The mass shell in the semi-relativistic Pauli-Fierz model

Oliver Matte (LMU Munich)
Joint work with
Martin Könenberg (U Vienna)

Spectral Days, Munich, 11.4.2012

Contents

Model

The semi-relativistic Pauli-Fierz Hamiltonian

for an electron (with spin) moving in $\mathbb{R}^3$ and interacting with the quantized radiation field is [Miyao-Spohn 2009]

$$\mathbb{H} := \sqrt{(\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} \otimes \mathbb{1} + \mathfrak{e}\,\mathbb{A}))^2 + \mathbb{1}} + \mathbb{1} \otimes H_{\mathrm{f}} \,.$$

It is acting in the Hilbert space $L^2(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^2) \otimes \mathscr{F}$, where $\mathscr{F}$ is the bosonic Fock space,

$$\mathscr{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathbb{R}^3_{\mathbf{k}} \times \mathbb{Z}_2)^n\big) \,.$$

- $\boldsymbol{\sigma}$: vector of Pauli matrices.
- $\mathbb{A}$: quantized, UV cutoff vector potential; $\mathfrak{e} > 0$.
- $H_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} |\mathbf{k}|\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$: radiation field energy.

The semi-relativistic Pauli-Fierz Hamiltonian

for an electron (with spin) moving in $\mathbb{R}^3$ and interacting with the quantized radiation field is [Miyao-Spohn 2009]

$$\mathbb{H} := \sqrt{(\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} \otimes \mathbb{1} + \mathfrak{e}\,\mathbb{A}))^2 + \mathbb{1}} + \mathbb{1} \otimes H_{\mathrm{f}}\,.$$

It is acting in the Hilbert space $L^2(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^2) \otimes \mathscr{F}$, where $\mathscr{F}$ is the bosonic Fock space,

$$\mathscr{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathbb{R}^3_{\mathbf{k}} \times \mathbb{Z}_2)^n\big)\,.$$

- $\boldsymbol{\sigma}$: vector of Pauli matrices.
- $\mathbb{A}$: quantized, UV cutoff vector potential; $\mathfrak{e} > 0$.
- $H_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} |\mathbf{k}|\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$: radiation field energy.

The semi-relativistic Pauli-Fierz Hamiltonian

for an electron (with spin) moving in $\mathbb{R}^3$ and interacting with the quantized radiation field is [Miyao-Spohn 2009]

$$\mathbb{H} := \sqrt{(\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} \otimes \mathbb{1} + \mathfrak{e}\,\mathbb{A}))^2 + \mathbb{1}} + \mathbb{1} \otimes H_{\mathrm{f}}\,.$$

It is acting in the Hilbert space $L^2(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^2) \otimes \mathscr{F}$, where $\mathscr{F}$ is the bosonic Fock space,

$$\mathscr{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathbb{R}^3_{\mathbf{k}} \times \mathbb{Z}_2)^n\big)\,.$$

- $\boldsymbol{\sigma}$: vector of Pauli matrices.
- $\mathbb{A}$: quantized, UV cutoff vector potential; $\mathfrak{e} > 0$.
- $H_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} |\mathbf{k}|\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$: radiation field energy.

The semi-relativistic Pauli-Fierz Hamiltonian

for an electron (with spin) moving in $\mathbb{R}^3$ and interacting with the quantized radiation field is [Miyao-Spohn 2009]

$$\mathbb{H} := \sqrt{(\boldsymbol{\sigma} \cdot (-i\nabla_{\mathbf{x}} \otimes \mathbb{1} + \mathfrak{e}\,\mathbb{A}))^2 + \mathbb{1}} + \mathbb{1} \otimes H_{\mathrm{f}} \,.$$

It is acting in the Hilbert space $L^2(\mathbb{R}^3_{\mathbf{x}}, \mathbb{C}^2) \otimes \mathscr{F}$, where $\mathscr{F}$ is the bosonic Fock space,

$$\mathscr{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathbb{R}^3_{\mathbf{k}} \times \mathbb{Z}_2)^n\big) \,.$$

- $\boldsymbol{\sigma}$: vector of Pauli matrices.
- $\mathbb{A}$: quantized, UV cutoff vector potential; $\mathfrak{e} > 0$.
- $H_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} |\mathbf{k}|\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$: radiation field energy.

Previous results in exterior potential

Thm (M.-Stockmeyer '10, Könenberg-M.-S. '11, K.-M. '1X)

For all $\mathfrak{e}, \kappa > 0$ and $\gamma \in [0, 2/\pi]$, there is a distinguished self-adjoint realization of

$$\mathbb{H}_\gamma := \mathbb{H} - \gamma/|\mathbf{x}| \,.$$

If $\gamma \in (0, 2/\pi]$, then $\inf \sigma(\mathbb{H}_\gamma)$ is a (degenerate) eigenvalue. If Φ is a corresponding eigenvector, and $a > 0$ satisfies

$$1-(1-a^2)^{1/2} < \inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H}_\gamma) \,, \quad \text{then} \quad e^{a|\mathbf{x}|}\,\Phi \in L^2 \otimes \mathscr{F} \,.$$

For $\gamma > 2/\pi$, the quadratic form of $\mathbb{H}_\gamma$ is unbounded below.

- NR case: Bach-Fröhlich-Sigal 1999, Griesemer-Lieb-Loss 2001.

Previous results in exterior potential

Thm (M.-Stockmeyer '10, Könenberg-M.-S. '11, K.-M. '1X)

For all $\mathfrak{e}, \kappa > 0$ and $\gamma \in [0, 2/\pi]$, there is a distinguished self-adjoint realization of

$$\mathbb{H}_\gamma := \mathbb{H} - \gamma/|\mathbf{x}| \, .$$

If $\gamma \in (0, 2/\pi]$, then $\inf \sigma(\mathbb{H}_\gamma)$ is a (degenerate) eigenvalue.
If Φ is a corresponding eigenvector, and $a > 0$ satisfies

$$1-(1-a^2)^{1/2} < \inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H}_\gamma) \, , \quad \text{then} \quad e^{a|\mathbf{x}|}\,\Phi \in L^2 \otimes \mathscr{F} \, .$$

For $\gamma > 2/\pi$, the quadratic form of $\mathbb{H}_\gamma$ is unbounded below.

- NR case: Bach-Fröhlich-Sigal 1999, Griesemer-Lieb-Loss 2001.

Previous results in exterior potential

Thm (M.-Stockmeyer '10, Könenberg-M.-S. '11, K.-M. '1X)

For all $\mathfrak{e}, \kappa > 0$ and $\gamma \in [0, {}^2/_\pi]$, there is a distinguished self-adjoint realization of

$$\mathbb{H}_\gamma := \mathbb{H} - \gamma/|\mathbf{x}| \,.$$

If $\gamma \in (0, {}^2/_\pi]$, then $\inf \sigma(\mathbb{H}_\gamma)$ is a (degenerate) eigenvalue. If Φ is a corresponding eigenvector, and $a > 0$ satisfies

$$1 - (1 - a^2)^{1/2} < \inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H}_\gamma)\,, \quad \text{then} \quad e^{a|\mathbf{x}|}\,\Phi \in L^2 \otimes \mathscr{F}\,.$$

For $\gamma > {}^2/_\pi$, the quadratic form of $\mathbb{H}_\gamma$ is unbounded below.

- NR case: Bach-Fröhlich-Sigal 1999, Griesemer-Lieb-Loss 2001.

Previous results in exterior potential

Thm (M.-Stockmeyer '10, Könenberg-M.-S. '11, K.-M. '1X)

For all $\mathfrak{e}, \kappa > 0$ and $\gamma \in [0, 2/\pi]$, there is a distinguished self-adjoint realization of

$$\mathbb{H}_\gamma := \mathbb{H} - \gamma/|\mathbf{x}| \, .$$

If $\gamma \in (0, 2/\pi]$, then $\inf \sigma(\mathbb{H}_\gamma)$ is a (degenerate) eigenvalue. If Φ is a corresponding eigenvector, and $a > 0$ satisfies

$$1-(1-a^2)^{1/2} < \inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H}_\gamma) \, , \quad \text{then} \quad e^{a|\mathbf{x}|}\,\Phi \in L^2 \otimes \mathscr{F} \, .$$

For $\gamma > 2/\pi$, the quadratic form of $\mathbb{H}_\gamma$ is unbounded below.

