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Tutorial Sheet 8
Algebra 2

Summer Semester 2026

Exercise 1. (i) Let A be a commutative ring and let M be an A-module. Try to give two different proofs, in particular for those of you who are attending the category theory lecture, of the isomorphism

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

as $S^{-1}A$-modules.

(ii) Conclude that localization preserves arbitrary direct sums. More precisely, prove that for every family of A-modules $(M_i)_{i\in I}$ there is a canonical isomorphism

$$S^{-1}\Big(\bigoplus_{i\in I} M_i\Big) \cong \bigoplus_{i\in I} S^{-1}M_i.$$

(iii) Deduce that free modules localize to free modules.

(iv) Deduce that projective modules localize to projective modules.

Exercise 2. Let A be a commutative ring. Prove that $\mathrm{Spec}(A)$ is quasi-compact.[1]

Hint: You may go back to Exercise Sheets 2 and 3.

Exercise 3. Let M be an A-module and let $m \in M$. Show that $m = 0$ if and only if $m/1 = 0$ in M_{f_i} for every i, where $(f_1, \ldots, f_n) = A$.

Exercise 4. Let A be a commutative ring. Then A is called reduced if it does not contain any non-trivial nilpotent elements.

Prove that A is reduced if and only if A is locally reduced.

[1] Recall that a topological space is called quasi-compact if every open cover admits a finite subcover.