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PDE II
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# **Excercise Sheet 4** for 29. 05. 2017

Let $d \in \mathbb{N}$.

**4.1.** For real-valued functions $f, g \in H^1(\mathbb{R}^d)$ prove that in the distributional sense $\nabla \max\{f, g\} = h$, with

$$h(x) := \begin{cases} (\nabla f)(x), & \text{for } f(x) \geqslant g(x); \\ (\nabla g)(x), & \text{for } f(x) < g(x). \end{cases}$$

**4.2.** Given $u_0 \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$ and $z \in \mathbb{R}^d \setminus \{0\}$, for $n \in \mathbb{N}$ let $u_n(x) := u_0(x - nz)$ for all $x \in \mathbb{R}^d$. Prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^d)$, but for any $p \in [1, \infty]$ it does not possess a subsequence convergent in $L^p(\mathbb{R}^d)$.

**4.3.** Let $V : \mathbb{R}^d \to [0, \infty)$ be a measurable function satisfying $\inf_{|x|\geqslant R} V(x) \underset{R\to\infty}{\longrightarrow} \infty$. Suppose that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $H^1(\mathbb{R}^d)$ converges to $f \in H^1(\mathbb{R}^d)$ weakly in $H^1(\mathbb{R}^d)$ and that $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} V(x)\big|f_n(x)\big|^2 \,\mathrm{d}x < \infty$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges to $f$ strongly in $L^2(\mathbb{R}^d)$.

**4.4.** For $j \in \mathbb{N}$ let $\Sigma^{(j)} := \big\{(a_k^{(j)}, b_k^{(j)}), k = 1, \ldots, 2^j\big\}$ be the collection of $2^j$ disjoint open intervals defined recursively by $\Sigma^{(1)} := \big\{(0,1), (2,3)\big\}$,

$$\Sigma^{(j+1)} := \left\{\left(a_k^{(j)}, \frac{2a_k^{(j)}}{3} + \frac{b_k^{(j)}}{3}\right),\ k = 1, \ldots, 2^j\right\} \cup \left\{\left(\frac{a_k^{(j)}}{3} + \frac{2b_k^{(j)}}{3}, b_k^{(j)}\right),\ k = 1, \ldots, 2^j\right\},$$

i.e. the passage from $\Sigma^{(j)}$ to $\Sigma^{(j+1)}$ consists in removing of $2^j$ disjoint closed intervals

$$I_k^{(j+1)} := \left[\frac{2a_k^{(j)}}{3} + \frac{b_k^{(j)}}{3}, \frac{a_k^{(j)}}{3} + \frac{2b_k^{(j)}}{3}\right],\ k = 1, \ldots, 2^j.$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} 0, & \text{for } x \in \mathbb{R} \setminus (0,3); \\ 1, & \text{for } x \in [1,2]; \\ \dfrac{f(a_k^{(j)}) + f(b_k^{(j)})}{2}, & \text{for } x \in I_k^{(j+1)},\ k = 1, \ldots, 2^j,\ j \in \mathbb{N} \end{cases}$$

is well-defined. Determine whether $f$ is continuous on $\mathbb{R}$ and whether $f \in H^1(\mathbb{R})$.