- NR case: Bach-Fröhlich-Sigal 1999, Griesemer-Lieb-Loss 2001.

Previous results in exterior potential

Thm (M.-Stockmeyer ’10, Könenberg-M.-S. ’11, K.-M. ’1X)

For all $\mathfrak{e}, \kappa > 0$ and $\gamma \in [0, 2/\pi]$, there is a distinguished self-adjoint realization of

$$\mathbb{H}_\gamma := \mathbb{H} - \gamma/|\mathbf{x}| \,.$$

If $\gamma \in (0, 2/\pi]$, then $\inf \sigma(\mathbb{H}_\gamma)$ is a (degenerate) eigenvalue. If Φ is a corresponding eigenvector, and $a > 0$ satisfies

$$1-(1-a^2)^{1/2} < \inf\sigma(\mathbb{H}) - \inf\sigma(\mathbb{H}_\gamma)\,, \quad \text{then} \quad e^{a|\mathbf{x}|}\,\Phi \in L^2 \otimes \mathscr{F}\,.$$

For $\gamma > 2/\pi$, the quadratic form of $\mathbb{H}_\gamma$ is unbounded below.

- NR case: Bach-Fröhlich-Sigal 1999, Griesemer-Lieb-Loss 2001.

Theorem (Könenberg-M.)

Let $\mathfrak{e}, \kappa > 0$, let $V : \mathbb{R}^3 \to [0, \infty)$ be a small form perturbation of $\sqrt{1-\Delta}$, and assume $\sqrt{1-\Delta} - 1 - V$ has negative eigenvalues $e_0 < e_1 < \ldots < 0$. Then the binding energy is increased in presence of the quantized radiation field, i.e.

$$\inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H} - V) > |e_0| . \qquad (\clubsuit)$$

Remarks.

- For non-zero $0 \leqslant V \in L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ one observes enhanced binding in the quantized radiation field at arbitrary $\mathfrak{e}, \kappa > 0$.
- NR case: Catto, Chen, Exner, Hainzl, Hiroshima, Linde, Spohn, Vougalter, Wugalter.
- $(\clubsuit)$ with $\geqslant$ has been shown first by Hiroshima-Sasaki, 2010.

Theorem (Könenberg-M.)

Let $\mathfrak{e}, \kappa > 0$, let $V : \mathbb{R}^3 \to [0, \infty)$ be a small form perturbation of $\sqrt{1-\Delta}$, and assume $\sqrt{1-\Delta} - 1 - V$ has negative eigenvalues $e_0 < e_1 < \ldots < 0$. Then the binding energy is increased in presence of the quantized radiation field, i.e.

$$\inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H} - V) > |e_0| \, . \qquad (\clubsuit)$$

Remarks.

- For non-zero $0 \leqslant V \in L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ one observes enhanced binding in the quantized radiation field at arbitrary $\mathfrak{e}, \kappa > 0$.
- NR case: Catto, Chen, Exner, Hainzl, Hiroshima, Linde, Spohn, Vougalter, Wugalter.
- ($\clubsuit$) with $\geqslant$ has been shown first by Hiroshima-Sasaki, 2010.

Theorem (Könenberg-M.)

Let $\mathfrak{e}, \kappa > 0$, let $V : \mathbb{R}^3 \to [0, \infty)$ be a small form perturbation of $\sqrt{1-\Delta}$, and assume $\sqrt{1-\Delta} - 1 - V$ has negative eigenvalues $e_0 < e_1 < \ldots < 0$. Then the binding energy is increased in presence of the quantized radiation field, i.e.

$$\inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H} - V) > |e_0| \,. \qquad (\clubsuit)$$

Remarks.

- For non-zero $0 \leqslant V \in L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ one observes enhanced binding in the quantized radiation field at arbitrary $\mathfrak{e}, \kappa > 0$.
- NR case: Catto, Chen, Exner, Hainzl, Hiroshima, Linde, Spohn, Vougalter, Wugalter.
- ($\clubsuit$) with $\geqslant$ has been shown first by Hiroshima-Sasaki, 2010.

Theorem (Könenberg-M.)

Let $\mathfrak{e}, \kappa > 0$, let $V : \mathbb{R}^3 \to [0, \infty)$ be a small form perturbation of $\sqrt{1-\Delta}$, and assume $\sqrt{1-\Delta} - 1 - V$ has negative eigenvalues $e_0 < e_1 < \ldots < 0$. Then the binding energy is increased in presence of the quantized radiation field, i.e.

$$\inf \sigma(\mathbb{H}) - \inf \sigma(\mathbb{H} - V) > |e_0| \,. \qquad (\clubsuit)$$

Remarks.

- For non-zero $0 \leqslant V \in L^{3/2}(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ one observes enhanced binding in the quantized radiation field at arbitrary $\mathfrak{e}, \kappa > 0$.
- NR case: Catto, Chen, Exner, Hainzl, Hiroshima, Linde, Spohn, Vougalter, Wugalter.
- ($\clubsuit$) with $\geqslant$ has been shown first by Hiroshima-Sasaki, 2010.

Intuitive picture:

A moving electron is surrounded by a cloud of soft photons, i.e. photons of low energy. The electron together with its photon cloud behaves like a particle having a larger mass than the electron alone. Heavier particles yield higher binding energies.

Aim

Study the electron and its photon cloud more precisely using Pizzo's iterative analytic perturbation theory [Pizzo 2003].

We shall establish results recently obtained in the *non*-relativistic case by [Chen-Fröhlich, 2007], [Chen-Fröhlich-Pizzo, 2009], [Fröhlich-Pizzo, 2010].

Intuitive picture:

A moving electron is surrounded by a cloud of soft photons, i.e. photons of low energy. The electron together with its photon cloud behaves like a particle having a larger mass than the electron alone. Heavier particles yield higher binding energies.

Aim

Study the electron and its photon cloud more precisely using Pizzo's iterative analytic perturbation theory [Pizzo 2003].

We shall establish results recently obtained in the *non*-relativistic case by [Chen-Fröhlich, 2007], [Chen-Fröhlich-Pizzo, 2009], [Fröhlich-Pizzo, 2010].

Intuitive picture:

A moving electron is surrounded by a cloud of soft photons, i.e. photons of low energy. The electron together with its photon cloud behaves like a particle having a larger mass than the electron alone. Heavier particles yield higher binding energies.

Aim

Study the electron and its photon cloud more precisely using Pizzo's iterative analytic perturbation theory [Pizzo 2003].

We shall establish results recently obtained in the *non*-relativistic case by [Chen-Fröhlich, 2007], [Chen-Fröhlich-Pizzo, 2009], [Fröhlich-Pizzo, 2010].

Intuitive picture:

A moving electron is surrounded by a cloud of soft photons, i.e. photons of low energy. The electron together with its photon cloud behaves like a particle having a larger mass than the electron alone. Heavier particles yield higher binding energies.

Aim

Study the electron and its photon cloud more precisely using Pizzo's iterative analytic perturbation theory [Pizzo 2003].

We shall establish results recently obtained in the *non*-relativistic case by [Chen-Fröhlich, 2007], [Chen-Fröhlich-Pizzo, 2009], [Fröhlich-Pizzo, 2010].

Fiber Hamiltonians

The semi-relativistic Pauli-Fierz Hamiltonian is unitarily equivalent to a direct integral,

$$\mathbb{H} \cong_U \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{P})\, d^3\mathbf{P}\,,$$

of fiber Hamiltonians,

$$H(\mathbf{P}) = \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{P}-\mathbf{p}_{\mathrm{f}}+\mathfrak{e}\,\mathbf{A}))^2+\mathbb{1}} + H_{\mathrm{f}}\,, \quad \mathbf{P}\in\mathbb{R}^3,\ \mathfrak{e}>0\,,$$

acting in the fiber Hilbert space $\mathbb{C}^2 \otimes \mathscr{F}$.

- $H_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} |\mathbf{k}|\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{p}_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} \mathbf{k}\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{A} = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0,1} \int_{|\mathbf{k}|<\kappa} \varepsilon_\lambda(\mathbf{k}) \left(a_\lambda^*(\mathbf{k}) + a_\lambda(\mathbf{k})\right) \frac{d^3\mathbf{k}}{(2|\mathbf{k}|)^{1/2}}$, $\kappa > 0$.

