $g \in L^2(\mathbb{R}^d)$.

From the Fourier transform, it is also easy to check that $u(\cdot,t) \to g$ in L^2 as $t \to 0$ (exercise)

To see the smoothness, note that $\forall t>0$, $\forall m \in \mathbb{N}$

$$(1+|2\pi k|^m)\,\hat{u}(k,t) = \underbrace{(1+|2\pi k|^m)\,e^{-t|2\pi k|^2}}_{\in L^\infty}\ \underbrace{\hat{g}(k)}_{\in L^2} \in L^2$$

$\Rightarrow u(\cdot,t) \in H^m(\mathbb{R}^d)$, $\forall m \geq 1$.

$\Rightarrow u(\cdot,t) \in C^\infty(\mathbb{R}^d)$ by Sobolev embedding (see below). This argument can be also used to show that $u \in C^\infty(\mathbb{R}^d \times (0,\infty))$ (exercise).

<u>Theorem</u> (Sobolev embedding) If $m > d/2$, then:

$$H^m(\mathbb{R}^d) \subset \left(C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \right)$$

<u>Proof</u>: We write $\forall u \in H^m(\mathbb{R}^d)$:

$$\hat{u}(k) = \underbrace{\hat{u}(k)\,(1+|2\pi k|^m)}_{\in L^2 \text{ as } u \in H^m} \cdot \underbrace{\frac{1}{1+|2\pi k|^m}}_{\in L^2 \text{ as } m > \frac{d}{2}}$$

$\Rightarrow \hat{u}(k) \in L^1(\mathbb{R}^d)$

$\Rightarrow u = (\hat{u})^\vee \in \left(C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \right)$.

Maximum principle:

Recall Poisson equation $-\Delta u \leq 0$ in Ω open, bounded

$$\Rightarrow \sup_{\overline{\Omega}} u(x) = \sup_{\partial\Omega} u(x)$$

Theorem: (Maximum principle for bounded sets)
Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let $T>0$ and define

$$\Omega_T = \Omega \times (0,T), \quad \partial^* \Omega_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0,T])$$

If $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega}_T)$ solves

$$\partial_t u - \Delta_x u \leq 0 \text{ in } \Omega_T$$

then:
$$\max_{\overline{\Omega}_T} u = \max_{\partial^* \Omega_T} u .$$

Proof: We will use Hopf's argument which is simpler than the mean-value theorem (there exists a mean-value theorem for heat equation, but it is complicated and we will not discuss).

Proof for Poisson eq: Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$

Step 1: Assume $\Delta u > 0$ in Ω. Since $\overline{\Omega}$ is compact, $\exists\, x_0 \in \overline{\Omega}$ s.t. $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$.

We prove that $x_0 \in \partial\Omega$. In fact, if $x_0 \in \Omega$, then since x_0 is a (local) maximizer of u in Ω, we have $\Delta u(x_0) \leq 0$, which contradicts to the assumption that $\Delta u > 0$ in Ω.

Thus $x_0 \in \partial\Omega$, and hence:

$$\max_{x \in \overline{\Omega}} u(x) = u(x_0) \leq \max_{x \in \partial\Omega} u(x).$$

Step 2: Now assume $\Delta u \geq 0$ in Ω. Define:

$$u_\varepsilon(x) = u(x) + \varepsilon |x|^2, \quad \varepsilon > 0.$$

Then $\Delta u_\varepsilon > 0$ in Ω, hence by Step 1 and

$$u \leq u_\varepsilon \leq u + \varepsilon \sup_{x \in \overline{\Omega}} |x|^2$$

we have:

$$\max_{x \in \overline{\Omega}} u(x) \leq \max_{x \in \overline{\Omega}} u_\varepsilon(x) \leq \max_{x \in \partial\Omega} u_\varepsilon(x) \leq$$

$$\leq \max_{x \in \partial\Omega} u(x) + \varepsilon\left(\sup_{x \in \overline{\Omega}} |x|^2\right) \xrightarrow[\varepsilon \to 0]{} \max_{x \in \partial\Omega} u(x).$$

Proof for heat equation:

<u>Step 1</u> Assume $u \in C^2_1(\Omega \times (0,T]) \cap C(\bar{\Omega} \times [0,T])$

and $\partial_t u - \Delta_x u < 0$ in $\Omega \times (0,T]$.

