Chapter 2: Convolution, Fourier transform and distributions

Definition (convolution).

If $f, g : \mathbb{R}^d \to \mathbb{R}$ (or $\mathbb{C}$), then $f * g : \mathbb{R}^d \to \mathbb{R}$

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

Remark:

•) $$f * g = g * f$$

Since $(f*g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y)\, dy$

$$\overset{z = x-y}{=} \int_{\mathbb{R}^d} f(z)\, g(x-z)\, dz$$

$$= (g * f)(x).$$

•) $(f * g) * h = f * (g * h)$ by Fubini

•) $\widehat{f * g} = \hat{f}\, \hat{g}$ (Fourier transform)

Theorem: given $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$. Then $f * g \in L^p(\mathbb{R}^d)$ and

$$\|f*g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}. \quad \text{(Young's inequality)}$$

More generally, if $f \in L^p$, $g \in L^q$, then $f*g \in L^r$ and

$$\|f*g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

provided that $1 \leq p, q, r \leq \infty$, $\boxed{\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}}$.

Proof: By Hölder inequality if $1 < p < \infty$

$$|(f*g)(x)| = \left| \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy \right|$$

$$\leq \left(\int_{\mathbb{R}^d} |f(x-y)|\, dy \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^d} |f(x-y)| |g(y)|^p\, dy \right)^{\frac{1}{p}}$$

$$= \|f\|_{L^1}^{\frac{1}{p'}} \left(\int_{\mathbb{R}^d} |f(x-y)|\, |g(y)|^p\, dy \right)^{\frac{1}{p}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$

$$\Rightarrow \quad |(f*g)(x)|^p \leq \|f\|_{L^1}^{\frac{p}{p'}} \int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|^p\,dy$$

$$\Rightarrow \int_{\mathbb{R}^d} |(f*g)(x)|^p\,dx \leq \|f\|_{L^1}^{\frac{p}{p'}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)|\,|g(y)|^p\,dy\,dx$$

$$= \|f\|_{L^1}^{\frac{p}{p'}} \int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d} |f(x-y)|\,dx\right)}_{\|f\|_{L^1}} |g(y)|^p\,dy$$

$$= \|f\|_{L^1}^{\frac{p}{p'}+1} \quad \|g\|_{L^p}^p$$

$$= \left(\|f\|_{L^1} \|g\|_{L^p}\right)^p \quad \text{since}$$

$$\frac{p}{p'}+1 = p\Big(\underbrace{\frac{1}{p'}+\frac{1}{p}}_{=1}\Big) = p$$

$$\Rightarrow \quad \|f*g\|_{L^p} \leq \|f\|_{L^1}\,\|g\|_{L^p}.$$

Cases $\begin{cases} p=1 \\ p=\infty \end{cases}$ trivial (triangle inequality).

Theorem: If $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ for $1 \leq p \leq \infty$. Then $f * g \in C^\infty(\mathbb{R}^d)$

and
$$D^\alpha(f*g) = (D^\alpha f)*g \quad , \forall \alpha.$$

Proof: First we prove that $f*g$ is continuous.

Take $\{y_n\} \subset \mathbb{R}^d$, $y_n \to y$ in $\mathbb{R}^d$. Then

$$(f*g)(y_n) = \int_{\mathbb{R}^d} f(y_n - x)\, g(x)\, dx$$

$$\longrightarrow \int_{\mathbb{R}^d} f(y-x)\, g(x)\, dx$$

by Dominated convergence. In fact:

$$f(y_n - x)\, g(x) \longrightarrow f(y-x)\, g(x)$$

and
$$|f(y_n - x)\, g(x)| \leq \underbrace{\|f\|_{L^\infty} \mathbb{1}_{B(0,R)}(x)\, |g(x)|}_{\in L^1(\mathbb{R}^d)}$$

where R is chosen s.t.

$$f(y_n - x) = 0 \quad \text{if} \quad |x| \geq R$$

which is doable since $y_n \to y$ & f is compactly supported.

Similarly: $e_i = (0, \dots, 1, \dots) \in \mathbb{R}^d$
ith

$$\partial_{x_i}(f*g)(x) = \lim_{h\to 0} \int \frac{f(x+he_i-y) - f(x-y)}{h} g(y)\,dy$$

Dominated Convergence

$$= \int \lim_{h\to 0} \frac{f(x+he_i-y) - f(x-y)}{h} g(y)\,dy$$

$$= \int (\partial_{x_i} f)(x-y)\, g(y)\,dy$$

$$= (\partial_{x_i} f) * g$$

And $\partial_{x_i} f * g \in C(\mathbb{R})$ by the previous step

Since $\partial_{x_i} f \in C_c^\infty(\mathbb{R}^d)$. The same argument gives

$$D^\alpha(f*g) = (D^\alpha f)*g \in C(\mathbb{R})$$

$\forall \alpha \rightsquigarrow$ conclusion.

Remark: $\exists$ <u>no</u> regular function ρ s.t.

$$\rho * g = g \qquad \text{(exercise)}$$

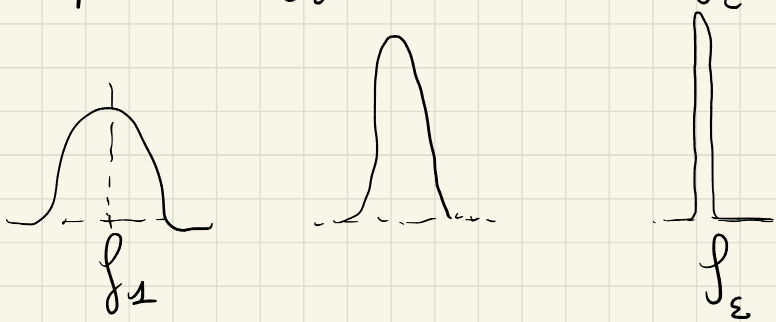
But actually we have:

$$\delta_0 * g = g \qquad \text{with } \delta_0 \text{ the delta function}$$

This "function" δ_0 can be defined properly as a distribution, i.e. $\delta_0 \in \left(C_c^\infty(\mathbb{R}^d)\right)'$.

