LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Tutorial Sheet 2
Algebra 2

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All rings in this tutorial sheet are assumed to be commutative and unital.

Exercise 1.

1) Let R, S be rings. Show that if R is noetherian and $f : R \to S$ is a ring homomorphism, then $\mathrm{im}(f)$ is noetherian.

2) Conclude that if R is noetherian and $\mathfrak{a} \subseteq R$ is an ideal, then $R/\mathfrak{a}$ is noetherian as well.

Exercise 2. Let R be a noetherian ring.

1) Prove that any ascending chain of ideals becomes stationary.

2) Prove that $R[X]$ is noetherian.

   Hint: If $I \subseteq R[X]$ is an ideal, consider the sets

$$I_n := \{\, a \in R \mid \exists f \in I \text{ with } f = aX^n + \dots \}.$$

   Show that the sets I_n are ideals and form an ascending chain. Then use (1). Next, use that the ideals I_n are finitely generated over R. Use this to find a finite generating set for I, and prove by induction that this indeed generates I.

3) Deduce that $R[X_1, \dots, X_n]$ is noetherian.

4) Show that if a ring A is a finitely generated R-algebra, then A is noetherian.

Exercise 3.

1) Let R be a ring and let $S \subseteq R$ be a multiplicative subset. Show that every ideal $I \subseteq S^{-1}R$ is of the form

$$I = S^{-1}\mathfrak{a} = \left\{ \frac{a}{s} \in S^{-1}R \mid a \in \mathfrak{a},\ s \in S \right\},$$

   where $\mathfrak{a}$ is an ideal of R.

   Hint: For an ideal $I \subseteq S^{-1}R$, define $\mathfrak{a} := \{\, r \in R \mid \frac{r}{1} \in I \,\}$. Show that $\mathfrak{a}$ is an ideal of R and that $I = S^{-1}\mathfrak{a}$.

2) Conclude that localization preserves the noetherian property.

Exercise 4. Let R be a ring. Show that

$$R[\mathbb{Z}] \cong R[X, X^{-1}]$$

as R-algebras, where $R[X, X^{-1}]$ denotes the localization of $R[X]$ at the multiplicative set $\{X^n \mid n \in \mathbb{N}_0\}$, and $R[\mathbb{Z}]$ denotes the group ring of $\mathbb{Z}$ over R. The ring $R[X, X^{-1}]$ is also called the ring of Laurent polynomials.

Conclude that if R is noetherian, then $R[\mathbb{Z}]$ is noetherian.