Then $\exists (x_0, t_0) \in \bar{\Omega} \times [0,T]$ s.t.

$$u(x_0, t_0) = \max_{(x,t) \in \bar{\Omega} \times [0,T]} u(x,t).$$

We prove that $(x_0, t_0) \in \partial^* \Omega_T$. Assume by contradiction that $(x_0, t_0) \notin \partial^* \Omega_T$, then

$$x_0 \in \Omega \text{ and } t_0 \in (0,T].$$

Since $x \mapsto u(x, t_0)$ has a (local) maximizer $x_0 \in \Omega \Rightarrow \Delta_x u(x_0, t_0) \leq 0$

Since $t \mapsto u(x_0, t)$ has a (local) maximizer $t_0 \in (0,T] \Rightarrow \partial_t u(x_0, t_0) \geq 0$

Thus: $(\partial_t u - \Delta_x u)(x_0, t_0) \geq 0$

which is a contradiction to the assumption.

Thus: $(x_0, t_0) \in \partial^* \Omega_T$, i.e. $\max_{\bar{\Omega}_T} u = \max_{\partial^* \Omega_T} u$.

Step 2. Assume $u \in C^2_1(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T])$

and $\partial_t u - \Delta_x u \le 0$ in $\Omega \times (0,T)$.

Let $\tilde{T} \in (0,T)$ and for $\varepsilon > 0$:

$$u_\varepsilon(x,t) = u(x,t) + \varepsilon |x|^2 .$$

Then: $u_\varepsilon \in C^2_1(\Omega \times (0,T']) \cap C(\overline{\Omega} \times [0,T'])$

and $\partial_t \tilde{u}_\varepsilon - \Delta_x \tilde{u}_\varepsilon < 0$ in $\Omega \times (0,T']$.

By Step 1: $$\max_{\overline{\Omega}_{T'}} \tilde{u}_\varepsilon \le \max_{\partial^* \Omega_{T'}} \tilde{u}_\varepsilon$$

$$\xrightarrow{\varepsilon \to 0} \quad \max_{\overline{\Omega}_{T'}} u \le \max_{\partial^* \Omega_{T'}} u$$

$$\xrightarrow{T' \to T} \quad \max_{\overline{\Omega}_T} u \le \max_{\partial^* \Omega_T} u . \qquad \square$$

Theorem (Maximum principle for $\Omega = \mathbb{R}^d$)

Let $\Omega_T = \mathbb{R}^d \times (0,T)$, $\overline{\Omega}_T = \mathbb{R}^d \times [0,T]$.

Let $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega}_T)$ such that

- $\partial_t u - \Delta_x u \leq 0$ in Ω_T
- $u(x,t) \leq M e^{M|x|^2}$, $\forall (x,t) \in \overline{\Omega}_T$.

Then

$$\sup_{\overline{\Omega}_T} u(x,t) = \sup_{x \in \mathbb{R}^d} u(x,0).$$

Proof: Step 1: $\forall y \in \mathbb{R}^d$ and $\varepsilon > 0$ define

$$v(x,t) = u(x,t) - \frac{\varepsilon}{(T+\varepsilon-t)^{d/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$$

$$\Rightarrow \partial_t v - \Delta_x v = \partial_t u - \Delta_x u \leq 0 \text{ in } \Omega_T$$

For $U = B(y,r)$, $U_T = U \times (0,T)$, $\overline{U}_T = \overline{U} \times [0,T]$

$$\partial^* U_T = (U \times \{0\}) \cup (\partial U \times [0,T])$$

By the maximum principle for U bounded,

$$\max_{\overline{U}_T} v \leq \max_{\partial^* U_T} v$$

let us bound $\max_{\partial^* U_T} v$.

.) On $U \times \{0\}$ we use $v \le u$, and hence

$$\max_{x \in \overline{U}} v(x,0) \le \max_{x \in \overline{U}} u(x,0) \le \max_{x \in \mathbb{R}^d} u(x,0)$$

.) On $\partial U \times [0,T]$ we use: $|x-y| = r \Rightarrow |x| \le |y| + r$

$$v(x,t) = u(x,t) - \frac{\varepsilon}{(T+\varepsilon-t)^{d/2}} \exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$$

$$\le M e^{M(|y|+r)^2} - \frac{\varepsilon}{(T+\varepsilon)^{d/2}} \exp\left(\frac{r^2}{4(T+\varepsilon)}\right)$$

$$\to -\infty \quad \text{as } r \to \infty \quad \text{if } M < \frac{1}{4(T+\varepsilon)}.$$

In particular, we can choose r large s.t.

$$\max_{\substack{x \in \partial U, \\ t \in [0,T]}} v(x,t) \le \max_{x \in \mathbb{R}^d} u(x,0).$$