Mathematically, we can approximate δ_0 by a sequence $\{\rho_\varepsilon\}_{\varepsilon \to 0}$ where $\rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x)$

with $\rho \in C_c(\mathbb{R}^d)$



$$\int \rho_\varepsilon = \int \rho = 1$$

Formally

$$\rho_\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} \delta_0(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

$$\int \delta_0 = 1$$

__Theorem__ (Approximation by convolution)

Let $f \in L^1(\mathbb{R}^d)$ s.t. $\int_{\mathbb{R}^d} f = 1$ and $f_\varepsilon(x) = \varepsilon^{-d} f(\varepsilon^{-1}x)$

Then for all $1 \leq p < \infty$ and $g \in L^p(\mathbb{R}^d)$:

$$f_\varepsilon * g \longrightarrow g \quad \text{in} \quad L^p(\mathbb{R}^d).$$

__Proof:__

__Step 1:__ Assume $f, g \in C_c(\mathbb{R}^d)$. Then:

$$(f_\varepsilon * g)(x) - g(x) = \int f_\varepsilon(y)\, g(x-y)\, dy - \int f_\varepsilon(y)\, g(x)\, dy$$

$$= \int_{\mathbb{R}^d} \varepsilon^{-d} f(\varepsilon^{-1}x) \big(g(x-y) - g(x) \big)\, dy$$

Assume $\operatorname{supp} f \subset B(0,R)$ i.e. $f(x) = 0$ if $|x| > R$

$\Rightarrow \quad \operatorname{supp} f_\varepsilon \subset B(0, R\varepsilon)$

$$\Rightarrow \left| (f_\varepsilon * g)(x) - g(x) \right| \leq \int_{|y| \leq R\varepsilon} \varepsilon^{-d} |f(\varepsilon^{-1}y)|\, \underbrace{|g(x-y) - g(x)|}\, dy$$

$$\leq \underbrace{\|f_\varepsilon\|_{L^1}}_{\|f\|_{L^1}} \sup_{|z| \leq R\varepsilon} |g(x-z) - g(x)| \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

Moreover, if $\operatorname{supp} g \subset B(0,R_1)$, then

$$\sup_{|z|\leq \frac{R}{n}} |g(x-z)-g(x)| \begin{cases} = 0 \text{ if } |x| > R_1 + R\varepsilon \\ \text{and bounded by } 2\|g\|_{L^\infty} \end{cases}$$

Thus:

$$|(\rho_n * g)(x) - g(x)| \longrightarrow 0 \quad \text{for all } x \in \mathbb{R}^d$$

$$|(\rho_n * g)(x) - g(x)| \leq 2\|g\|_{L^\infty} \mathbb{1}_{B(0,R_1+1)}(x)$$

for all $|\varepsilon| < 1$

$$\Rightarrow \quad \rho_n * g - g \longrightarrow 0 \quad \text{in } L^p(\mathbb{R}^d)$$

by Dominated convergence.

Next we remove the technical assumption $f, g \in C_c(\mathbb{R}^d)$. We will use the fact that

$$\boxed{C_c(\mathbb{R}^d) \text{ is dense in } L^p(\mathbb{R}^d),\ \forall 1 \leq p < \infty}$$

which can be proved without using convolution.

<u>Step 2:</u> Let $f \in C_c(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$.
Then we can find a sequence $g_m \in C_c(\mathbb{R}^d)$ s.t. $g_m \longrightarrow g$ in $L^p(\mathbb{R}^d)$. Then:

$$\|f_\varepsilon * g - g\|_{L^p} \leq \|f_\varepsilon * (g - g_m)\|_{L^p} + \|f_\varepsilon * g_m - g_m\|_{L^p}$$
$$+ \|g_m - g\|_{L^p}$$

$$\leq \underbrace{\|f_\varepsilon\|_{L^1}}_{=\|f\|_{L^1}} \|g - g_m\|_{L^p} + \|f_\varepsilon * g_m - g_m\|_{L^p}$$
$$+ \|g_m - g\|_{L^p}$$

$$\leq (\|f\|_{L^1} + 1) \|g_m - g\|_{L^p} + \underbrace{\|f_\varepsilon * g_m - g_m\|_{L^p}}_{\xrightarrow{\varepsilon \to 0} 0}$$

$$\Rightarrow \limsup_{\varepsilon \to \infty} \|f_\varepsilon * g - g\|_{L^p} \leq (\|f\|_{L^1} + 1) \|g_m - g\|_{L^p} \xrightarrow{m \to \infty} 0$$

Step 3. Let $\rho \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$.

$\exists\, F_m \in C_c(\mathbb{R}^d)$ s.t.

$$\begin{cases} F_m \to \rho \text{ in } L^1(\mathbb{R}^d) \text{ when } m \to \infty \\ \int_{\mathbb{R}^d} F_m = 1, \ \forall m \geq 1 \quad \left(\text{note: } \int_{\mathbb{R}^d} \rho = 1\right) \end{cases}$$

Define $F_{m,\varepsilon}(x) = \varepsilon^{-d} F_m(\varepsilon^{-1} x)$. Then:

Then by the triangle & Young inequalities

$$\|\rho_\varepsilon * g - g\|_{L^p} \leq \|(\rho_\varepsilon - F_{m,\varepsilon}) * g\|_{L^p}$$

$$+ \|F_{m,\varepsilon} * g - g\|_{L^p} \xrightarrow{\varepsilon \to 0} 0$$

$$\leq \underbrace{\|\rho_\varepsilon - F_{m,\varepsilon}\|_{L^1}}_{=\ \|\rho - F_m\|_{L^1}} \|g\|_{L^p} + \|F_{m,n} * g - g\|_{L^p}$$

$\|\rho - F_m\|_{L^1} \to 0$ as $m \to \infty$. $\square$

(App. C4 Evans)

<u>Theorem:</u> Let Ω be open in $\mathbb{R}^d$ and define

$$\Omega_\varepsilon = \{ x \in \Omega : \ \operatorname{dist}(x, \Omega^c) > \varepsilon \}.$$

Let $f \in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} f = 1$, $f = 0$ in $|x| \geq 1$.