Fiber Hamiltonians

The semi-relativistic Pauli-Fierz Hamiltonian is unitarily equivalent to a direct integral,

$$\mathbb{H} \cong_U \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{P})\, d^3\mathbf{P}\,,$$

of fiber Hamiltonians,

$$H(\mathbf{P}) = \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{P}-\mathbf{p}_{\mathrm{f}}+\mathfrak{e}\,\mathbf{A}))^2+\mathbb{1}} + H_{\mathrm{f}}\,, \quad \mathbf{P}\in\mathbb{R}^3,\ \mathfrak{e}>0\,,$$

acting in the fiber Hilbert space $\mathbb{C}^2\otimes\mathscr{F}$.

- $H_{\mathrm{f}} = \sum_{\lambda=0,1}\int_{\mathbb{R}^3}|\mathbf{k}|\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{p}_{\mathrm{f}} = \sum_{\lambda=0,1}\int_{\mathbb{R}^3}\mathbf{k}\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{A} = \frac{1}{(2\pi)^{3/2}}\sum_{\lambda=0,1}\int_{|\mathbf{k}|<\kappa}\varepsilon_\lambda(\mathbf{k})\left(a_\lambda^*(\mathbf{k})+a_\lambda(\mathbf{k})\right)\frac{d^3\mathbf{k}}{(2|\mathbf{k}|)^{1/2}}$, $\kappa>0$.

Fiber Hamiltonians

The semi-relativistic Pauli-Fierz Hamiltonian is unitarily equivalent to a direct integral,

$$\mathbb{H} \cong_U \int_{\mathbb{R}^3}^{\oplus} H(\mathbf{P})\, d^3\mathbf{P}\,,$$

of fiber Hamiltonians,

$$H(\mathbf{P}) = \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{P} - \mathbf{p}_{\mathrm{f}} + \mathfrak{e}\,\mathbf{A}))^2 + \mathbb{1}} + H_{\mathrm{f}}\,, \quad \mathbf{P} \in \mathbb{R}^3,\ \mathfrak{e} > 0\,,$$

acting in the fiber Hilbert space $\mathbb{C}^2 \otimes \mathscr{F}$.

- $H_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} |\mathbf{k}|\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{p}_{\mathrm{f}} = \sum_{\lambda=0,1} \int_{\mathbb{R}^3} \mathbf{k}\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{A} = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0,1} \int_{|\mathbf{k}|<\kappa} \boldsymbol{\varepsilon}_\lambda(\mathbf{k})\, \big(a_\lambda^*(\mathbf{k}) + a_\lambda(\mathbf{k})\big)\, \frac{d^3\mathbf{k}}{(2|\mathbf{k}|)^{1/2}}$, $\kappa > 0$.

Main Result

Define the mass shell / ground state energies,

$$E(\mathbf{P}) := \inf \sigma(H(\mathbf{P})), \quad \mathbf{P} \in \mathbb{R}^3.$$

Theorem (Könenberg-M.)

For all $\kappa, \mathfrak{p} > 0$, there exists $\mathfrak{e}_0 > 0$ such that, for all $\mathfrak{e} \in (0, \mathfrak{e}_0]$, the ground state energy E is twice continuously differentiable and strictly convex on $\mathcal{B}_{\mathfrak{p}} := \{\mathbf{P} \in \mathbb{R}^3 : |\mathbf{P}| < \mathfrak{p}\}$. Moreover, $E(\mathbf{0}) = \min E$.

- NR case: Fröhlich-Pizzo 2010.
- Bounds on E'' in the NR case: Bach-Chen-Fröhlich-Sigal 2007, Chen 2008.

Define the mass shell / ground state energies,

$$E(\mathbf{P}) := \inf \sigma(H(\mathbf{P}))\,, \quad \mathbf{P} \in \mathbb{R}^3.$$

Theorem (Könenberg-M.)

For all $\kappa, \mathfrak{p} > 0$*, there exists* $\mathfrak{e}_0 > 0$ *such that, for all* $\mathfrak{e} \in (0, \mathfrak{e}_0]$*, the ground state energy* E *is twice continuously differentiable and strictly convex on* $\mathcal{B}_\mathfrak{p} := \{\mathbf{P} \in \mathbb{R}^3 : |\mathbf{P}| < \mathfrak{p}\}$*.*
Moreover, $E(\mathbf{0}) = \min E$*.*

- NR case: Fröhlich-Pizzo 2010.
- Bounds on E'' in the NR case: Bach-Chen-Fröhlich-Sigal 2007, Chen 2008.

Define the mass shell / ground state energies,

$$E(\mathbf{P}) := \inf \sigma(H(\mathbf{P})) , \quad \mathbf{P} \in \mathbb{R}^3.$$

Theorem (Könenberg-M.)

For all $\kappa, \mathfrak{p} > 0$, there exists $\mathfrak{e}_0 > 0$ such that, for all $\mathfrak{e} \in (0, \mathfrak{e}_0]$, the ground state energy E is twice continuously differentiable and strictly convex on $\mathcal{B}_{\mathfrak{p}} := \{\mathbf{P} \in \mathbb{R}^3 : |\mathbf{P}| < \mathfrak{p}\}$. Moreover, $E(\mathbf{0}) = \min E$.

- NR case: Fröhlich-Pizzo 2010.
- Bounds on E'' in the NR case: Bach-Chen-Fröhlich-Sigal 2007, Chen 2008.

Strategy

Obstacle

$E(\mathbf{P})$ is not an isolated eigenvalue of $H(\mathbf{P})$;
analytic perturbation theory is not applicable.

$\rightsquigarrow$ Introduce IR cutoff fiber Hilbert spaces, $\mathbb{C}^2 \otimes \mathscr{F}_j$, $j \in \mathbb{N}_0$,

$$\mathscr{F}_j := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)\,, \qquad \mathcal{A}_j := \{|\mathbf{k}| \geqslant \kappa\,(1/2)^j\}\,;$$

define $H_j(\mathbf{P})$ in the same way as $H(\mathbf{P})$ on the IR cutoff space,

$$H_j(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_{\mathrm{f}}^{(j)} + \mathfrak{e}\,\mathbf{A}^{(j)})^2 + \mathbb{1}} + H_{\mathrm{f}}^{(j)}\,.$$

- $H_{\mathrm{f}}^{(j)} = \sum_{\lambda=0,1} \int_{\mathcal{A}_j} |\mathbf{k}|\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{p}_{\mathrm{f}}^{(j)} = \sum_{\lambda=0,1} \int_{\mathcal{A}_j} \mathbf{k}\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{A}^{(j)} = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0,1} \int_{|\mathbf{k}|<\kappa} \varepsilon_\lambda(\mathbf{k})\big(a^*_\lambda(\mathbf{k}) + a_\lambda(\mathbf{k})\big)\, \frac{\mathbb{1}_{\mathcal{A}_j}(\mathbf{k})\, d^3\mathbf{k}}{(2|\mathbf{k}|)^{1/2}}$

Obstacle

$E(\mathbf{P})$ is not an isolated eigenvalue of $H(\mathbf{P})$;
analytic perturbation theory is not applicable.

$\rightsquigarrow$ Introduce IR cutoff fiber Hilbert spaces, $\mathbb{C}^2 \otimes \mathscr{F}_j$, $j \in \mathbb{N}_0$,

$$\mathscr{F}_j := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)\,, \qquad \mathcal{A}_j := \{|\mathbf{k}| \geqslant \kappa\,(1/2)^j\}\,;$$

define $H_j(\mathbf{P})$ in the same way as $H(\mathbf{P})$ on the IR cutoff space,

$$H_j(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{P}-\mathbf{p}_{\mathrm{f}}^{(j)} + \mathfrak{e}\,\mathbf{A}^{(j)})^2 + \mathbb{1}} + H_{\mathrm{f}}^{(j)}\,.$$

- $H_{\mathrm{f}}^{(j)} = \sum_{\lambda=0,1} \int_{\mathcal{A}_j} |\mathbf{k}|\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{p}_{\mathrm{f}}^{(j)} = \sum_{\lambda=0,1} \int_{\mathcal{A}_j} \mathbf{k}\, a_\lambda^*(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{A}^{(j)} = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0,1} \int_{|\mathbf{k}|<\kappa} \varepsilon_\lambda(\mathbf{k})\big(a_\lambda^*(\mathbf{k}) + a_\lambda(\mathbf{k})\big) \frac{\mathbb{1}_{\mathcal{A}_j}(\mathbf{k})\, d^3\mathbf{k}}{(2|\mathbf{k}|)^{1/2}}$

Obstacle

$E(\mathbf{P})$ is not an isolated eigenvalue of $H(\mathbf{P})$;
analytic perturbation theory is not applicable.