In summary, if $M < \frac{1}{4(T+\varepsilon)}$, then:

$$\begin{array}{c} v(y,t) \le \max\limits_{U_T} v \le \max\limits_{x\in\mathbb{R}^d} u(x,0) \\ \| \\ u(y,t) - \frac{\varepsilon}{(T+\varepsilon-t)^{d/2}} \end{array}$$

This holds for all $(y,t) \in \mathbb{R}^d \times [0,T]$

$$\Rightarrow \max_{\mathbb{R}^d\times[0,T]} u \le \frac{\varepsilon}{(T+\varepsilon-t)^{d/2}} + \max_{x\in\mathbb{R}^d} u(x,0)$$

Taking $\varepsilon \to 0$ we conclude that if $M < \frac{1}{4T}$,

$$\max_{\mathbb{R}^d\times[0,T]} u \le \max_{x\in\mathbb{R}^d} u(x,0)$$

<u>Step 2:</u> For general T, we denote:

$$T_1 = \frac{T}{N}, \quad N \in \mathbb{N} \text{ s.t. } M < \frac{1}{4T_1}.$$

Then by Step 1:

$$\max_{\mathbb{R}^d\times[0,T_1]} u \le \max_{x\in\mathbb{R}^d} u(x,0)$$

$$\max_{\mathbb{R}^d\times[T_1,2T_1]} u \le \max_{x\in\mathbb{R}^d} u(x,T_1) \le \max_{x\in\mathbb{R}^d} u(x,0)$$

;

$$\max_{\mathbb{R}^d \times [(N-1)T_1, NT_1]} u \le \max_{x\in\mathbb{R}^d} u(x,(N-1)T_1) \le \max_{x\in\mathbb{R}^d} u(x,0)$$

$$\rightsquigarrow \quad \max_{\mathbb{R}^d\times[0,T]} u \le \max_{x\in\mathbb{R}^d} u(x,0).$$

Remark: The condition $u \le Me^{M|x|^2}$ is necessary; otherwise, $\exists$ solutions $u \neq 0$ s.t. $u(x,0)=0$.

Theorem (Uniqueness)

If $u \in C^2_1(\mathbb{R}^d\times(0,T)) \cap C(\mathbb{R}^d\times[0,T])$

$$u(x,t) \le Me^{M|x|^2} \quad \text{in } \mathbb{R}^d\times[0,T]$$

$$\partial_t u - \Delta_x u = 0 \quad \text{in } \mathbb{R}^d\times(0,T)$$

$$u(x,0) = 0 \quad \text{in } \mathbb{R}^d$$

Then $u=0$ in $\mathbb{R}^d\times[0,T]$.

Proof: Use the maximum principle for u & $-u$.

Remark: If we know that $u(\cdot,t) \in L^2(\mathbb{R}^d)$, then the proof of the uniqueness is easier.

Heuristically:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(x,t)|^2 dt = 2 \int_{\mathbb{R}^d} \partial_t u \cdot u \, dx$$

$$= 2 \int_{\mathbb{R}^d} \Delta_x u \cdot u \, dx = - \int_{\mathbb{R}^d} |\nabla_x u|^2 dx \leq 0$$

$\Rightarrow$ $e(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx$ is decreasing

Hence, if $e(0) = 0 \Rightarrow e(t) = 0 \quad \forall t \geq 0$.

This argument will be helpful below for the heat backward equation.

Remark: The heat equation $\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=0) = g \end{cases}$

is a well-posed problem (unknown $u(\cdot,t)$, $t>0$)

- Existence
- Uniqueness
- Stability (solution depends continuously on data)

For the latter issue, by the maximum principle we have

$$\| u(\cdot,t)\|_{L^\infty} \leq \| u(\cdot,0)\|_{L^\infty}, \forall t$$

or in the L^2-situation

$$\| u(\cdot,t)\|_{L^2} \leq \| u(\cdot,0)\|_{L^2} \forall t.$$

On the other hand, the heat back-ward equation

$$\begin{cases} \partial_t u - \Delta_x u = 0 \\ u(t=T) = g \end{cases}$$

(unknown $u(\cdot,t)$, $t<T$)

is <u>not</u> well-posed.

- Non-existence: in general, the existence require some special property on g, e.g. g is very smooth (only $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ or $g \in L^2(\mathbb{R}^d)$ is not enough).

- Uniqueness: on the other hand, the uniqueness still holds.

Thm: If $u \in C^2_1(\mathbb{R}^d \times [0,T]) \cap C^1(H^1(\mathbb{R}^d), [0,T])$

and
$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,T) = 0 \end{cases}$$

Then $u = 0$ in $\mathbb{R}^d \times [0,T]$.

