LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Tutorial Sheet 4
Algebra 2

Summer Semester 2026

Exercise 1. 1) Let R be a ring and let P be an R-module. Prove that P is projective if and only if the functor $\operatorname{Hom}_R(P, -)$ is exact.

2) Let E be an R-module. Prove that $\operatorname{Hom}_R(-, E)$ is a contravariant left exact functor, i.e. if

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow \operatorname{Hom}_R(N'', E) \xrightarrow{g^*} \operatorname{Hom}_R(N, E) \xrightarrow{f^*} \operatorname{Hom}_R(N', E)$$

is an exact sequence, where $f^*(h) = h \circ f$, and analogously for g^*.

3) Prove that E is an injective R-module if and only if $\operatorname{Hom}_R(-, E)$ is an exact contravariant functor.

Exercise 2. Let R be a ring and let P be an R-module. Prove that the following are equivalent:

1) P is a direct summand of a free R-module, i.e. there exists an R-module Q such that $P \oplus Q \cong R^{(I)}$.

2) P is projective.

3) If M is an R-module and $f\colon M \to P$ is a surjective R-linear map, then there exists a section $s\colon P \to M$, i.e.

$$M \underset{s}{\overset{f}{\rightleftarrows}} P \longrightarrow 0$$

with $f \circ s = \operatorname{id}_P$.

Hint: To prove 1) $\Rightarrow$ 2), consider the following commutative diagram (see the last tutorial sheet for the definition of g_*):

$$\begin{array}{ccc}
\operatorname{Hom}_R(R^{(I)}, M) & \xrightarrow{\quad g_* \quad} & \operatorname{Hom}_R(R^{(I)}, N) \\
{\scriptstyle \iota^*}\big\downarrow & & \big\downarrow{\scriptstyle \iota^*} \\
\operatorname{Hom}_R(P, M) & \xrightarrow{\quad g_* \quad} & \operatorname{Hom}_R(P, N)
\end{array}$$

and use that we proved in the last tutorial that every free module is projective.

Here, $\iota\colon P \hookrightarrow R^{(I)}$ denotes the inclusion. Furthermore, prove that

$$\iota^*\colon \operatorname{Hom}_R(R^{(I)}, N) \to \operatorname{Hom}_R(P, N)$$

is surjective. For this, it may be useful to consider the projection $\pi\colon R^{(I)} \to P$.

Exercise 3. 1) Let

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{u_1} & M_2 & \xrightarrow{u_2} & M_3 & \xrightarrow{u_3} & M_4 & \xrightarrow{u_4} & M_5 \\
{\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow & & {\scriptstyle f_5}\downarrow \\
N_1 & \xrightarrow{v_1} & N_2 & \xrightarrow{v_2} & N_3 & \xrightarrow{v_3} & N_4 & \xrightarrow{v_4} & N_5
\end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms with exact rows. Prove the following:

i) If f_1 is surjective and f_2, f_4 are injective, then f_3 is injective.

ii) If f_5 is injective and f_2, f_4 are surjective, then f_3 is surjective.

This is called the 5-Lemma.

2) Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram with exact columns. Prove that if the top two rows are exact, then the bottom row is exact. Prove also that if the bottom two rows are exact, then the top row is exact.

This is called the 3×3-Lemma.