$\rightsquigarrow$ Introduce IR cutoff fiber Hilbert spaces, $\mathbb{C}^2 \otimes \mathscr{F}_j$, $j \in \mathbb{N}_0$,

$$\mathscr{F}_j := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)\,, \qquad \mathcal{A}_j := \{|\mathbf{k}| \geqslant \kappa\,(1/2)^j\}\,;$$

define $H_j(\mathbf{P})$ in the same way as $H(\mathbf{P})$ on the IR cutoff space,

$$H_j(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma}\cdot(\mathbf{P}-\mathbf{p}_{\mathrm{f}}^{(j)} + \mathfrak{e}\,\mathbf{A}^{(j)})^2 + \mathbb{1}} + H_{\mathrm{f}}^{(j)}\,.$$

- $H_{\mathrm{f}}^{(j)} = \sum_{\lambda=0,1} \int_{\mathcal{A}_j} |\mathbf{k}|\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{p}_{\mathrm{f}}^{(j)} = \sum_{\lambda=0,1} \int_{\mathcal{A}_j} \mathbf{k}\, a^*_\lambda(\mathbf{k})\, a_\lambda(\mathbf{k})\, d^3\mathbf{k}$
- $\mathbf{A}^{(j)} = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda=0,1} \int_{|\mathbf{k}|<\kappa} \boldsymbol{\varepsilon}_\lambda(\mathbf{k})\big(a^*_\lambda(\mathbf{k}) + a_\lambda(\mathbf{k})\big)\, \frac{\mathbb{1}_{\mathcal{A}_j}(\mathbf{k})\, d^3\mathbf{k}}{(2|\mathbf{k}|)^{1/2}}$

Strategy

⇝ Introduce IR cutoff fiber Hilbert spaces, $\mathbb{C}^2 \otimes \mathscr{F}_j$, $j \in \mathbb{N}_0$,

$$\mathscr{F}_j := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)\,, \qquad \mathcal{A}_j := \{|\mathbf{k}| \geqslant \kappa\,(1/2)^j\}\,,$$

and define $H_j(\mathbf{P})$ in the same way as $H(\mathbf{P})$, but on the IR cutoff space,

$$H_j(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_{\mathrm{f}}^{(j)} + \mathfrak{e}\,\mathbf{A}^{(j)})^2 + \mathbb{1}} + H_{\mathrm{f}}^{(j)}.$$

⇒ $H_{\mathrm{f}}^{(j)} \geqslant \kappa\,(1/2)^j$ on $L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)$.

⇒ If $\mathfrak{e} > 0$ is small, depending on $|\mathbf{P}|$ and κ, then

- $E_j(\mathbf{P}) := \inf \sigma(H_j(\mathbf{P}))$ is an isolated, two-fold degenerate eigenvalue.
- $\mathrm{gap}_j := \inf\big\{\sigma(H_j(\mathbf{P}) - E_j(\mathbf{P})) \setminus \{0\}\big\} \geqslant (1/2)^j/\mathfrak{c}$.

Strategy

⇝ Introduce IR cutoff fiber Hilbert spaces, $\mathbb{C}^2 \otimes \mathscr{F}_j$, $j \in \mathbb{N}_0$,

$$\mathscr{F}_j := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)\,, \qquad \mathcal{A}_j := \{|\mathbf{k}| \geqslant \kappa\,(1/2)^j\}\,,$$

and define $H_j(\mathbf{P})$ in the same way as $H(\mathbf{P})$, but on the IR cutoff space,

$$H_j(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_{\mathrm{f}}^{(j)} + \mathfrak{e}\,\mathbf{A}^{(j)})^2 + \mathbb{1}} + H_{\mathrm{f}}^{(j)}.$$

⇒ $H_{\mathrm{f}}^{(j)} \geqslant \kappa\,(1/2)^j$ on $L^2_{\mathrm{sym}}\big((\mathcal{A}_j \times \mathbb{Z}_2)^n\big)$.

⇒ If $\mathfrak{e} > 0$ is small, depending on $|\mathbf{P}|$ and κ, then

- $E_j(\mathbf{P}) := \inf \sigma(H_j(\mathbf{P}))$ is an isolated, two-fold degenerate eigenvalue.
- $\mathrm{gap}_j := \inf\big\{\sigma(H_j(\mathbf{P}) - E_j(\mathbf{P})) \setminus \{0\}\big\} \geqslant (1/2)^j/\mathfrak{c}$.

How to treat the square root?

The two-fold direct sum of

$$\mathcal{T} := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_{\mathrm{f}} + \mathfrak{e}\mathbf{A}))^2 + \mathbb{1}}$$

can be written as

$$\mathcal{T} \oplus \mathcal{T}\,\psi = \lim_{\tau \to \infty} \int_{-\tau}^{\tau} \left(\mathbb{1} + iy\,(D - iy)^{-1}\right) \psi\,\frac{dy}{\pi}\,,$$

for $\psi \in \mathcal{D}(D)$, where

$$D := \boldsymbol{\alpha} \cdot (\mathbf{P} - \mathbf{p}_{\mathrm{f}} + \mathfrak{e}\mathbf{A}) + \beta\,.$$

α_1, α_2, α_3, and β are the Dirac matrices.

Strategy

So, $E_j(\mathbf{P})$ is an eigenvalue of $H_j(\mathbf{P})$. Let

$$\Pi_j \equiv \Pi_j(\mathbf{P}) := \mathbb{1}_{\{E_j(\mathbf{P})\}}(H_j(\mathbf{P}))$$

be the corresponding spectral projection.

Since $\mathrm{gap}_j > 0$, the resolvent

$$\mathcal{R}_j^{\perp} \equiv \mathcal{R}_j^{\perp}(\mathbf{P}) := \left(H_j(\mathbf{P})\,\Pi_j(\mathbf{P})^{\perp} - E_j(\mathbf{P})\right)^{-1}\Pi_j(\mathbf{P})^{\perp}$$

is well-defined.

Strategy

So, $E_j(\mathbf{P})$ is an eigenvalue of $H_j(\mathbf{P})$. Let

$$\Pi_j \equiv \Pi_j(\mathbf{P}) := \mathbb{1}_{\{E_j(\mathbf{P})\}}(H_j(\mathbf{P}))$$

be the corresponding spectral projection.

Since $\text{gap}_j > 0$, the resolvent

$$\mathcal{R}_j^\perp \equiv \mathcal{R}_j^\perp(\mathbf{P}) := \big(H_j(\mathbf{P})\,\Pi_j(\mathbf{P})^\perp - E_j(\mathbf{P})\big)^{-1}\,\Pi_j(\mathbf{P})^\perp$$

is well-defined.

Strategy

For $j \in \mathbb{N}_0$, Hellmann-Feynman formulas are valid,

$$\partial_{\mathbf{h}} E_j = \mathrm{Tr}[\Pi_j \, \partial_{\mathbf{h}} H_j \, \Pi_j]/2 \,,$$
$$\partial_{\mathbf{h}}^2 E_j = \mathrm{Tr}[\Pi_j \, \partial_{\mathbf{h}}^2 H_j \, \Pi_j]/2 - \|(\mathcal{R}_j^{\perp})^{1/2} \partial_{\mathbf{h}} H_j \, \Pi_j\|_{\mathrm{HS}}^2 .$$

Use these formulas to show that

$$E = E_0 + \sum_{j=0}^{\infty} (E_{j+1} - E_j) \quad \text{converges absolutely in } C_b^2(\mathcal{B}_{\mathfrak{p}}) \,.$$

More precisely, show

$$\sup_{\mathcal{B}_{\mathfrak{p}}} |\partial_{\mathbf{h}}^{\nu} E_{j+1} - \partial_{\mathbf{h}}^{\nu} E_j| \leqslant \mathfrak{c}\, \mathfrak{e}\, (1 + \mathfrak{c}\, \mathfrak{e})^j \begin{cases} (1/2)^j, & \nu = 0, 1 \,, \\ (1/2)^{j/2}, & \nu = 2 \,. \end{cases}$$

Since $E_0(\mathbf{P}) = \sqrt{\mathbf{P}^2 + 1}$, this implies $E \in C^2$, $E'' > 0$, on $B_{\mathfrak{p}}$.

