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PDE II
SoSe 2017
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Excercise Sheet 7 for 26. 06. 2017

7.1 (6 points)**.** Prove the following result in two dimensions related to the Newton's theorem: for any finite non-negative radial Borel measure μ on $\mathbb{R}^2$ the inequality

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}\mu(y)}{|x-y|} \geqslant \int_{\mathbb{R}^2} \frac{\mathrm{d}\mu(y)}{\max\big\{|x|,|y|\big\}}$$

holds for all $x \in \mathbb{R}^2$.
Hint: Prove that the map $x \mapsto |x|^{-1}$ is subharmonic on $\mathbb{R}^2 \setminus \{0\}$.

7.2 (6 points)**.** Let μ be a finite, compactly supported, non-negative Borel measure on $\mathbb{R}^3$. Prove that

$$u(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathrm{d}\mu(y)}{|x-y|}$$

is the unique distributional solution of $-\Delta u = \mu$ satisfying $u(x) \underset{|x|\to\infty}{\longrightarrow} 0$.

7.3 (6 points)**.** For $Z > 0$ let

$$\mathcal{E}(u) := \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{Z\big|u(x)\big|^2}{|x|} \,\mathrm{d}x + \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\big|u(x)\big|^2\big|u(y)\big|^2}{|x-y|} \,\mathrm{d}x \,\mathrm{d}y$$

be the Hartree functional and let $E(\lambda) := \inf\Big\{\mathcal{E}(u) : \ u \in H^1(\mathbb{R}^3),\ \int_{\mathbb{R}^3} \big|u(x)\big|^2 \,\mathrm{d}x = \lambda\Big\}$.
Prove that if u_0 is a minimizer of $E(\lambda)$, then it satisfies

$$-\Delta u_0 - \frac{Z}{|x|} u_0 + \big(|u_0|^2 * |x|^{-1}\big) u_0 = \mu u_0$$

with $\mu \leqslant 0$.