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PDE II
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Excercise Sheet 8 for 03. 07. 2017

For $Z > 0$ let

$$\mathcal{E}(u) := \int_{\mathbb{R}^3} \big|\nabla u(x)\big|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{Z}{|x|}\big|u(x)\big|^2 \,\mathrm{d}x + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\big|u(x)\big|^2\big|u(y)\big|^2}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

be the Hartree functional and

$$E(\lambda) := \inf\big\{\mathcal{E}(u) :\ u \in H^1(\mathbb{R}^3),\ \|u\|_2^2 = \lambda\big\}.$$

8.1. Assume that u_n is a minimizing sequence for $E(\lambda)$, that u_0 is a minimizer for $E(\lambda)$ and that $u_n \to u_0$ weakly in $H^1(\mathbb{R}^3)$. Prove that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^3)$.

8.2. Prove that the inequality

$$\mathcal{E}(u) + \mathcal{E}(v) \geqslant 2\mathcal{E}\left(\sqrt{\frac{u^2+v^2}{2}}\right)$$

holds for all non-negative $u, v \in H^1(\mathbb{R}^3)$ and that the inequality is strict unless $u = v$. Deduce that $E(\lambda)$ has at most one non-negative minimizer.

8.3. Prove that the function $\lambda \mapsto E(\lambda)$ is convex and that there exists $\lambda^* \in [Z, 2Z]$ such that E is strictly decreasing on $[0, \lambda^*]$ and $E(\lambda) = E(\lambda^*)$ for all $\lambda \geqslant \lambda^*$.

8.4. Prove that $E(\lambda)$ has a minimizer if $\lambda \leqslant \lambda^*$ and has no minimizer if $\lambda > \lambda^*$.