Strategy

For $j \in \mathbb{N}_0$, Hellmann-Feynman formulas are valid,

$$\partial_{\mathbf{h}} E_j = \mathrm{Tr}[\Pi_j \, \partial_{\mathbf{h}} H_j \, \Pi_j]/2 \,,$$
$$\partial_{\mathbf{h}}^2 E_j = \mathrm{Tr}[\Pi_j \, \partial_{\mathbf{h}}^2 H_j \, \Pi_j]/2 - \|(\mathcal{R}_j^{\perp})^{1/2} \partial_{\mathbf{h}} H_j \, \Pi_j\|_{\mathrm{HS}}^2 .$$

Use these formulas to show that

$$E = E_0 + \sum_{j=0}^{\infty} (E_{j+1} - E_j) \quad \text{converges absolutely in } C_b^2(\mathcal{B}_{\mathfrak{p}}) \,.$$

More precisely, show

$$\sup_{\mathcal{B}_{\mathfrak{p}}} |\partial_{\mathbf{h}}^{\nu} E_{j+1} - \partial_{\mathbf{h}}^{\nu} E_j| \leqslant \mathfrak{c} \, \mathfrak{e} \, (1 + \mathfrak{c} \, \mathfrak{e})^j \begin{cases} (1/2)^j, & \nu = 0, 1 \,, \\ (1/2)^{j/2}, & \nu = 2 \,. \end{cases}$$

Since $E_0(\mathbf{P}) = \sqrt{\mathbf{P}^2 + 1}$, this implies $E \in C^2$, $E'' > 0$, on $\mathcal{B}_{\mathfrak{p}}$.

Strategy

For $j \in \mathbb{N}_0$, Hellmann-Feynman formulas are valid,

$$\begin{aligned} \partial_{\mathbf{h}} E_j &= \mathrm{Tr}[\Pi_j \, \partial_{\mathbf{h}} H_j \, \Pi_j]/2 \,, \\ \partial_{\mathbf{h}}^2 E_j &= \mathrm{Tr}[\Pi_j \, \partial_{\mathbf{h}}^2 H_j \, \Pi_j]/2 - \|(\mathcal{R}_j^{\perp})^{1/2} \partial_{\mathbf{h}} H_j \, \Pi_j\|_{\mathrm{HS}}^2 . \end{aligned}$$

Use these formulas to show that

$$E = E_0 + \sum_{j=0}^{\infty} (E_{j+1} - E_j) \quad \text{converges absolutely in } C_b^2(\mathcal{B}_{\mathfrak{p}}) \,.$$

More precisely, show

$$\sup_{\mathcal{B}_{\mathfrak{p}}} |\partial_{\mathbf{h}}^{\nu} E_{j+1} - \partial_{\mathbf{h}}^{\nu} E_j| \leqslant \mathfrak{c}\, \mathfrak{e}\, (1 + \mathfrak{c}\, \mathfrak{e})^j \begin{cases} (1/2)^j, & \nu = 0, 1 \,, \\ (1/2)^{j/2}, & \nu = 2 \,. \end{cases}$$

Since $E_0(\mathbf{P}) = \sqrt{\mathbf{P}^2 + 1}$, this implies $E \in C^2$, $E'' > 0$, on $B_{\mathfrak{p}}$.

Strategy

For $j \in \mathbb{N}_0$, Hellmann-Feynman formulas are valid,

$$\partial_{\mathbf{h}} E_j = \mathrm{Tr}[\Pi_j\, \partial_{\mathbf{h}} H_j\, \Pi_j]/2\,,$$
$$\partial_{\mathbf{h}}^2 E_j = \mathrm{Tr}[\Pi_j\, \partial_{\mathbf{h}}^2 H_j\, \Pi_j]/2 - \|(\mathcal{R}_j^{\perp})^{1/2} \partial_{\mathbf{h}} H_j\, \Pi_j\|_{\mathrm{HS}}^2.$$

Use these formulas to show that

$$E = E_0 + \sum_{j=0}^{\infty} (E_{j+1} - E_j) \quad \text{converges absolutely in } C_b^2(\mathcal{B}_{\mathfrak{p}})\,.$$

More precisely, show

$$\sup_{\mathcal{B}_{\mathfrak{p}}} |\partial_{\mathbf{h}}^{\nu} E_{j+1} - \partial_{\mathbf{h}}^{\nu} E_j| \leqslant \mathfrak{c}\, \mathfrak{e}\, (1 + \mathfrak{c}\, \mathfrak{e})^j \begin{cases} (1/2)^j, & \nu = 0, 1\,, \\ (1/2)^{j/2}, & \nu = 2\,. \end{cases}$$

Since $E_0(\mathbf{P}) = \sqrt{\mathbf{P}^2 + 1}$, this implies $E \in C^2$, $E'' > 0$, on $B_{\mathfrak{p}}$.

Strategy

To compare operators acting in the same Hilbert space, $\mathbb{C}^2 \otimes \mathscr{F}_{j+1}$, we introduce

$$H_j^{j+1}(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_\mathrm{f}^{(j+1)} + \mathfrak{e}\,\mathbf{A}^{(j)}))^2 + \mathbb{1}} + H_\mathrm{f}^{(j+1)} .$$

$\Rightarrow E_j(\mathbf{P}) = \inf \sigma(H_j^{j+1}(\mathbf{P}))$ is again an isolated, two-fold degenerate eigenvalue of $H_j^{j+1}(\mathbf{P})$ and

$$\partial_\mathbf{h} E_j = \mathrm{Tr}[\Pi_j^{j+1}\, \partial_\mathbf{h} H_j^{j+1}\, \Pi_j^{j+1}]/2 \,,$$

where

$$\Pi_j^{j+1} \equiv \Pi_j^{j+1}(\mathbf{P}) := \mathbb{1}_{\{E_j(\mathbf{P})\}}(H_j^{j+1}(\mathbf{P})) \,,$$

and similarly for $\partial_\mathbf{h}^2 E_j$.

Strategy

To compare operators acting in the same Hilbert space, $\mathbb{C}^2 \otimes \mathscr{F}_{j+1}$, we introduce

$$H_j^{j+1}(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_\mathrm{f}^{(j+1)} + \mathfrak{e}\,\mathbf{A}^{(j)}))^2 + \mathbb{1}} + H_\mathrm{f}^{(j+1)}.$$

$\Rightarrow$ $E_j(\mathbf{P}) = \inf \sigma(H_j^{j+1}(\mathbf{P}))$ is again an isolated, two-fold degenerate eigenvalue of $H_j^{j+1}(\mathbf{P})$ and

$$\partial_\mathbf{h} E_j = \mathrm{Tr}[\Pi_j^{j+1}\, \partial_\mathbf{h} H_j^{j+1}\, \Pi_j^{j+1}]/2\,,$$

where

$$\Pi_j^{j+1} \equiv \Pi_j^{j+1}(\mathbf{P}) := \mathbb{1}_{\{E_j(\mathbf{P})\}}(H_j^{j+1}(\mathbf{P}))\,,$$

and similarly for $\partial_\mathbf{h}^2 E_j$.

Strategy: The dressing transform

In order to find a good bound on $|\partial_{\mathbf{h}}^{\nu} E_{j+1}(\mathbf{P}) - \partial_{\mathbf{h}}^{\nu} E_j(\mathbf{P})|$, for $\mathbf{P} \neq \mathbf{0}$, we must not compare $\Pi_{j+1}(\mathbf{P})$ directly with $\Pi_j^{j+1}(\mathbf{P})$, which is of the form

$$\Pi_j^{j+1}(\mathbf{P}) = \Pi_j(\mathbf{P}) \otimes P_{\Omega_j^{j+1}} ,$$

$$P_{\Omega_j^{j+1}} := \text{ projection onto the vacuum sector in}$$

$$\mathscr{F}_j^{j+1} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\text{sym}}\left(([\mathcal{A}_{j+1} \setminus \mathcal{A}_j] \times \mathbb{Z}_2)^n\right) .$$

(Recall $\mathscr{F}_{j+1} = \mathscr{F}_j \otimes \mathscr{F}_j^{j+1}$.)

In fact, if the total system is moving with total momentum $\mathbf{P} \neq \mathbf{0}$, then the electron should be dressed into a cloud of soft photons. Hence, $\Pi_j^{j+1}(\mathbf{P})$ is not a good approximation of $\Pi_{j+1}(\mathbf{P})$, since it contains no photons with frequences in $\mathcal{A}_{j+1} \setminus \mathcal{A}_j$.

