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Tutorial Sheet 1
Algebra 2

Summer Semester 2026

Exercise 1. Let E and E' be subfields of a field L. Their compositum EE' is defined as the smallest subfield of L containing both E and E'.

Now consider the following situation: let L/K be a field extension, and suppose that E and E' are intermediate fields of L/K.

(i) Show that if E/K and E'/K are algebraic extensions, then EE'/K is algebraic.

(ii) Show that if E/K and E'/K are normal extensions, then EE'/K is normal.

(iii) Show that if E/K and E'/K are separable extensions, then EE'/K is separable.

Hint: You may use the fact that if $\alpha_1, \dots, \alpha_n$ are separable over K, then $K(\alpha_1, \dots, \alpha_n)/K$ is separable. Then conclude that the same holds for arbitrary families of separable elements.

(iv) Show that if E/K and E'/K are finite extensions, then EE'/K is finite.

(v) Assume now that E/K and E'/K are finite Galois extensions. Then, by parts (ii)–(iv), EE'/K is again a finite Galois extension. Show that the homomorphism

$$\mathrm{Gal}(EE'/E) \longrightarrow \mathrm{Gal}(E'/E \cap E'), \quad \sigma \longmapsto \sigma|_{E'}$$

is an isomorphism.

(vi) Show that the homomorphism

$$\mathrm{Gal}(EE'/K) \longrightarrow \mathrm{Gal}(E/K) \times \mathrm{Gal}(E'/K), \quad \sigma \longmapsto \big(\sigma|_E, \sigma|_{E'}\big)$$

is injective. Show that if, in addition, $E \cap E' = K$, then the map is also surjective.

Remark: If you are familiar with infinite Galois theory, note that in (v) we obtain an isomorphism in the category **TopGrp**, and in (vi), if moreover $E \cap E' = K$, the map is also an isomorphism in **TopGrp**.

(vii) Let $n, m \in \mathbb{N}$ with $\gcd(m, n) = 1$. Show that the homomorphism

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n, \zeta_m)/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}), \quad \sigma \longmapsto \big(\sigma|_{\mathbb{Q}(\zeta_n)}, \sigma|_{\mathbb{Q}(\zeta_m)}\big)$$

is an isomorphism.

Hint: Apply (vi) and recall that we have already proved in the tutorials last semester that $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m) = \mathbb{Q}$ in the special case where n and m are distinct primes.

(viii) Conclude that if $n = p^k$ and $m = q^\ell$, where p and q are distinct prime numbers and $k, \ell \in \mathbb{N}$, then

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{nm})/\mathbb{Q})$$

is generated by two elements, one of order $(p-1)p^{k-1}$ and one of order $(q-1)q^{\ell-1}$.

(ix) Show that in the situation of (vii) we have

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n, \zeta_m)/\mathbb{Q}(\zeta_m)) \cong \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}).$$