Denote $f_\varepsilon(x) = \varepsilon^{-d} f(\varepsilon^{-1} x)$. Then $\forall g \in L^p_{loc}(\Omega)$, $(1 \leq p < \infty)$

$$g_\varepsilon = (f_\varepsilon * g)(x) = \int_{\mathbb{R}^d} f_\varepsilon(x-y) g(y) \, dy$$

is well-defined in Ω_ε and:

(a) $g_\varepsilon \in C^\infty(\Omega_\varepsilon)$

(b) $g_\varepsilon \to g$ in $L^p_{loc}(\Omega)$ (and a.e.)

(c) If $g \in C(\Omega)$, then $g_\varepsilon \to g$ uniformly in any compact subset of Ω.

<u>Proof.</u> (a) $D^\alpha(g_\varepsilon) = (D^\alpha f_\varepsilon) * g \in C(\Omega_\varepsilon)$

(b) Replace g by $\chi_U g$ for $U \subset\subset \Omega$.

(c) Already in the proof of $\mathbb{R}^d$ case.
(g is uniformly continuous in compact set)

Fourier transform:

Def: Given $f : \mathbb{R}^d \to \mathbb{C}$, define

$$(\mathcal{F}f)(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i k\cdot x}\, dx$$

$\llbracket$ Here $k \cdot x = \sum_{i=1}^{d} k_i \cdot x_i$ with $k = (k_i)$, $x = (x_i)$ $\rrbracket$

Theorem (Basic properties)

(a) If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^\infty(\mathbb{R}^d)$ and

$$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

(b) The mapping $\mathcal{F} : L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ can be extended to be a unitary transformation $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ s.t.

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2}, \quad \forall f \in L^2(\mathbb{R}^d). \quad \text{(Plancherel)}$$

(c) The inverse mapping $\mathcal{F}^{-1}$ is well defined as

$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = \int_{\mathbb{R}^d} f(k)\, e^{2\pi i k\cdot x}\, dk, \quad \forall f \in L^1 \cap L^2$$

(d) $\widehat{D^\alpha f} = (2\pi i k)^\alpha \hat{f}(k)$ if $(2\pi i k)^\alpha \hat{f}(k) \in L^2(\mathbb{R}^d)$

(e) $\widehat{f * g} = \hat{f}\,\hat{g}$ if $f \,\&\, g$ "nice enough"

(we will make rigorous later)

Theorem (Hausdorff-Young inequality)

If $1 \leq p \leq 2$, then $\mathcal{F} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ with

$$\frac{1}{p} + \frac{1}{q} = 1$$

and

$$\|\mathcal{F} f\|_{L^q} \leq \|f\|_{L^p}, \ \forall f \in L^p(\mathbb{R}^d).$$

Proof: No easy proof. We can deduce it from the Riesz-Thorin interpolation theorem:

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let Ω be open in $\mathbb{R}^d$.
Let:

$$T : L^{p_0}(\Omega) + L^{p_1}(\Omega) \to L^{q_0}(\Omega) + L^{q_1}(\Omega)$$

be a linear map such that $T : L^{p_i} \to L^{q_i}$ and

$$\|T\|_{L^{p_i} \to L^{q_i}} \leq 1,$$

Then $T : L^{p_\theta} \to L^{q_\theta}$ and $\|T\|_{L^{p_\theta} \to L^{q_\theta}} \leq 1$ for all $0 < \theta < 1$ where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Theorem (Fourier transform & convolution)

Let $1 \leq p, q, r \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Recall: we know that if $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$. The new statement here is that if we assume further $1 \leq p, q, r \leq 2$, then:

$$\widehat{f * g} = \hat{f} \cdot \hat{g}$$

Proof: If $f, g \in C_c(\mathbb{R}^d)$, then by Fubini

$$\widehat{f*g}(k) = \iint f(x-y)\, g(y)\, e^{-2\pi i k \cdot x}\, dx\, dy$$

$$= \iint f(x-y)\, e^{-2\pi i k(x-y)} \cdot g(y)\, e^{-2\pi i k \cdot y}\, dx\, dy$$

$$= \left(\int f(z) e^{-2\pi i k \cdot z}\, dz \right) \left(\int g(y) e^{-2\pi i k \cdot y}\, dy \right)$$

$$= \hat{f}(k)\, \hat{g}(k).$$

General case: density argument (exercise).

Fundamental solution of Laplace equation

Consider $\qquad -\Delta u = f \quad \text{in } \mathbb{R}^d$

$$\Rightarrow \quad |2\pi k|^2 \quad \hat{u}(k) = \hat{f}(k)$$

$$\Rightarrow \quad \hat{u}(k) = \frac{1}{|2\pi k|^2} \hat{f}(k)$$

If we can find G s.t. $\hat{G}(k) = \frac{1}{|2\pi k|^2}$, then

$$\hat{u}(k) = \hat{G}(k) \hat{f}(k) = \widehat{G * f}(k)$$

$$\Rightarrow \quad u = G * f$$

Thus we need to compute

$$G(x) = \left(\frac{1}{|2\pi k|^2} \right)^{\vee}$$

It turns out that for $d \geq 3$,

$$G(x) = \frac{1}{d(d-2)|B_1||x|^{d-2}} = \text{the fundamental sol. of Laplace eq.}$$

To make it rigorous, we need to compute the Fourier transform of $\frac{1}{|x|^\alpha}$, $0 < \alpha < d$.

Theorem: (Fourier transform of $|x|^{-\alpha}$, $0<\alpha<d$)

a) The function $|x|^{-\alpha}$, $x \in \mathbb{R}^d$, with $0<\alpha<d$, satisfies formally:

$$\widehat{\frac{c_\alpha}{|x|^\alpha}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}}$$

where

$$c_\alpha = \pi^{-\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2}\right) = \pi^{-\frac{\alpha}{2}} \int_0^\infty \lambda^{\frac{\alpha}{2}-1} e^{-\lambda} \, d\lambda .$$

b) More precisely, $\forall f \in C_c^\infty(\mathbb{R}^d)$:

$$\frac{c_\alpha}{|x|^\alpha} * f = \Bigg(\underbrace{\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k)}_{\in L^1(\mathbb{R}^d)} \Bigg)^\vee .$$

(c) Moreover, if $d > \alpha > d/2$, then:

$$\widehat{\underbrace{\frac{c_\alpha}{|x|^\alpha} * f}_{\in L^2(\mathbb{R}^d)}} = \frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) .$$

Remark: If $\int f \neq 0$, then $\frac{c_\alpha}{|x|^\alpha} * f \sim \frac{c_\alpha (\int f)}{|x|^\alpha}$ as $|x| \to \infty$, which is not in any $L^p(\mathbb{R}^d)$ with $1 \leq p \leq 2$ when $\alpha \leq d/2$.

