Prof. Dr. Fabien Morel
Laurenz Wiesenberger

Tutorial sheet 10
Algebra

Winter term 25/26

Exercise 1. Determine the degree of the following field extensions over $\mathbb{Q}$ and give an explicit $\mathbb{Q}$-basis in each case.

(i) $\mathbb{Q}(\sqrt{2}, i)/\mathbb{Q}$,

(ii) $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$, where ζ_3 denotes a primitive third root of unity,

(iii) $\mathbb{Q}(\sqrt[n]{p})/\mathbb{Q}$, where p is a prime number and $n \geq 2$.

Exercise 2.

(i) Let $d_1, \ldots, d_n \in \mathbb{Z}$ be pairwise distinct squarefree integers and set

$$K := \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n}).$$

Show that $[K : \mathbb{Q}] \leq 2^n$. and determine precisely when equality holds.

(ii) Determine the degree of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(iii) Determine the degree of $\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{10})$.

Exercise 3. Let

$$L := \{ \alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q} \} \subseteq \mathbb{C}.$$

(i) Show that L is a subfield of $\mathbb{C}$.

(ii) Show that the field extension $L/\mathbb{Q}$ has infinite degree, i.e. $[L : \mathbb{Q}] = \infty$.

(iii) *Bonus:* Show that L is countable and conclude that $[\mathbb{C} : L] = \infty$.