Strategy: The dressing transform

In order to find a good bound on $|\partial_{\mathbf{h}}^{\nu} E_{j+1}(\mathbf{P}) - \partial_{\mathbf{h}}^{\nu} E_j(\mathbf{P})|$, for $\mathbf{P} \neq \mathbf{0}$, we must not compare $\Pi_{j+1}(\mathbf{P})$ directly with $\Pi_j^{j+1}(\mathbf{P})$, which is of the form

$$\Pi_j^{j+1}(\mathbf{P}) = \Pi_j(\mathbf{P}) \otimes P_{\Omega_j^{j+1}} ,$$

$$P_{\Omega_j^{j+1}} := \text{ projection onto the vacuum sector in}$$

$$\mathscr{F}_j^{j+1} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\text{sym}}\big(([\mathcal{A}_{j+1} \setminus \mathcal{A}_j] \times \mathbb{Z}_2)^n\big) .$$

(Recall $\mathscr{F}_{j+1} = \mathscr{F}_j \otimes \mathscr{F}_j^{j+1}$.)

In fact, if the total system is moving with total momentum $\mathbf{P} \neq \mathbf{0}$, then the electron should be dressed into a cloud of soft photons. Hence, $\Pi_j^{j+1}(\mathbf{P})$ is not a good approximation of $\Pi_{j+1}(\mathbf{P})$, since it contains no photons with frequences in $\mathcal{A}_{j+1} \setminus \mathcal{A}_j$.

Strategy: The dressing transform

⇝ Define a dressing transform (compare [Chen-Fröhlich, 2007]),

$$U_j(\mathbf{P}) := e^{-i\mathfrak{e}\varpi(f_j(\mathbf{P}))},$$

$$\varpi(f_j) := \frac{i}{2^{1/2}} \sum_{\lambda=0,1} \int_{\mathcal{A}_{j+1}\setminus\mathcal{A}_j} f_j(\mathbf{P};\mathbf{k},\lambda) \left(a^*(\mathbf{k}) - a(\mathbf{k})\right) d^3\mathbf{k}\,,$$

$$f_j(\mathbf{P};\mathbf{k},\lambda) := \frac{1}{(2\pi)^{3/2}} \frac{1}{|\mathbf{k}|^{1/2}} \frac{\varepsilon_\lambda(\mathbf{k})\cdot\nabla E_j(\mathbf{P})}{|\mathbf{k}| - \mathbf{k}\cdot\nabla E_j(\mathbf{P})}\,,$$

and seek for a bound on

$$\left\|\check{\Pi}_{j+1}(\mathbf{P}) - \Pi_j^{j+1}(\mathbf{P})\right\|_{\mathrm{HS}},$$

where

$$\check{\Pi}_{j+1}(\mathbf{P}) := U_j(\mathbf{P})\,\Pi_{j+1}(\mathbf{P})\,U_j(\mathbf{P})^*.$$

Strategy: The dressing transform

⇝ Define a dressing transform (compare [Chen-Fröhlich, 2007]),

$$U_j(\mathbf{P}) := e^{-i\mathfrak{e}\varpi(f_j(\mathbf{P}))},$$

$$\varpi(f_j) := \frac{i}{2^{1/2}} \sum_{\lambda=0,1} \int_{\mathcal{A}_{j+1}\setminus\mathcal{A}_j} f_j(\mathbf{P};\mathbf{k},\lambda) \left(a^*(\mathbf{k}) - a(\mathbf{k})\right) d^3\mathbf{k}\,,$$

$$f_j(\mathbf{P};\mathbf{k},\lambda) := \frac{1}{(2\pi)^{3/2}} \frac{1}{|\mathbf{k}|^{1/2}} \frac{\varepsilon_\lambda(\mathbf{k})\cdot\nabla E_j(\mathbf{P})}{|\mathbf{k}| - \mathbf{k}\cdot\nabla E_j(\mathbf{P})}\,,$$

and seek for a bound on

$$\left\|\check{\Pi}_{j+1}(\mathbf{P}) - \Pi_j^{j+1}(\mathbf{P})\right\|_{\mathrm{HS}}\,,$$

where

$$\check{\Pi}_{j+1}(\mathbf{P}) := U_j(\mathbf{P})\,\Pi_{j+1}(\mathbf{P})\,U_j(\mathbf{P})^*.$$

Strategy

It turns out that

$$\big\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\big\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^j\,\big\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}} + \mathfrak{c}\,\mathfrak{e}\,(1/2)^j.$$

Guiding theme in Pizzo's iterative perturbation theory:
(Note that $\|\mathcal{R}_j^{\perp}\| \sim 2^j$.) $\rightsquigarrow$ Estimate expressions like

$$K_j^{(1/2)} := \big\|(\mathcal{R}_j^{\perp})^{1/2}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}}\,, \quad K_j^{(1)} := \big\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}}\,,$$

by relating them to their precessors at scale $j-1$.
It turns out that

$$K_{j+1}^{(1)} \leqslant (1+\mathfrak{c}\,\mathfrak{e})\,K_j^{(1)} + \mathfrak{c}\,\mathfrak{e}\,,$$

$$\big|K_{j+1}^{(1/2)} - K_j^{(1/2)}\big| \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^{j/2}\,(K_j^{(1)}+1)\,,$$

$$\big\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\big\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1+\mathfrak{c}\,\mathfrak{e})^j\,(1/2)^j.$$

This implies the desired bounds on $\big|\partial_{\mathbf{h}}^{\nu}E_{j+1}(\mathbf{P}) - \partial_{\mathbf{h}}^{\nu}E_j(\mathbf{P})\big|$.

Strategy

It turns out that

$$\big\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\big\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^j\,\big\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}} + \mathfrak{c}\,\mathfrak{e}\,(1/2)^j.$$

Guiding theme in Pizzo's iterative perturbation theory:
(Note that $\|\mathcal{R}_j^{\perp}\| \sim 2^j$.) $\rightsquigarrow$ Estimate expressions like

$$K_j^{(1/2)} := \big\|(\mathcal{R}_j^{\perp})^{1/2}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}}\,, \quad K_j^{(1)} := \big\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}}\,,$$

by relating them to their precessors at scale $j-1$.

It turns out that

$$K_{j+1}^{(1)} \leqslant (1+\mathfrak{c}\,\mathfrak{e})\,K_j^{(1)} + \mathfrak{c}\,\mathfrak{e}\,,$$
$$\big|K_{j+1}^{(1/2)} - K_j^{(1/2)}\big| \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^{j/2}\,(K_j^{(1)} + 1)\,,$$
$$\big\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\big\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1+\mathfrak{c}\,\mathfrak{e})^j\,(1/2)^j.$$

This implies the desired bounds on $|\partial_{\mathbf{h}}^{\nu} E_{j+1}(\mathbf{P}) - \partial_{\mathbf{h}}^{\nu} E_j(\mathbf{P})|$.

Strategy

It turns out that

$$\big\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\big\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^j \,\big\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}} + \mathfrak{c}\,\mathfrak{e}\,(1/2)^j.$$

Guiding theme in Pizzo's iterative perturbation theory:
(Note that $\|\mathcal{R}_j^{\perp}\| \sim 2^j$.) ⇝ Estimate expressions like

$$K_j^{(1/2)} := \big\|(\mathcal{R}_j^{\perp})^{1/2}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}}\,, \quad K_j^{(1)} := \big\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\big\|_{\mathrm{HS}}\,,$$

by relating them to their precessors at scale $j-1$.
It turns out that

$$K_{j+1}^{(1)} \leqslant (1+\mathfrak{c}\,\mathfrak{e})\,K_j^{(1)} + \mathfrak{c}\,\mathfrak{e}\,,$$
$$\big|K_{j+1}^{(1/2)} - K_j^{(1/2)}\big| \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^{j/2}\,(K_j^{(1)}+1)\,,$$
$$\big\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\big\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1+\mathfrak{c}\,\mathfrak{e})^j\,(1/2)^j.$$

This implies the desired bounds on $\big|\partial_{\mathbf{h}}^{\nu} E_{j+1}(\mathbf{P}) - \partial_{\mathbf{h}}^{\nu} E_j(\mathbf{P})\big|$.