Proof: Lemma: $\mathcal{F}\left(e^{-\pi |x|^2}\right) = e^{-\pi |k|^2}$.

More generally:

$$\mathcal{F}\left(e^{-\pi \lambda^2 |x|^2}\right) = \lambda^{-d} e^{-\pi \frac{|k|^2}{\lambda^2}}, \ \forall \lambda > 0$$

(Proof: exercise)

(a) Formally:

$$\frac{c_\alpha}{|x|^\alpha} = \pi^{-\frac{\alpha}{2}} \int_0^\infty e^{-\lambda} \lambda^{\frac{\alpha}{2}-1} d\lambda \cdot \frac{1}{|x|^\alpha}$$

$$= \int_0^\infty e^{-\pi |x|^2 \lambda} \lambda^{\alpha/2-1} d\lambda$$

$$\Rightarrow \widehat{\frac{c_\alpha}{|x|^\alpha}} = \int_0^\infty \widehat{e^{-\pi |x|^2 \lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

$$\overset{(a)}{=} \int_0^\infty \lambda^{-\frac{d}{2}} e^{-\pi \frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha}{2}-1} d\lambda$$

$$= \int_0^\infty e^{-\pi \frac{|k|^2}{\lambda}} \lambda^{\frac{\alpha-d}{2}-1} d\lambda$$

$$\overset{\lambda \mapsto \frac{1}{\lambda}}{=} \int_0^\infty e^{-\pi |k|^2 \lambda} \lambda^{\frac{d-\alpha}{2}+1} \cdot \frac{1}{\lambda^2} d\lambda$$

$$= \int_0^\infty e^{-\pi |k|^2 \lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda = \frac{c_{d-\alpha}}{|k|^{d-\alpha}}$$

(b) Rigorously: $f \in C_c^\infty(\mathbb{R}^d)$. Then $\hat{f} \in L^1(\mathbb{R}^d) \cap L^\infty$ (exercise), and hence $\frac{1}{|k|^{d-\alpha}} \hat{f}(k) \in L^1(\mathbb{R}^d)$.

This allows us to compute

$$\left(\frac{c_{d-\alpha}}{|k|^{d-\alpha}} \hat{f}(k) \right)^\vee = \int_{\mathbb{R}^d} e^{2\pi i k \cdot x} \left(\int_0^\infty e^{-\pi k^2 \lambda} \lambda^{\frac{d-\alpha}{2}-1} d\lambda \right) \hat{f}(k)\, dk$$

$$= \int_0^\infty \left[\int_{\mathbb{R}^d} e^{2\pi i k \cdot x} e^{-\pi k^2 \lambda} \hat{f}(k)\, dk \right] \lambda^{\frac{d-\alpha}{2}-1} d\lambda$$

$$= \int_0^\infty \left(e^{-\pi k^2 \lambda} \hat{f}(k) \right)^\vee \lambda^{\frac{d-\alpha}{2}-1} d\lambda$$

$$= \int_0^\infty \left(\lambda^{-\frac{d}{2}} e^{-\pi \frac{x^2}{\lambda}} * f \right) \lambda^{\frac{d-\alpha}{2}-1} d\lambda$$

$$= \left(\int_0^\infty \lambda^{-\frac{\alpha}{2}-1} e^{-\pi \frac{x^2}{\lambda}} d\lambda \right) * f$$

$$\overset{\lambda \mapsto \frac{1}{\lambda}}{=} \left(\int_0^\infty \lambda^{\frac{\alpha}{2}-1} e^{-\lambda x^2 \lambda} d\lambda \right) * f = \frac{c_\alpha}{|x|^\alpha} * f$$

(c) If $\alpha > \frac{d}{2}$, then $\frac{1}{|x|^\alpha} * f \in L^2$ ⇝ the Fourier transform is well-defined.

Theory of distributions:

Let Ω open $\subset \mathbb{R}^d$. We denote:

- Test functions $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$
- $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ if $\forall K$ compact $\subset \Omega$,

$$\begin{cases} \operatorname{supp}(\varphi_n - \varphi) \subset K, \ \forall n \\ \| D^\alpha(\varphi_n - \varphi) \|_{L^\infty(K)} \to 0 \text{ as } n \to \infty, \ \forall \alpha \end{cases}$$

- Distributions $\mathcal{D}'(\Omega) = \{ T : \mathcal{D}(\Omega) \to \mathbb{R} \text{ or } \mathbb{C} \}$

linear & continuous

i.e. $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$

$\Rightarrow T(\varphi_n) \to T(\varphi)$.

Example: Regular functions are distributions:

If $f \in L^1_{loc}(\Omega)$, define

$$T_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx$$

Exercise: $T_f \in \mathcal{D}'(\Omega)$

Question: Why $f \mapsto T_f$ is injective, i.e.

why $T_f = 0 \overset{?}{\Rightarrow} f = 0$.

<u>Theorem:</u> (Fundamental theorem of calculus)

Let Ω be an open set in $\mathbb{R}^d$. Let $f \in L^1_{loc}(\Omega)$

If

$$\int_\Omega f\varphi = 0 \quad , \forall \varphi \in C_c^\infty(\Omega)$$

then $\quad f = 0$. a.e.

<u>Proof:</u> Take $\Omega_\varepsilon \overset{open}{\subset\subset} \Omega$ and $g \in C_c^\infty(\Omega_\varepsilon)$. Then

$$\int_\Omega f \underbrace{g\varphi}_{\in C_c^\infty(\Omega)} = 0 \quad , \forall \varphi \in C_c^\infty(\Omega)$$

Note: $\quad fg \in L^1(\Omega_\varepsilon)$ and supported in Ω_ε.

Take $h \in C_c^\infty(B(0,1))$, $\int h = 1$, $h_\varepsilon(x) = \varepsilon^{-d} h(\varepsilon^{-1}x)$.

