LMU

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MATHEMATISCHES INSTITUT

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TUTORIAL SHEET 7
ALGEBRA 2

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Exercise 1. 1) Let A be a commutative ring and let B be an A-algebra. Prove that for every integer $n \geq 1$ we have

$$B \otimes_A M_n(A) \cong M_n(B)$$

as B-algebras.

2) Conclude that for two A-algebras B and C we have

$$M_n(B) \otimes_A M_m(C) \cong M_{nm}(B \otimes_A C)$$

as A-algebras.

Hint: An exercise from Exercise Sheet 6 may be useful.

Exercise 2. Let R be a ring and G be a group.

1) The ring homomorphism

$$\varepsilon : R[G] \longrightarrow R, \qquad \sum_{g \in G} a_g g \longmapsto \sum_{g \in G} a_g$$

is called the *augmentation map.* Its kernel $I_G := \ker(\varepsilon)$ is called the *augmentation ideal.* Prove that I_G is a free R-module with basis $\{\, g - 1 \mid g \in G,\ g \neq 1 \,\}$.

2) Let S be a G-set and consider the free R-module $R[S]$. Show that the action of G on S extends uniquely to an R-linear action on $R[S]$ via

$$g \cdot \Big(\sum_{s \in S} \lambda_s s \Big) = \sum_{s \in S} \lambda_s (g \cdot s), \qquad g \in G.$$

Verify that

$$g \cdot (\theta_1 + \theta_2) = g \cdot \theta_1 + g \cdot \theta_2, \qquad g \cdot (r\theta) = r(g \cdot \theta)$$

for all $\theta, \theta_1, \theta_2 \in R[S]$ and $r \in R$, and conclude that $R[S]$ becomes naturally an $R[G]$-module.

3) Define $R[S]_G := R[S]/I_G R[S]$. Show that $I_G R[S] = \big\langle g \cdot s - s \;\big|\; g \in G,\ s \in S \big\rangle_R$.

Let

$$\pi : S \longrightarrow S/G, \qquad s \longmapsto G \cdot s,$$

be the orbit map. Prove that π induces a canonical isomorphism $R[S]_G \cong R[S/G]$. In particular, $R[S]_G$ is a free R-module with basis indexed by the set of orbits S/G.

4) Define

$$R[S]^G := \{\theta \in R[S] \mid g \cdot \theta = \theta \text{ for all } g \in G\}.$$

Assume that all orbits are finite. For $\alpha \in S/G$, define $s_\alpha := \sum_{t \in \alpha} t$. Show that

$$R[S]^G = \bigoplus_{\alpha \in S/G} R\, s_\alpha.$$

In particular, $R[S]^G$ is a free R-module with basis $\{\, s_\alpha \mid \alpha \in S/G \,\}$. Deduce that there is a canonical isomorphism $R[S]^G \cong R[S/G]$. Finally, show that the natural map

$$R[S]^G \hookrightarrow R[S] \twoheadrightarrow R[S]_G$$

is in general *not* an isomorphism.

5) As an application, we give another proof of Burnside's Lemma, which we already proved last semester in the tutorials. Recall that Burnside's Lemma states that if G is a finite group acting on a finite set S, then

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} |S^g|\,.$$

Prove this by following the steps below.

   (i) Set $V := \mathbb{Q}[S]$. By 4), we have $\dim_{\mathbb{Q}} V^G = |S/G|$. Define the averaging operator

   $$P := \frac{1}{|G|} \sum_{g \in G} g \in \operatorname{End}_{\mathbb{Q}}(V).$$

   Note that by part 2), every $g \in G$ induces a $\mathbb{Q}$-linear endomorphism of V via the induced action on V. Hence $g(v) := g \cdot v$ defines an element of $\operatorname{End}_{\mathbb{Q}}(V)$. Prove that P is a projection onto V^G, i.e. $P^2 = P$ and $\operatorname{im}(P) = V^G$.

   (ii) Conclude from (i) that $\dim_{\mathbb{Q}} V^G = \operatorname{tr}(P)$.

   Hint: Recall from linear algebra that for a projection P one has $V = \ker(P) \oplus \operatorname{im}(P)$.

   (iii) Conclude Burnside's Lemma from (ii).

   Hint: Use the $\mathbb{Q}$-linearity of the trace and prove that $\operatorname{tr}(g) = |S^g|$ for every $g \in G$.

6) Let G be a finite group. Apply Burnside's Lemma to prove that there exists no proper subgroup $H \subsetneq G$ such that

$$G = \bigcup_{g \in G} gHg^{-1}.$$

In other words, a finite group cannot be written as the union of the conjugates of a proper subgroup.

Hint: Consider the action of G on $S := G/H$ by left multiplication. Compute $|S/G|$ and apply Burnside's Lemma.