Strategy

It turns out that

$$\left\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\right\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^j \left\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\right\|_{\mathrm{HS}} + \mathfrak{c}\,\mathfrak{e}\,(1/2)^j.$$

Guiding theme in Pizzo's iterative perturbation theory: (Note that $\|\mathcal{R}_j^{\perp}\| \sim 2^j$.) ⇝ Estimate expressions like

$$K_j^{(1/2)} := \left\|(\mathcal{R}_j^{\perp})^{1/2}\,\nabla H_j\,\Pi_j\right\|_{\mathrm{HS}}, \quad K_j^{(1)} := \left\|\mathcal{R}_j^{\perp}\,\nabla H_j\,\Pi_j\right\|_{\mathrm{HS}},$$

by relating them to their precessors at scale $j-1$.
It turns out that

$$K_{j+1}^{(1)} \leqslant (1+\mathfrak{c}\,\mathfrak{e})\,K_j^{(1)} + \mathfrak{c}\,\mathfrak{e}\,,$$

$$\left|K_{j+1}^{(1/2)} - K_j^{(1/2)}\right| \leqslant \mathfrak{c}\,\mathfrak{e}\,(1/2)^{j/2}\,(K_j^{(1)}+1)\,,$$

$$\left\|\check{\Pi}_{j+1} - \Pi_j^{j+1}\right\|_{\mathrm{HS}} \leqslant \mathfrak{c}\,\mathfrak{e}\,(1+\mathfrak{c}\,\mathfrak{e})^j\,(1/2)^j.$$

This implies the desired bounds on $|\partial_{\mathbf{h}}^{\nu}E_{j+1}(\mathbf{P}) - \partial_{\mathbf{h}}^{\nu}E_j(\mathbf{P})|$. ■

Related results

Existence and multiplicity of ground states

Define operators on the *original* fiber Hilbert space, $\mathbb{C}^2 \otimes \mathscr{F}$,

$$H_j^\infty(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_\mathrm{f} + \mathfrak{e}\mathbf{A}^{(j)}))^2 + \mathbb{1}} + H_\mathrm{f}\,,$$

$$\widetilde{H}_j^\infty(\mathbf{P}) := W_j(\mathbf{P})\, H_j^\infty(\mathbf{P})\, W_j(\mathbf{P})^*\,, \qquad W_j(\mathbf{P}) := \prod_{\ell=0}^{j-1} U_j(\mathbf{P})\,,$$

$$\widetilde{H}(\mathbf{P}) := \text{norm-res.-}\lim_{j\to\infty} \widetilde{H}_j^\infty(\mathbf{P})\,.$$

Theorem (Könenberg-M.)

For all $\mathfrak{p}, \kappa > 0$, there exist $\mathfrak{e}_0, \mathfrak{c} > 0$ such that, for all $\mathbf{P} \in \mathcal{B}_\mathfrak{p}$ and $\mathfrak{e} \in (0, \mathfrak{e}_0]$, the ground state energy $E(\mathbf{P})$ is an exactly two-fold degenerate eigenvalue of $\widetilde{H}(\mathbf{P})$, and

$$\left\| \mathbb{1}_{E(\mathbf{P})}(\widetilde{H}(\mathbf{P})) - \mathbb{1}_{E_j(\mathbf{P})}(\widetilde{H}_j^\infty(\mathbf{P})) \right\| \leqslant \mathfrak{c}\,\mathfrak{e}\,(1 + \mathfrak{c}\,\mathfrak{e})^j\,(1/2)^j.$$

(Notice that $\widetilde{H}(\mathbf{0}) = H(\mathbf{0})$.)

Existence and multiplicity of ground states

Define operators on the *original* fiber Hilbert space, $\mathbb{C}^2 \otimes \mathscr{F}$,

$$H_j^{\infty}(\mathbf{P}) := \sqrt{(\boldsymbol{\sigma} \cdot (\mathbf{P} - \mathbf{p}_{\mathrm{f}} + \mathfrak{e}\mathbf{A}^{(j)}))^2 + \mathbb{1}} + H_{\mathrm{f}}\,,$$

$$\widetilde{H}_j^{\infty}(\mathbf{P}) := W_j(\mathbf{P})\, H_j^{\infty}(\mathbf{P})\, W_j(\mathbf{P})^*\,, \qquad W_j(\mathbf{P}) := \prod_{\ell=0}^{j-1} U_j(\mathbf{P})\,,$$

$$\widetilde{H}(\mathbf{P}) := \underset{j\to\infty}{\text{norm-res.-lim}}\, \widetilde{H}_j^{\infty}(\mathbf{P})\,.$$

Theorem (Könenberg-M.)

For all $\mathfrak{p}, \kappa > 0$, *there exist* $\mathfrak{e}_0, \mathfrak{c} > 0$ *such that, for all* $\mathbf{P} \in \mathcal{B}_{\mathfrak{p}}$ *and* $\mathfrak{e} \in (0, \mathfrak{e}_0]$, *the ground state energy* $E(\mathbf{P})$ *is an exactly two-fold degenerate eigenvalue of* $\widetilde{H}(\mathbf{P})$, *and*

$$\big\| \mathbb{1}_{E(\mathbf{P})}(\widetilde{H}(\mathbf{P})) - \mathbb{1}_{E_j(\mathbf{P})}(\widetilde{H}_j^{\infty}(\mathbf{P})) \big\| \leqslant \mathfrak{c}\, \mathfrak{e}\, (1 + \mathfrak{c}\,\mathfrak{e})^j\, (1/2)^j.$$

(Notice that $\widetilde{H}(\mathbf{0}) = H(\mathbf{0})$.*)*

Absence of ground states at non-zero momenta

However:

Theorem (Könenberg-M.)

For all $\mathfrak{p}, \kappa > 0$, *there exists* $\mathfrak{e}_0 > 0$ *such that, for all* $\mathbf{P} \in \mathcal{B}_{\mathfrak{p}} \setminus \{\mathbf{0}\}$ *and* $\mathfrak{e} \in (0, \mathfrak{e}_0]$, *the ground state energy* $E(\mathbf{P})$ *is not an eigenvalue of* $H(\mathbf{P})$.

This follows from the bound

$$\Big\| a_\lambda(\mathbf{k})\, \phi_j(\mathbf{P}) + \mathfrak{e}\, \underbrace{f_j(\mathbf{P}; \mathbf{k}, \lambda)}_{\sim |\mathbf{k}|^{-3/2}}\, \phi_j(\mathbf{P}) \Big\| \leqslant \mathfrak{c}\mathfrak{e} \frac{\mathbb{1}_{|\mathbf{k}|<\kappa}}{|\mathbf{k}|^{1/2}}, \quad \mathbf{k} \in \mathcal{A}_j,$$

for every normalized ground state eigenvector, $\phi_j(\mathbf{P})$, of $H_j(\mathbf{P})$.

- NR case: [Schroer, 1963], [Fröhlich 1973], [Chen-Fröhlich, 2007], [Hasler-Herbst, 2008].

Absence of ground states at non-zero momenta

However:

Theorem (Könenberg-M.)

For all $\mathfrak{p}, \kappa > 0$, *there exists* $\mathfrak{e}_0 > 0$ *such that, for all* $\mathbf{P} \in \mathcal{B}_{\mathfrak{p}} \setminus \{\mathbf{0}\}$ *and* $\mathfrak{e} \in (0, \mathfrak{e}_0]$, *the ground state energy* $E(\mathbf{P})$ *is not an eigenvalue of* $H(\mathbf{P})$.

This follows from the bound

$$\Big\| a_\lambda(\mathbf{k})\, \phi_j(\mathbf{P}) + \mathfrak{e}\, \underbrace{f_j(\mathbf{P};\mathbf{k},\lambda)}_{\sim |\mathbf{k}|^{-3/2}}\, \phi_j(\mathbf{P}) \Big\| \leqslant \mathfrak{c}\,\mathfrak{e}\, \frac{\mathbb{1}_{|\mathbf{k}|<\kappa}}{|\mathbf{k}|^{1/2}}, \quad \mathbf{k} \in \mathcal{A}_j,$$

for every normalized ground state eigenvector, $\phi_j(\mathbf{P})$, of $H_j(\mathbf{P})$.

- NR case: [Schroer, 1963], [Fröhlich 1973], [Chen-Fröhlich, 2007], [Hasler-Herbst, 2008].

Absence of ground states at non-zero momenta

However:

Theorem (Könenberg-M.)

