LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Prof. Dr. Fabien Morel
Laurenz Wiesenberger

Tutorial sheet 11
Algebra

Winter term 25/26

Exercise 1.

(i) Show that every field extension L/K of degree 2 is normal.

(ii) Consider the tower of field extensions $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. Prove that normality is not a transitive property.

(iii) Let $f = X^4 - 4X^2 - 5 \in \mathbb{Q}[X]$. Determine the splitting field of f over $\mathbb{Q}$ and compute the degree of the corresponding field extension.

(iv) Let $f = X^3 - 2 \in \mathbb{Q}[X]$. Determine the splitting field of f over $\mathbb{Q}$ and compute the degree of the corresponding field extension.

Exercise 2.

(i) Provide a proof of the following statement by combining results from the lecture and previous exercises:

Let p be a prime number. Show that for every $n \in \mathbb{N}$ there exists, up to isomorphism, exactly one field $\mathbb{F}$ with $|\mathbb{F}| = p^n$.

(ii) Let $\mathbb{F}'/\mathbb{F}$ be a finite field extension. Assume that $|\mathbb{F}| = p^n =: q$ and $|\mathbb{F}'| = q^m$ for some $n, m \in \mathbb{N}$. Prove that

$$\mathrm{Aut}_{\mathbb{F}}(\mathbb{F}') = \langle \phi_q \rangle,$$

where $\phi_q : \mathbb{F}' \to \mathbb{F}'$, $\alpha \mapsto \alpha^q$, denotes the Frobenius automorphism.

Exercise 3. Let $p \neq q$ be prime numbers. Consider the cyclotomic extensions $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ and $\mathbb{Q}(\zeta_q)/\mathbb{Q}$, where ζ_p and ζ_q denote primitive pth and qth roots of unity, respectively.

(i) Show that $\mathbb{Q}(\zeta_p, \zeta_q) = \mathbb{Q}(\zeta_p \zeta_q) = \mathbb{Q}(\zeta_{pq})$.

(ii) Use the degree formula to conclude that $\mathbb{Q}(\zeta_p) \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$.

Hint: You may use that $[\mathbb{Q}(\zeta_{pq}) : \mathbb{Q}] = (p-1)(q-1)$.

Exercise 4. Let K be a field of characteristic 0, and let $\overline{K}$ be an algebraic closure of K.

(i) Show that an element $\alpha \in \overline{K}$ is a multiple root of a polynomial $f \in K[X]$ if and only if α is a common root of f and its derivative f', i.e. $f(\alpha) = 0$ and $f'(\alpha) = 0$.

(ii) Deduce that every irreducible polynomial in $K[X]$ has no multiple roots in $\overline{K}$.

Definition. We say that a non-constant polynomial $f \in K[X]$ is *separable* if it has no multiple roots in an algebraic closure $\overline{K}$ of K.

In particular, part (ii) implies that every irreducible polynomial over a field of characteristic 0 is separable.