Then: $$h_\varepsilon * (fg) \underset{\varepsilon\to 0}{\longrightarrow} fg \quad \text{in } L^1(\Omega_\varepsilon)$$

but

$$h_\varepsilon * (fg) = \int h_\varepsilon(x-y) f(y) g(y) = 0$$

$$\Rightarrow fg = 0 \quad , \forall g \in C_c^\infty(\Omega_\varepsilon)$$

$$\Rightarrow f(x) = 0, \quad \text{a.e. } x \in \Omega_\varepsilon$$

$$\Rightarrow f(x) = 0 \quad , \quad \text{a.e. } x \in \Omega. \qquad \square$$

Example: **Dirac delta function:**

Let Ω open $\subset \mathbb{R}^n$ and $x \in \Omega$. Let $\delta_x : \mathcal{D}(\Omega) \to \mathbb{C}$ defined by $\delta_x(\varphi) = \varphi(x)$, $\forall \varphi \in \mathcal{D}(\Omega)$.

Exercise: $\delta_x \in \mathcal{D}'(\Omega)$ and $\delta_x \notin L^1_{loc}(\Omega)$.

Example: (Principle value) The function

$$f(x) = \frac{1}{x}$$

is not in $L^1_{loc}(\mathbb{R})$. However,

$$\int_{\mathbb{R}} f\varphi \text{ is well-defined } \forall \varphi \in C_c^\infty(\mathbb{R}) \text{ s.t. } \varphi(0) = 0.$$

Question: $\exists ?\, T \in \mathcal{D}'(\mathbb{R})$ s.t.

$$T(\varphi) = \int_{\mathbb{R}} f\varphi, \ \forall \varphi \in C_c^\infty(\mathbb{R})$$

Exercise. Define

$$T(\varphi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} f\varphi \quad , \forall \varphi \in C_c^\infty(\mathbb{R})$$

Prove that $T \in \mathcal{D}'(\mathbb{R})$.

Hint: $$\frac{\varphi(x) - \varphi(-x)}{x} \to 2\varphi'(0) \quad \text{as } x \to 0$$

Def: (Derivatives of distributions) Let Ω open $\subset \mathbb{R}^d$.
If $T \in \mathcal{D}'(\Omega)$, we define $D^\alpha T$ as

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|}\, T(D^\alpha \varphi)$$

Exercise: Prove that if $T \in \mathcal{D}'(\Omega)$, then all distributional derivatives $D^\alpha T$ exist and belong to $\mathcal{D}(\Omega)$.

Motivation:

If $T = T_f$ with $f \in C_c^\infty(\Omega)$, then: $\forall \varphi \in C_c^\infty$

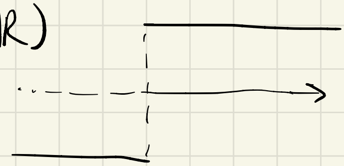
$$\begin{aligned}(D^\alpha T_f)(\varphi) &= (-1)^{|\alpha|}\, T_f(D^\alpha \varphi) \\ &= (-1)^\alpha \int_\Omega f \cdot (D^\alpha \varphi) \\ &= \int_\Omega (D^\alpha f) \cdot \varphi = (T_{D^\alpha f})\varphi\end{aligned}$$

$$\Rightarrow \quad D^\alpha T_f = T_{D^\alpha f}$$

i.e. the distributional derivatives = the classical derivatives if both exist.

Example: Consider $f(x) = |x|$ with $x \in \mathbb{R}$. Then $f \notin C^1(\mathbb{R})$ but the distributional derivative exists:

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \in L^1_{loc}(\mathbb{R})$$



Moreover,

$$f''(x) = 2\delta_0(x) \in D'(\mathbb{R})$$

Exercise: Prove that

$$(D^\alpha \delta_x)(\varphi) = (-1)^{|\alpha|} (D^\alpha \varphi)(x).$$

Def (Convergence of distributions)

We say that $T_n \to T$ in $D'(\Omega)$ if

$$T_n(\varphi) \to T(\varphi), \quad \forall \varphi \in D(\Omega).$$

Exercise: Let $f \in L^1(\mathbb{R}^d)$, $\int f = 1$, $f_\varepsilon(x) = \varepsilon^{-d} f(\varepsilon^{-1} x)$

Prove that $f_\varepsilon \to \delta_0$ in $D'(\mathbb{R}^d)$ as $\varepsilon \to 0$

Exercise: Let $T_n \to T$ in $D'(\Omega)$. Prove that

$$D^\alpha T_n \to D^\alpha T \text{ in } D'(\Omega), \ \forall \alpha.$$

Def: (Distributions & convolutions)

Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Define:

$(T * \tilde{f})(y) = T(f_y)$, $f_y(x) = f(x-y)$, $\tilde{f}(x) = f(-x)$.

Theorem: $\forall T \in \mathcal{D}'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$, then:

(a) $T * \tilde{f} \in C^\infty(\mathbb{R}^d)$ and

$$D_y^\alpha (T * \tilde{f}) = (D^\alpha T)(f_y) = (-1)^{|\alpha|} T((D^\alpha f)_y).$$

(b) $\forall g \in L^1(\mathbb{R}^d)$ & compactly supported, Then:

$$\int_{\mathbb{R}^d} g(y)\, T(f_y) = T(\underbrace{f * g}_{\in C_c^\infty(\mathbb{R}^d)}).$$

Proof: (a) Let $f \in C_c^\infty(\mathbb{R}^d)$. If $y_n \to y$ in $\mathbb{R}^d$, then:

$$|f_{y_n}(x) - f_y(x)| = |f(x-y_n) - f(x-y)|$$

$$\le \|\nabla f\|_{L^\infty} |y_n - y| \xrightarrow{n\to\infty} 0$$

uniformly in x

Similarly: $D_x^\alpha f_{y_n} \xrightarrow{n\to\infty} D_x^\alpha f_y$ uniformly

Hence

$$T \in \mathcal{D}'(\mathbb{R}^d) \to T(f_{y_n}) \xrightarrow{n\to\infty} T(f_y) \Rightarrow \underset{\text{cont.}}{y \mapsto T(f_y)}$$