Proof: Recall $e(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx$ satisfies

$$e'(t) = 2\int_{\mathbb{R}^d} u \cdot \partial_t u \, dx = 2\int_{\mathbb{R}^d} u \, \Delta_x u \, dx = -2\int_{\mathbb{R}^d} |\nabla_x u|^2 dx$$

Moreover,

$$e''(t) = -4\int_{\mathbb{R}^d} \nabla_x u \cdot \nabla_x(\partial_t u) = 4\int_{\mathbb{R}^d} \Delta_x u \cdot \partial_t u \, dx$$
$$= 4\int_{\mathbb{R}^d} |\Delta_x u|^2 dx \geq 0$$

and hence

$$|e'(t)|^2 = 4\left|\int_{\mathbb{R}^d} u \, \Delta_x u \, dx\right|^2 \leq 4\left(\int_{\mathbb{R}^d} |u|^2 dx\right)\left(\int_{\mathbb{R}^d} |\Delta_x u|^2 dx\right)$$
$$= e(t)\, e''(t).$$

Lemma: If $e \in C^2(0,T)$, $e(t) \geq 0$, $e'(t) \leq 0$, $e''(t) \geq 0$ and $|e(t)|^2 \leq e(t) e''(t)$ for $t \in [0,T]$ and $e(T) = 0$, then $e \equiv 0$.

Proof: Since $e\downarrow$ and $e(T)=0$, $\exists\, t_0\in[0,T]$ s.t.

$$e(t_0)=0 \quad \text{and} \quad e(t)>0 \text{ if } t<t_0.$$

We need to prove that $t_0=0$. If not, $0<t_0\leq T$, then define

$$f(t)=\log e(t), \quad t\in(0,t_0)$$

$$\Rightarrow f'(t)=\frac{e'(t)}{e(t)}$$

$$\Rightarrow f''(t)=\frac{e''(t)\,e(t)-|e'(t)|^2}{e(t)^2}\geq 0$$

$\Rightarrow f$ is convex

$$\Rightarrow f\big(\tau t_1+(1-\tau)t_2\big)\leq \tau f(t_1)+(1-\tau)f(t_2)$$

$\forall\, t_1,t_2\in(0,t_0)$ and $\tau\in(0,1)$

$$\Rightarrow e\big(\tau t_1+(1-\tau)t_2\big)\leq e(t_1)^{\tau}\,e(t_2)^{1-\tau}$$

$\forall\, t_1,t_2\in(0,t_0)$ and $\tau\in(0,1)$.

Taking $t_2\to t_0 \Rightarrow e\big(\tau t_1+(1-\tau)t_0\big)=0$

Taking $\tau\to 1 \Rightarrow e(t_1)=0$, $\forall\, t_1\in(0,t_0)$

which is a contradiction. $\square$

Instability of the back-ward heat equation

<u>Theorem</u> (Instability) There exist functions

$$u_\varepsilon \in C_1^2(\mathbb{R}^d \times (0,T)) \cap C^1(H^1(\mathbb{R}^d) \times [0,T]) \text{ s.t.}$$

$$\partial_t u - \Delta_x u = 0 \text{ in } \mathbb{R}^d \times (0,T)$$

such that when $\varepsilon \to 0^+$

$$\| u_\varepsilon(\cdot, T) \|_{L^2(\mathbb{R}^d)} \to 0, \quad \| u(\cdot, 0) \|_{L^2(\mathbb{R}^d)} \to \infty.$$

<u>Remark:</u> This means that a small error of the data at $t = T$ may cause a large error of the output $t = 0$.

<u>Proof:</u> Recall by Fourier transform

$$\partial_t \hat{u}(k,t) = -|2\pi k|^2 \hat{u}(k,t)$$

$$\Rightarrow \quad \hat{u}(k,t) = e^{-t|2\pi k|^2} \hat{u}(k,0)$$

$$\Rightarrow \quad \hat{u}(k,0) = e^{T|2\pi k|^2} \hat{u}(k,T).$$

We can take $\hat{u}_\varepsilon(k,T) = \mathbb{1}(|k| \le \varepsilon^{-1}) \, \varepsilon^{\frac{d+1}{2}}$

$$\Rightarrow \quad \| u_\varepsilon(\cdot, T) \|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathbb{1}(|k| \le \varepsilon^{-1}) \, \varepsilon^{d+1} \, dk$$

$$\sim \varepsilon \to 0$$

and $\|u_\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} e^{2T(2\pi k)^2} \mathbb{1}(|k| \leq \varepsilon^{-1}) \, \varepsilon^{d+1} \, dk$

$$\geq \int_{\frac{\varepsilon^{-1}}{2} \leq |k| \leq \varepsilon^{-1}} \cdots \gtrsim e^{2T\varepsilon^{-2}} \varepsilon \to \infty \quad \text{as } \varepsilon \to 0$$

