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Tutorial Sheet 6
Algebra 2

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Exercise 1. Let M be a free R-module with basis $(e_i)_{i\in I}$ and let N be an arbitrary R-module. Prove that every element of $M \otimes_R N$ can be uniquely written as a finite sum

$$\sum_i e_i \otimes n_i, \qquad n_i \in N.$$

Hint: Exercise 2.1 from the previous tutorial sheet may be useful.

Exercise 2. (i) Give two different proofs that for a ring extension $A \subseteq B$ one has

$$B \otimes_A A[X_1, \dots, X_n] \cong B[X_1, \dots, X_n]$$

as B-algebras.

(ii) Let $A \subseteq B$ be a ring extension and let C be an A-module. Prove that for an A-submodule C' of C we have an isomorphism of B-modules

$$B \otimes_A (C/C') \cong (B \otimes_A C)/\operatorname{im}(B \otimes_A C').$$

If C is in addition an A-algebra and C' is an ideal of C, prove that this isomorphism is an isomorphism of B-algebras.

Hint: You may find a useful hint on Exercise Sheet 5.

(iii) Conclude that for a ring extension $A \subseteq B$ and an ideal $I \subseteq A[X_1, \dots, X_n]$ we have an isomorphism of B-algebras

$$B \otimes_A \big(A[X_1, \dots, X_n]/I\big) \cong B[X_1, \dots, X_n]/I.B[X_1, \dots, X_n].$$

Exercise 3. 1) Prove (in one line) that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ as $\mathbb{C}$-algebras.

2) Let $K/\mathbb{Q}$ be a number field, i.e. a finite field extension over $\mathbb{Q}$. Prove that

$$\mathbb{R} \otimes_{\mathbb{Q}} K \cong \mathbb{R}^r \times \mathbb{C}^s,$$

where r is the number of real embeddings and s is the number of pairs of complex embeddings.

Hint: You may use the primitive element theorem from algebra.

3) Conclude that for $n \geq 3$ we have

$$\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_n) \cong \mathbb{C}^{\varphi(n)/2},$$

where φ denotes Euler's totient function.