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Tutorial Sheet 5
Algebra 2

Summer Semester 2026

Exercise 1. Compute the following tensor products:

1) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$
2) For an abelian torsion group A: $A \otimes_{\mathbb{Z}} \mathbb{Q}$
3) $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$
4) $(\mathbb{Z}/14\mathbb{Z})[X] \otimes_{\mathbb{Z}[X]} (\mathbb{Z}/8\mathbb{Z})[X]$
5) $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$

Exercise 2. 1) Prove using the universal property of the tensor product that for two families of R-modules $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$ one has

$$\left(\bigoplus_{i\in I} M_i\right) \otimes_R \left(\bigoplus_{j\in J} N_j\right) \cong \bigoplus_{(i,j)\in I\times J} (M_i \otimes_R N_j).$$

Hint: To construct a map $\bigoplus_{(i,j)}(M_i \otimes_R N_j) \to (\bigoplus_i M_i) \otimes_R (\bigoplus_j N_j)$, you may use the universal property of the direct sum.

2) Conclude that if M is a free R-module of rank n and N is a free R-module of rank m, then $M \otimes_R N$ is a free R-module of rank nm.

3) Now prove 2) directly by showing more generally that if $(m_i)_{i\in I}$ is an R-basis of M and $(n_j)_{j\in J}$ is an R-basis of N, then $(m_i \otimes n_j)_{(i,j)\in I\times J}$ is an R-basis of $M \otimes_R N$.

Hint: Consider for $i_0 \in I$ the R-module morphism $\varphi_{i_0} : M \to R$ given by $m_i \mapsto \delta_{i,i_0}$, where δ_{i,i_0} denotes the Kronecker delta. In the same way, construct for $j_0 \in J$ the R-module morphism $\phi_{j_0} : N \to R$. Then consider any finite linear combination

$$\sum_{i,j} c_{i,j}\, m_i \otimes n_j = 0$$

and apply $\varphi_{i_0} \otimes \phi_{j_0}$.

4) Conclude that $R[X_1, \dots, X_n] \cong R[X]^{\otimes n}$.

Exercise 3.

Definition. Let M be an R-module. A kth exterior power of M is an R-module T together with an alternating k-fold multilinear map $\Lambda : M^k \longrightarrow T$ satisfying the following universal property: For every alternating k-fold multilinear map $f : M^k \longrightarrow N$ to another R-module N, there exists a unique linear map $g : T \longrightarrow N$ such that $f = g \circ \Lambda$, i.e. the following diagram commutes:

$$\begin{array}{ccc} M^k & \longrightarrow & N \\ \big\downarrow & \nearrow & \\ T & & \end{array}$$

1) Prove that for any R-module M and any $k \in \mathbb{N}$, there exists a kth exterior power and that it is unique up to unique isomorphism. We will write $\Lambda^k M := T$ and $m_1 \wedge \cdots \wedge m_k := \Lambda(m_1, \ldots, m_k)$.

   Hint: Consider the R-module $M^{\otimes k}$ and define the R-submodule

   $$L := \langle m_1 \otimes \cdots \otimes m_k \mid \exists\, i \neq j : m_i = m_j \rangle\,.$$

   Now set $T := M^{\otimes k}/L$ and use the universal property of the tensor product.

2) Let M and N be R-modules. Prove that there is a natural isomorphism

   $$\mathrm{Hom}_R(\Lambda^k M, N) \cong \mathrm{Alt}^k_R(M, N),$$

   where $\mathrm{Alt}^k_R(M, N)$ denotes the set of alternating k-fold R-multilinear maps $M^k \to N$.

3) Prove that

   $$\Lambda^k : R\text{-}\mathbf{Mod} \longrightarrow R\text{-}\mathbf{Mod}$$

   defines a (covariant) functor

   $$M \longmapsto \Lambda^k M$$
   $$M \xrightarrow{f} N \longmapsto \Lambda^k M \xrightarrow{\Lambda^k f} \Lambda^k N$$

   such that

   $$\Lambda^k f(m_1 \wedge \cdots \wedge m_k) = f(m_1) \wedge \cdots \wedge f(m_k).$$

   In particular, if $M \cong N$, then $\Lambda^k M \cong \Lambda^k N$.

4) Let M be a free R-module of rank n. Prove that $\mathrm{rk}_R\, \Lambda^k M = \binom{n}{k}$.

   Hint: Let $(e_1, \ldots, e_n)$ be an R-basis of M. Consider first the case $n < k$. Then, for $k \leq n$, prove that $(e_{i_1} \wedge \cdots \wedge e_{i_k})_{1 \leq i_1 < \cdots < i_k \leq n}$ is an R-basis of $\Lambda^k M$.

5) Let M be a free R-module of rank n. Then by 3), every endomorphism $f : M \to M$ induces an endomorphism $\Lambda^n f : \Lambda^n M \to \Lambda^n M$. Since $\Lambda^n M$ is free of rank 1 by 4), conclude that

   $$\Lambda^n f(\omega) = \det(f)\, \omega \qquad \forall \omega \in \Lambda^n M.$$