$\square$

<u>Theorem</u> (Regularized solution) Assume that

$u \in C^2_1(\mathbb{R}^d \times (0,T)) \cap C^1(H^1(\mathbb{R}^d), [0,T])$

$$\begin{cases} \partial_t u - \Delta_x u = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ u(x,T) = g(x) & \text{in } \mathbb{R}^d. \end{cases}$$

Then from given data $g_\varepsilon \in L^2(\mathbb{R}^d)$ s.t.

$$\|g_\varepsilon - g\|_{L^2(\mathbb{R}^d)} \leq \varepsilon$$

we can construct a sol $\tilde{u}_\varepsilon$ s.t.

$$\sup_{t \in [0,T]} \|\tilde{u}_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0.$$

<u>Remark:</u> For applications, both u and g are unknown, only g_ε is given. So we have to construct $\tilde{u}_\varepsilon$ using only information from g_ε.

<u>Proof:</u> Clearly we should not choose $\tilde{u}_\varepsilon$ to solve the eq

$$\begin{cases} \partial_t u_\varepsilon - \Delta_x u_\varepsilon = 0 \\ u_\varepsilon(t=T) = g_\varepsilon \end{cases}$$

i.e. $\widehat{u_\varepsilon}(k,t) = e^{(T-t)|2\pi k|^2} \hat{g}_\varepsilon(k)$

Rather we take

$$\widehat{\tilde{u}_\varepsilon}(k,t) = e^{(T-t)|2\pi k|^2} \hat{g}_\varepsilon(k) \, \mathbb{1}(|k| \leq \delta_\varepsilon^{-1})$$

where $\delta_\varepsilon \to 0$ (chosen later).

Then we have for all $t \in [0,T]$

$$\| \tilde{u}_\varepsilon(\cdot,t) - u(\cdot,t) \|^2_{L^2(\mathbb{R}^d)}$$

$g = u(\cdot,T)$

$$= \int_{\mathbb{R}^d} e^{2(T-t)|2\pi k|^2} \left| \hat{g}_\varepsilon(k) \mathbb{1}(|k| \leq \delta_\varepsilon^{-1}) - \hat{g}(k) \right|^2 dk$$

$$\leq 2 \int_{\mathbb{R}^d} e^{2T|2\pi k|^2} |\hat{g}_\varepsilon(k) - \hat{g}(k)| \, \mathbb{1}(|k| \leq \delta_\varepsilon^{-1}) \, dk$$

$$+ 2 \int_{\mathbb{R}^d} \underbrace{e^{2T|2\pi k|^2} |\hat{g}(k)|^2}_{|\hat{u}(k,0)|^2} \mathbb{1}(|k| > \delta_\varepsilon^{-1}) \, dk$$

$$= (\mathrm{I}) + (\mathrm{II})$$

We have:

$$(I) \leq 2\int_{\mathbb{R}^d} e^{c\delta_\varepsilon^{-2}} |\hat{g}_\varepsilon(k) - \hat{g}(k)|^2 \, dk$$

$$= 2 e^{c\delta_\varepsilon^{-2}} \varepsilon^{-2} \to 0 \text{ if } \delta_\varepsilon \gg \frac{1}{\sqrt{|\ln \varepsilon|}}$$

$$(II) = 2\int_{\mathbb{R}^d} |\hat{u}(k,0)|^2 \, \mathbb{1}(|k| \geq \delta_\varepsilon^{-1}) \, dk$$

$$\leq 2\int_{\mathbb{R}^d} |k|^2 \delta_\varepsilon^2 \, |\hat{u}(k,0)|^2 \, dk$$

$$\leq 2 \delta_\varepsilon^2 \, \|u(\cdot,0)\|_{H^1(\mathbb{R}^d)} \to 0 \text{ as } \delta_\varepsilon \to 0.$$

Thus choosin $\frac{1}{\sqrt{|\ln \varepsilon|}} \ll \delta_\varepsilon \ll 1$, e.g.

$\delta_\varepsilon = (|\ln \varepsilon|)^{-1/4}$ we find that

$$\sup_{t\in[0,T]} \|u_\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\mathbb{R}^d)} \leq (I) + (II) \to 0$$

as $\varepsilon \to 0$.

$\square$