For all $\mathfrak{p}, \kappa > 0$, *there exists* $\mathfrak{e}_0 > 0$ *such that, for all* $\mathbf{P} \in \mathcal{B}_{\mathfrak{p}} \setminus \{\mathbf{0}\}$ *and* $\mathfrak{e} \in (0, \mathfrak{e}_0]$, *the ground state energy* $E(\mathbf{P})$ *is not an eigenvalue of* $H(\mathbf{P})$.

This follows from the bound

$$\Big\| a_\lambda(\mathbf{k})\, \phi_j(\mathbf{P}) + \mathfrak{e} \underbrace{f_j(\mathbf{P}; \mathbf{k}, \lambda)}_{\sim |\mathbf{k}|^{-3/2}} \phi_j(\mathbf{P}) \Big\| \leqslant \mathfrak{c}\mathfrak{e} \frac{\mathbb{1}_{|\mathbf{k}|<\kappa}}{|\mathbf{k}|^{1/2}}, \quad \mathbf{k} \in \mathcal{A}_j,$$

for every normalized ground state eigenvector, $\phi_j(\mathbf{P})$, of $H_j(\mathbf{P})$.

- NR case: [Schroer, 1963], [Fröhlich 1973], [Chen-Fröhlich, 2007], [Hasler-Herbst, 2008].

Coherent state representation space

The unitaries $W_j(\mathbf{P})$ do not have limit.

However, consider (as in [Fröhlich 1973]) the incomplete direct product space in the sense of von Neumann,

$$\mathscr{H}_{\mathbf{P}}^{\text{ren}} := \mathbb{C}^2 \otimes \mathscr{F}_0 \otimes \bigotimes_{j \in \mathbb{N}}^{\widetilde{\Omega}_{\mathbf{P}}} \mathscr{F}_j^{j+1} ,$$

containing the coherent state

$$\widetilde{\Omega}_{\mathbf{P}} := v \otimes \Omega_0 \otimes U_0^*(\mathbf{P})\, \Omega_0^1 \otimes U_1^*(\mathbf{P})\, \Omega_1^2 \otimes \dots ,$$

where v may be any vector in $\mathbb{C}^2$. One can construct a unitary map $W(\mathbf{P})^* : \mathbb{C}^2 \otimes \mathscr{F} \to \mathscr{H}_{\mathbf{P}}^{\text{ren}}$, so that

$$W(\mathbf{P})^* v \otimes \Omega = \widetilde{\Omega}_{\mathbf{P}} , \quad W(\mathbf{P})\, a(g_j)\, W(\mathbf{P})^* = a(g_j) - \mathfrak{e} \langle g_j \,|\, f_j \rangle ,$$

where $g_j \in L^2((\mathcal{A}_{j+1} \setminus \mathcal{A}_j) \times \mathbb{Z}_2)$.

Coherent state representation space

The unitaries $W_j(\mathbf{P})$ do not have limit.

However, consider (as in [Fröhlich 1973]) the incomplete direct product space in the sense of von Neumann,

$$\mathscr{H}_{\mathbf{P}}^{\text{ren}} := \mathbb{C}^2 \otimes \mathscr{F}_0 \otimes \bigotimes_{j\in\mathbb{N}}^{\widetilde{\Omega}_{\mathbf{P}}} \mathscr{F}_j^{j+1} ,$$

containing the coherent state

$$\widetilde{\Omega}_{\mathbf{P}} := v \otimes \Omega_0 \otimes U_0^*(\mathbf{P})\,\Omega_0^1 \otimes U_1^*(\mathbf{P})\,\Omega_1^2 \otimes \dots ,$$

where v may be any vector in $\mathbb{C}^2$. One can construct a unitary map $W(\mathbf{P})^* : \mathbb{C}^2 \otimes \mathscr{F} \to \mathscr{H}_{\mathbf{P}}^{\text{ren}}$, so that

$$W(\mathbf{P})^* v \otimes \Omega = \widetilde{\Omega}_{\mathbf{P}} , \quad W(\mathbf{P})\, a(g_j)\, W(\mathbf{P})^* = a(g_j) - \mathfrak{c}\,\langle\, g_j \,|\, f_j \,\rangle ,$$

where $g_j \in L^2((\mathcal{A}_{j+1} \setminus \mathcal{A}_j) \times \mathbb{Z}_2)$.

Coherent state representation space

The unitaries $W_j(\mathbf{P})$ do not have limit.

However, consider (as in [Fröhlich 1973]) the incomplete direct product space in the sense of von Neumann,

$$\mathscr{H}_{\mathbf{P}}^{\mathrm{ren}} := \mathbb{C}^2 \otimes \mathscr{F}_0 \otimes \bigotimes_{j\in\mathbb{N}}^{\widetilde{\Omega}_{\mathbf{P}}} \mathscr{F}_j^{j+1} ,$$

containing the coherent state

$$\widetilde{\Omega}_{\mathbf{P}} := v \otimes \Omega_0 \otimes U_0^*(\mathbf{P})\,\Omega_0^1 \otimes U_1^*(\mathbf{P})\,\Omega_1^2 \otimes \dots ,$$

where v may be any vector in $\mathbb{C}^2$. One can construct a unitary map $W(\mathbf{P})^* : \mathbb{C}^2 \otimes \mathscr{F} \to \mathscr{H}_{\mathbf{P}}^{\mathrm{ren}}$, so that

$$W(\mathbf{P})^* v \otimes \Omega = \widetilde{\Omega}_{\mathbf{P}} , \quad W(\mathbf{P})\, a(g_j)\, W(\mathbf{P})^* = a(g_j) - \mathfrak{e}\,\langle\, g_j \,|\, f_j \,\rangle ,$$

where $g_j \in L^2((\mathcal{A}_{j+1} \setminus \mathcal{A}_j) \times \mathbb{Z}_2)$.

Coherent state representation space

Define

$$H^{\mathrm{ren}}(\mathbf{P}) := W(\mathbf{P})^* \widetilde{H}(\mathbf{P}) W(\mathbf{P}) .$$

Then $E(\mathbf{P})$ is an exactly two-fold degenerate eigenvalue of $H^{\mathrm{ren}}(\mathbf{P})$ and

$$\lim_{j\to\infty} \mathrm{Tr}\left[\mathbb{1}_{E_j(\mathbf{P})}(H_j^\infty(\mathbf{P}))\, A\right] = \mathrm{Tr}\left[\mathbb{1}_{E(\mathbf{P})}(H^{\mathrm{ren}}(\mathbf{P}))\, \pi_{\mathbf{P}}(A)\right] ,$$

for every

$$A \in \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}_j) \cong \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}_j) \otimes \mathbb{1} \subset \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}) ,$$

and $\pi_{\mathbf{P}} : \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}_j) \longrightarrow \mathcal{B}(\mathscr{H}_{\mathbf{P}}^{\mathrm{ren}})$ is a natural embedding.

Coherent state representation space

Define

$$H^{\text{ren}}(\mathbf{P}) := W(\mathbf{P})^* \, \widetilde{H}(\mathbf{P}) \, W(\mathbf{P}) \, .$$

Then $E(\mathbf{P})$ is an exactly two-fold degenerate eigenvalue of $H^{\text{ren}}(\mathbf{P})$ and

$$\lim_{j\to\infty} \mathrm{Tr}\big[\mathbb{1}_{E_j(\mathbf{P})}(H_j^{\infty}(\mathbf{P}))\, A\big] = \mathrm{Tr}\big[\mathbb{1}_{E(\mathbf{P})}(H^{\text{ren}}(\mathbf{P}))\, \pi_{\mathbf{P}}(A)\big] \, ,$$

for every

$$A \in \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}_j) \cong \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}_j) \otimes \mathbb{1} \subset \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}) \, ,$$

and $\pi_{\mathbf{P}} : \mathcal{B}(\mathbb{C}^2 \otimes \mathscr{F}_j) \to \mathcal{B}(\mathscr{H}_{\mathbf{P}}^{\text{ren}})$ is a natural embedding.

On the renormalized electron mass

Theorem (Könenberg-M.)

Let $\kappa, \mathfrak{e} > 0$ be arbitrary. If *E is twice continuously differentiable near* $\mathbf{0}$*, then the renormalized electron mass is strictly larger than its bare mass, i.e.*

$$1/\partial_{\mathbf{h}}^2 E(\mathbf{0}) > 1\,, \qquad |\mathbf{h}| = 1\,.$$

Thank you for your attention!