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Tutorial sheet 8
Algebra

Winter term 25/26
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Exercise 1.

(1) Prove that $\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}$.

(2) Prove that for $n \in \mathbb{N}$, $K[[X]]/(X^n) \;\cong\; K[X]/(X^n)$.

(3) Prove that for any commutative ring R and any ideal $\mathfrak{a} \subseteq R$ we have an isomorphism

$$R[X]/\mathfrak{a}R[X] \;\cong\; (R/\mathfrak{a})[X],$$

where $\mathfrak{a}R[X]$ denotes the ideal generated by $\mathfrak{a}$ in $R[X]$, i.e.

$$\mathfrak{a}R[X] := \left\{ \sum_{i=1}^{n} a_i f_i \;\middle|\; n \in \mathbb{N},\ a_i \in \mathfrak{a},\ f_i \in R[X] \right\} \subseteq R[X].$$

Conclude that if $\mathfrak{a}$ is a prime ideal, then $\mathfrak{a}R[X]$ is a prime ideal in $R[X]$. Does the same implication hold if $\mathfrak{a}$ is maximal?

(4) Convince yourself that the same isomorphism holds also for group rings, i.e. let R be a ring and G an abelian group. Then

$$R[G]/\mathfrak{a}R[G] \;\cong\; (R/\mathfrak{a})[G].$$

(5) Prove that for any prime number p and any polynomial $f(X) \in \mathbb{Z}[X]$ we have an isomorphism

$$\mathbb{Z}[X]/(p, f(X)) \;\cong\; \mathbb{F}_p[X]/(\overline{f}(X)),$$

where $\overline{f}(X)$ denotes the reduction of f modulo p.

In particular, the ideal $(p, f(X))$ is maximal in $\mathbb{Z}[X]$ if and only if $\overline{f}(X)$ is irreducible in $\mathbb{F}_p[X]$.

Exercise 2. Let K be a field. A formal *Laurent series* over K is a formal series of the form

$$\sum_{n=m}^{\infty} a_n X^n, \qquad m \in \mathbb{Z},\ a_n \in K.$$

We denote the ring of Laurent series by $K((T))$.

(a) Show that the field $K((X))$ is the field of fractions of the ring of formal power series $K[[X]]$.

(b) Show that the valuation v from the last tutorial sheet extends to the Laurent series field $K((X))$ and satisfies the same properties (i.e. for all non-zero $f, g \in K((X))$, $v(fg) = v(f) + v(g)$ and $v(f+g) \geq \min\{v(f), v(g)\}$, with the usual convention $v(0) = \infty$).