Similarly

$$\left| \frac{f(x+he_i-y) - f(x-y)}{h} - \partial_{x_i} f(x-y) \right| \le C h$$

$$\Rightarrow \quad \frac{f(x+he_i-y) - f(x-y)}{h} \xrightarrow[h\to 0]{} \partial_{x_i} f(x-y) \quad \text{uniformly in } x$$

namely

$$\frac{f_{y-he_i} - f_y}{h} \xrightarrow[h\to 0]{} (\partial_{x_i} f)_y \quad \text{uniformly}$$

Similarly:

$$D_x^\alpha\left(\frac{f_{y-he_i} - f_y}{h} \right) \xrightarrow{h\to 0} D_x^\alpha (\partial_{x_i} f)_y \quad \text{uniformly } \forall \alpha$$

$$\Rightarrow \quad T\left(\frac{f_{y-he_i} - f_y}{h} \right) \xrightarrow{h\to 0} T((\partial_{x_i} f)_y)$$

$$\Rightarrow \quad \partial_{y_i} T(f_y) = - T((\partial_{x_i} f)_y) \in C(\mathbb{R}^d)$$

$$\Rightarrow \quad y \mapsto T(f_y) \in C^1(\mathbb{R}^d)$$

By induction we find that

$$D_y^\alpha T(\rho_y) = (-1)^{|\alpha|} T\big((D^\alpha \rho)_y\big)$$
$$= (D^\alpha T)(\rho_y) \qquad \forall \alpha$$

$\Rightarrow y \mapsto T(\rho_y) \in C^\infty(\mathbb{R})$.

(b) Step 1: First consider $g \in C_c^\infty(\mathbb{R})$. Then:

$$\int_{\mathbb{R}^d} g(y)\, T(\rho_y)\, dy \underset{\text{Riemann}}{=} \lim_{\Delta_m \to 0} \Delta_m \sum_{j=1}^m g(y_j)\, T(\rho_{y_j})$$

$$= \lim_{\Delta_m \to 0} T\Big(\Delta_m \sum_{j=1}^m g(y_j)\, \rho_{y_j}\Big) = T(g * \rho)$$

Since

$$\lim_{\Delta_m \to 0} \Delta_m \sum_{j=1}^m g(y_j)\, \rho_{y_j}(x) = \lim_{\Delta_m \to 0} \Delta_m \sum_{j=1}^m g(y_j)\, \rho(x - y_j)$$
$$= \int_{\mathbb{R}^d} g(y)\, \rho(x-y)\, dy = (\rho * g)(x)$$

in $D(\mathbb{R}^d)$.

Step 2: Now take $g \in L^1(\mathbb{R}^d)$, g is compactly supported (so $g * \rho \in C_c^\infty(\mathbb{R}^d)$).

Then we can approximate g by $g_n \in C_c^\infty(\mathbb{R}^d)$ s.t. $\{g_n\}, \{g\}$ all supported in a compact set K and $\|g_n - g\|_{L^1} \to 0$. This implies that

$$f * g_n \longrightarrow f * g \quad \text{in } D(\mathbb{R}^d),$$

$$\Rightarrow \quad T(f * g_n) \xrightarrow[n\to\infty]{} T(f*g)$$

$$\overset{\text{Step 1}}{\|}$$

$$\int_{\mathbb{R}^d} g_n(y)\, T(f_y)\, dy \longrightarrow \int_{\mathbb{R}^d} g(y)\,(T*f)(y)\, dy$$

as $g_n \to g$ in L^1 & $T*f \in C_c^\infty$.

<u>Theorem</u> Let Ω open $\subset \mathbb{R}^d$, $f \in C_c^\infty(\Omega)$ and

$$\Omega_f = \{ y \in \mathbb{R}^d : \operatorname{supp} f_y \subset \Omega \}.$$

Then $\forall T \in D'(\Omega)$:

(a) $y \mapsto T(f_y) \in C^\infty(\Omega_f)$ and

$$D_y^\alpha\left(T(f_y)\right) = (D^\alpha T)(f_y) = (-1)^{|\alpha|}\, T\left((D^\alpha f)_y\right).$$

(b) $\forall g \in L^1(\Omega_f)$ and compactly supported

$$\int_{\Omega_f} g(y)\, T(f_y)\, dy = T(\underbrace{f*g}_{\in C_c^\infty(\Omega)})$$

(Exercise)

Theorem: (Taylor expansion for distributions)

Let Ω open $\subset \mathbb{R}^d$, $T \in \mathcal{D}'(\Omega)$ and $f \in \mathcal{D}(\Omega)$.
Let $y \in \mathbb{R}^d$ s.t. $f_{ty} \in \mathcal{D}(\Omega)$, $\forall 0 \leq t \leq 1$. Then:

$$T(f_y) = T(f) + \int_0^1 \sum_{j=1}^d y_j (\partial_j T)(f_{ty})\, dt.$$

Proof: Since $y \mapsto T(f_y)$ is $C^\infty(\Omega_f)$ and

$$\frac{d}{dt}\left[T(f_{ty})\right] = y \cdot \left[\nabla T(f_{ty})\right] = \sum_{j=1}^d y_j (\partial_j T)(f_{ty})$$

$$\Rightarrow \quad T(f_y) - T(f) = \int_0^1 \frac{d}{dt}\left[T(f_{ty})\right] dt$$

$$= \int_0^1 \sum_{j=1}^d y_j \cdot \partial_j T(f_{ty})\, dt$$

Corollary: If $g \in L^1_{loc}(\mathbb{R}^d)$ and $\partial_{x_i} g \in L^1_{loc}(\mathbb{R}^d)$ for all $i = 1, 2, \dots, d$ (i.e. $g \in W^{1,1}_{loc}(\mathbb{R}^d)$), then $\forall y$:

$$g(x+y) = g(x) + \int_0^1 y \cdot \nabla g(x+ty)\, dt \quad \text{for a.e } x$$

Proof:

If $g \in W^{1,1}_{loc}$, then $\forall f \in C_c^\infty$ we have:

$$\int f(x)\left[g(x+y) - g(x)\right] dx = g(f_y) - g(f)$$

$$= \int_0^1 \sum_{j=1}^d y_j \cdot (\partial_j g)(f_{ty})\, dt$$

$$= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} (\partial_j g)(x)\ f(x-ty)\, dx \quad dt$$

$$= \int_0^1 \sum_{j=1}^d y_j \int_{\mathbb{R}^d} \partial_j g\,(x+ty)\ f(x)\, dx\, dt$$

$$= \int_{\mathbb{R}^d} f(x) \left[\int_0^1 \sum_{j=1}^d y_j \cdot \partial_j g\,(x+ty)\, dt \right] dx$$

$\forall f$

$$\Rightarrow \quad g(x+y) - g(x) = \int_0^1 \sum_{j=1}^d y_j \cdot \partial_j\, g(x+ty)\, dt \quad \text{a.e. } x$$

<u>Corollary:</u> If $T \in \mathcal{D}'(\Omega)$ and $\nabla T = 0$, then $T = \text{const}$.

