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Tutorial sheet 9
Algebra

Winter term 25/26

Exercise 1. Let R be a unique factorization domain and let $a, b \in R$ be coprime. Let $q \in \mathbb{N}$ and assume that ab is a qth power in R. Show that a and b are qth powers up to units.

Exercise 2. Let R be an integral domain with only finitely many ideals. Show that R is a field.

Hint: You may use that every descending chain of ideals becomes stationary.

Exercise 3. Let

$$R := \{ f \in \mathbb{Q}[X] \mid f(0) \in \mathbb{Z} \}.$$

(i) Show that R is a subring of $\mathbb{Q}[X]$.

(ii) Show that R is not Noetherian.

Exercise 4 (Dedekind–Hasse-Kriterium)**.** Let R be an integral domain. Show that R is a principal ideal domain if and only if there exists a function

$$\delta\colon R \setminus \{0\} \longrightarrow \mathbb{N}$$

such that for all $0 \neq a \in R$ and $b \in R$ either $a \mid b$ or there exists a nonzero element

$$c \in (a, b) := aR + bR$$

satisfying

$$\delta(c) < \delta(a).$$

Note: Using this criterion, one can show that there exist principal ideal domains which are not Euclidean domains; a famous example is $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$.