Prof. Dr. Fabien Morel
Laurenz Wiesenberger

Tutorial sheet 7
Algebra

Winter term 25/26
December 1, 2025

Exercise 1.

(i) Let R and S be rings, and let $\varphi : R \to S$ be a ring homomorphism. Show that φ is injective if and only if $\ker(\varphi) = 0$.

(ii) Show that every ring homomorphism $\varphi : K \to L$, where K and L are fields, is injective.

Exercise 2.

(1) Let k be a field with $|k| = \infty$. Consider the evaluation map from the lecture

$$\mathrm{ev} : k[X] \longrightarrow \mathrm{Map}(k, k), \quad P \longmapsto (x \mapsto P(x)).$$

Show that ev is a ring monomorphism.

Note: The same statement remains true for the polynomial ring $k[X_1, \dots, X_n]$ in n variables.

(2) Does the statement in (1) remain true if k is a finite field?

Exercise 3.

(1) Let R be a ring, and let

$$f(X) = \sum_{n=0}^{\infty} a_n X^n \in R[[X]].$$

Show that f is invertible in $R[[X]]$ if and only if $a_0 \in R^\times$.

(2) Deduce from (1) that the power series $1 - X$ is invertible in $R[[X]]$, and compute its inverse explicitly.

(3) Let R be an integral domain, and let

$$f(X) = \sum_{n=0}^{\infty} a_n X^n.$$

Define the valuation $v : R[[X]] \to \mathbb{N}_0 \cup \{\infty\}$ by

$$v(f) := \begin{cases} \min\{\, n \in \mathbb{N}_0 \mid a_n \neq 0 \,\}, & \text{if } f \neq 0, \\ \infty, & \text{if } f = 0. \end{cases}$$

Show that for all $f, g \in R[[X]]$:

$$v(fg) = v(f) + v(g), \qquad v(f + g) \geq \min\{v(f), v(g)\}.$$

Bonus Exercise (Not relevant for the final exam)**.** Let **Grp** denote the category of groups and **Ring** the category of rings. Let R be a commutative ring. For a group G, let $R[G]$ denote the group ring of G over R.

(1) Let $\varphi : G \to H$ be a group homomorphism. Show that

$$R[\varphi] : R[G] \longrightarrow R[H], \quad \sum_{g\in G} a_g g \mapsto \sum_{g \in G} a_g\, \varphi(g).$$

is a ring homomorphism.

(2) Show that

$$R[\mathrm{id}_G] = \mathrm{id}_{R[G]} \qquad \text{and} \qquad R[\psi \circ \varphi] = R[\psi] \circ R[\varphi].$$

(3) Conclude that the assignment

$$G \longmapsto R[G], \qquad \varphi \longmapsto R[\varphi]$$

defines a functor

$$R[-] : \mathbf{Grp} \longrightarrow \mathbf{Ring}.$$

Furthermore, using Exercise 3 from Tutorial Sheet 3, we obtain the following sequence of functors:

$$\mathbf{Grp} \xrightarrow{(-)^{\mathrm{ab}}} \mathbf{AbGrp} \xrightarrow{R[-]} \mathbf{cRing}.$$