<u>Proof:</u> $(\Omega = \mathbb{R}^d)$

$$T(f_y) = T(f), \quad \forall f \in C_c^\infty, \quad \forall y \in \mathbb{R}^d$$

$$\Rightarrow \left(\int g\right) T(f) = \int_{\mathbb{R}^d} T(f_y)\, g(y)\, dy = T(f * g)$$

$$= \left(\int f\right) T(g), \quad \forall f, g \in C_c^\infty$$

(due to the symmetry between $f \leftrightarrow g$)

Thus. $T(f) / \left(\int f\right)$ is independent of f, namely

$$T(f) = \text{const.} \int f \quad \Rightarrow T = \text{const.} \qquad \square$$

Theorem: (Equivalence of classical & distributional derivatives) Let Ω open $\subset \mathbb{R}^d$, $T \in \mathcal{D}'(\Omega)$, $g_i = \partial_{x_i} T$, $\forall i = 1, \dots, d$. Then TFAE:

1) $T = G$ with $F \in C^1(\Omega)$ and $g_i = \partial_{x_i} G$

2) $g_i \in C(\Omega)$, $\forall i = 1, 2, \dots, d$.

Proof: $\boxed{(1) \Rightarrow (2)}$ By integration by part

$$g_i(\varphi) = - \quad T(\partial_i \varphi) = -\int G(\partial_i \varphi) = \int (\partial_i G)\varphi$$
$$\forall \varphi \in C_c^\infty(\Omega)$$

$$\longrightarrow \quad g_i = \partial_i G \in C(\Omega)$$

$\boxed{(2) \Rightarrow (1)}$ $(\Omega = \mathbb{R}^d)$ Assume $g_i = \partial_{x_i} T \in C(\Omega)$. $\forall \rho \in C_c^\infty$

$$T(\rho_y) - T(\rho) = \int_0^1 \sum_{j=1}^d y_j \, (\partial_{x_j} T)(\rho_{ty})\, dt$$

$$= \int_0^1 \left(\sum_{j=1}^d y_j \int_{\mathbb{R}^d} g_j(x)\, \rho(x - ty)\, dx \right) dt$$

Fubini

$$= \int_{\mathbb{R}^d} \left(\int_0^1 \sum_{j=1}^d g_j(x + ty) \cdot y_j \, dt \right) \rho(x)\, dx$$

Integrating against $\varphi(y)$ with $\varphi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \left(T(f_y)\varphi(y) - T(f)\varphi(y) \right) dy = T(f*\varphi) - T(f)\int\varphi$$

$$= \int_{\mathbb{R}^d} f(x)\, T(\varphi_x) dy - T(f)\int\varphi$$

$$\|$$

$$\int_{\mathbb{R}^d} \underbrace{\left(\int_{\mathbb{R}^d}\int_0^1 \sum_{j=1}^d g_j(x+ty)\cdot y_j\, \varphi(y)\, dy\, dt \right)}_{F_\varphi(x)} f(x)\, dx$$

We can take $\int\varphi = 1$ and find that

$$T(f) = \int_{\mathbb{R}^d} \underbrace{\left[T(\varphi_x) - \int_{\mathbb{R}^d}\int_0^1 \sum_{j=1}^d g_j(x+ty)\cdot y_j\, \varphi(y)\, dy\, dt \right]}_{G(x)} f(x)\, dx$$

$$\Rightarrow T = G \in C(\mathbb{R}^d)$$

Since $T = G \in C(\mathbb{R}^d)$ and $\partial_{x_i} T = g_i \in C(\mathbb{R}^d)$

$$\Rightarrow G \in W^{1,1}_{loc}$$

$$\Rightarrow \quad G(x+y) - G(x) = \int_0^1 \sum_{j=1}^d g_j(x+ty)\cdot y_j\, dt$$

$$\Rightarrow \quad \frac{G(x+he_i)-G(x)}{h} = \int_0^1 g_i(x+the_i)\,dt \xrightarrow[h\to 0]{} g_i(x)$$

$$\Rightarrow \quad \partial_{x_i} G = g_i \in C(\mathbb{R}^d) \quad \Rightarrow \quad G \in C^1(\mathbb{R}^d).$$

<u>Remark:</u> From the beginning, can we simply take

$$G(y) = G(0) + \int_0^1 \sum_i g_i(ty)\cdot y_i\,dt \quad ?$$

Since $g_i \in C(\mathbb{R}^d) \Rightarrow G \in C(\mathbb{R}^d)$. Why $\partial_{x_i} G = g_i$?

$$\frac{G(y+he_i)-G(y)}{h} = \frac{1}{h}\int_0^1 \sum_j \Big[g_j(ty+the_j)\cdot(y_j+\delta_{ij}h) - g_j(ty)\cdot y_j\Big]\,dt$$

$$= \underbrace{\int_0^1 g_i(ty+the_i)\,dt}_{\downarrow h\to 0 \atop \int_0^1 g_i(ty)\,dt} + \underbrace{\int_0^1 \sum_{j\neq i} \frac{g_j(ty+the_j)-g_j(ty)}{h}\,y_j\,dt}_{\downarrow h\to 0 \atop 0 \text{ (? not easy!)}}$$

Sobolev spaces: $W^{1,p}(\Omega)$ and $W^{1,p}_{loc}(\Omega)$.

Exercise: (Approximation of $W^{1,p}_{loc}(\Omega)$ by $C^\infty(\Omega)$) $1 \le p < \infty$

Let $f \in L^p_{loc}(\Omega)$ s.t. $\nabla f \in L^p_{loc}(\Omega)$ (i.e. $f \in W^{1,p}_{loc}(\Omega)$)

Then there exists $\{f_n\} \subset C^\infty(\Omega)$ s.t. for every compact set $K \subset \Omega$ we have:

$$\|f_n - f\|_{L^p(K)} + \|\nabla(f_n - f)\|_{L^p(K)} \xrightarrow[n\to\infty]{} 0.$$

Hint: Use the convolution.

Theorem: (Chain rule) Let $G \in C^1(\mathbb{R}^d)$ with ∇G is bounded. If $f = \{f_i\}_{i=1}^d \subset W^{1,p}_{loc}(\Omega)$, then $G(f) \in W^{1,p}_{loc}(\Omega)$ and

$$\frac{\partial}{\partial x_i} G(f) = \sum_{k=1}^{d} \partial_k G \cdot \frac{\partial f_k}{\partial x_i} \quad \text{in } D'(\Omega).$$

Moreover, if $\{f_i\} \subset W^{1,p}(\Omega) \Rightarrow G(f) \in W^{1,p}(\Omega)$ provided that either $|\Omega| < \infty$ or ($|\Omega| = \infty$ and $G(0) = 0$).

Proof: Since G is b.d. in compact sets, $G(\rho) \in L^p_{loc}$.
To compute the derivatives, we find

$$\rho^{(n)} = \left(\rho_i^{(n)}\right)_{i=1}^d \subset C^\infty(\Omega)^d \quad \text{s.t.}$$

$\rho^{(n)} \to \rho$ in $W^{1,p}_{loc}(\Omega)$, and pointwise a.e. Using $\|\nabla G\|_{L^\infty} < \infty$, we get $G(\rho^{(n)}) \to G(\rho)$ in L^p_{loc}.
Using the normal C^1-chain rule we get: $\forall \varphi \in D(\Omega)$

$$\int_\Omega (\partial_i \varphi)\, G(\rho^{(n)}) = -\int \varphi\, \partial_i (G(\rho^n))$$

$$= -\int \varphi \sum_{k=1}^d \underbrace{\partial_k G(\rho^n)}_{\text{b.d.}} \cdot \partial_i \rho_k^{(n)}\, dx$$

Take $n \to \infty$

$\downarrow$ pointwise / L^p_{loc}

$$\int_\Omega (\partial_i \varphi)\, G(\rho) = -\int \varphi \sum_{k=1}^d \partial_k G(\rho)\, \partial_i \rho_k\, dx.$$

$$\Rightarrow \qquad \partial_i G(\rho) = \sum_{k=1}^d \underbrace{\partial_k G(\rho)}_{\text{b.d}}\, \underbrace{\partial_i \rho_k}_{\in L^p_{loc}} \in L^p_{loc}.$$

If $\nabla \rho \in L^p \Rightarrow \nabla G \in L^p$. However, to get $G(\rho) \in L^p$ from $\rho \in L^p$ we need.

$$|G(\rho) - G(0)| \le \|\nabla G\|_{L^\infty} |\rho| \in L^p$$

and $G(0) \in L^p \overset{\text{ok if}}{\rightsquigarrow}$ either $|\Omega| < \infty$ or $G(0) = 0$. $\square$

Exercise: (Derivative of absolute value)

Let $f \in W^{1,p}(\Omega)$. Prove that $|f| \in W^{1,p}(\Omega)$ and

$$(\nabla|f|)(x) = \begin{cases} \frac{1}{|f(x)|} (\mathrm{Re} f)\nabla(\mathrm{Re} f) + (\mathrm{Im} f)\nabla(\mathrm{Im} f) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

In particular, if f is real-valued, then:

$$(\nabla|f|)(x) = \begin{cases} \nabla f(x) \cdot \mathrm{sign}(f(x)) & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Consequently: $|\nabla|f|(x)| \leq |\nabla f(x)|$ a.e. x

(called the diamagnetic inequality)

Exercise: Let $f \in W^{1,1}_{loc}(\Omega)$. Let $A \subset \mathbb{R}$, $|A| = 0$.

Then $\nabla f(x) = 0$ for a.e. $x \in f^{-1}(A)$.

Theorem: (Positive distributions are measures)

Let Ω open $\subset \mathbb{R}^d$. Let $T \in \mathcal{D}'(\Omega)$ s.t. $T \geq 0$,

i.e. $T(\varphi) \geq 0$, $\forall \varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$. Then:

$\exists!$ positive Borel measure μ on Ω s.t. $\mu(K) < \infty$ for all compact sets K, and

$$T(\varphi) = \int \varphi(x)\, d\mu(x), \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Reversely, $\forall$ positive Borel measure μ on Ω s.t. $\mu(K) < \infty$ for all compact $K \subset \Omega$, then $\exists!\, T \in D'(\Omega)$ s.t. $T(\varphi) = \int_\Omega \varphi(x)\, d\mu(x)$, $\forall \varphi \in D(\Omega)$.

<u>Proof:</u> "$\Leftarrow$" Easy.

"$\Rightarrow$" We define the measure μ by $\forall U$ open $\subset \Omega$

$$\begin{cases} \mu(U) = \sup\{T(\varphi) : \varphi \in D(\Omega),\ 0 \le \varphi(x) \le 1,\ \operatorname{supp}\varphi \subset U\} \\ \mu(\emptyset) = 0 \end{cases}$$

Then $\forall A \subset \Omega$, define

$$\mu(A) = \inf\{\mu(O) : O \text{ open},\ A \subset O\}$$

$\Rightarrow \mu$ is an outer measure, i.e.

$$\begin{cases} \mu(\emptyset) = 0 \\ \mu(A) \le \mu(B) \text{ if } A \subset B \\ \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i) \end{cases}$$

$\Rightarrow$ we can define a sigma algebra Σ such that μ is a measure on Σ & a set E is measurable iff $\mu(E) = \mu(E \cap A) + \mu(E \cap A^c)$ for all A.

We can show that all open sets are measurable, and that the measure is regular.

<u>Example:</u> The Dirac-delta function is a Borel probability measure on $\mathbb{R}^d$.