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Tutorial sheet 6
Algebra
Suggested Solutions

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In this tutorial sheet we do some further exercises about group actions and the Sylow Theorems. There will probably be more exercises that we will discuss in the tutorial, nevertheless the ones given here are good for practice.

Exercise 1. Decide whether the following statements are true or false and justify your answers.

1) Let p be a prime number, G be a finite group and $N \trianglelefteq G$ be a normal subgroup. Assume that $p \nmid [G : N]$, then all Sylow p-subgroups of G lie in N.

   Suggested solution. Let $|G| = p^r m$ with $p \nmid m$, and assume $r \geq 1$ (otherwise the statement is trivial). By assumption we have $p \nmid |G : N|$, and hence, by Lagrange's theorem, $|N| = p^r n$. By Sylow I there exists a Sylow p-subgroup $U \leq N$, and therefore $|U| = p^r$. Since the subgroup relation is transitive, U is also a Sylow p-subgroup of G. Now let U' be any Sylow p-subgroup of G. By Sylow II (respectively the corollary following it), there exists some $g \in G$ such that

   $$U' = gUg^{-1} \subseteq gNg^{-1} \overset{(*)}{=} N;$$

   for $(*)$ we use that N is a normal subgroup. Hence $U' \leq N$.

   □

2) Let G be a group of order $|G| = p^r$, where p is a prime number and $r \geq 1$. Then:

   (i) There exists an element $g \in G \setminus \{1\}$ such that $hg = gh$ for all $h \in G$.

   Suggested solution. True; see for example Exercise 2(b) on Tutorial Sheet 4. This was also proven in the lecture.

   □

   (ii) G is solvable.

   Suggested solution. True, since by the lecture every finite p-group is nilpotent, and by Exercise 4 on Exercise Sheet 4 we know that nilpotent groups are solvable. □

   (iii) G has precisely one Sylow p-subgroup.

   Suggested solution. True, since G is itself a Sylow p-subgroup. □

3) Groups of the following order are abelian:

(i) $|G| = 4$.

Suggested solution. True. If $|G| = 4$, then we know from lecture that $G \cong \mathbb{Z}/4\mathbb{Z}$ or $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Both groups are abelian. Of course, one may also argue that $|G| = 2^2$, which is the square of a prime.

$\square$

(ii) $|G| = 6$.

Suggested solution. False. We have $|S_3| = 6$, and from the lecture we know that S_3 is not abelian. For instance,

$$(12)(23) = (123) \neq (132) = (23)(12),$$

$\square$

(iii) $|G| = 12$.

Suggested solution. In general, a group of order 12 need not be abelian. As a counterexample one may take A_4. For example, in A_4 one has

$$(1\,2\,3)(1\,3\,4) = (2\,3\,4) \neq (1\,2\,4) = (1\,3\,4)(1\,2\,3),$$

$\square$

(iv) $|G| = 17$.

Suggested solution. If $|G| = 17$, then G is cyclic (by Lagrange's Theorem) and therefore abelian.

$\square$

(v) $|G| = 121$.

Suggested solution. If $|G| = 121 = 11^2$, then by Tutorial Sheet 4, Exercise 2(c) (or Exercise Sheet 3), the group G is abelian.

$\square$

(vi) $|G| = 143$.

Suggested solution. If $|G| = 143 = 11 \cdot 13$, then by Exercise 3 on Exercise Sheet 4, we have

$$G \cong \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z},$$

and therefore G is abelian.

$\square$

Exercise 2. 1) Let G be a group of order 22 acting on a set X of size 11 with no fixed points. Show that the action is transitive.

Suggested solution. From the lecture we know that

$$X = {}^{G}X \ \sqcup \bigsqcup_{|G\cdot x|\geq 2} G\cdot x.$$

Since X has no fixed points, we obtain

$$11 = |X| = \sum_{|G\cdot x|\geq 2} |G\cdot x|.$$

By the orbit–stabilizer theorem we have

$$|G\cdot x| = |G/I_x| \mid 22.$$

Since $2 \leq |G\cdot x| \leq 11$, every orbit size is in

$$|G\cdot x| \in \{2, 11\}.$$

As the sum of all orbit sizes equals 11, and 11 cannot be written as a multiple of 2, we must have

$$|G\cdot x| = 11.$$

Thus there is exactly one orbit of size 11.

□

2) The canonical action from $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{R}^2 \setminus \{0\}$ is transitive.

Suggested solution. Consider the action

$$GL_2(\mathbb{R}) \times \big(\mathbb{R}^2 \setminus \{0\}\big) \longrightarrow \mathbb{R}^2 \setminus \{0\}, \qquad (A, v) \longmapsto Av.$$

Let $e_1 = (1,0)^T$, and let $y \in \mathbb{R}^2 \setminus \{0\}$ be an arbitrary vector. Now choose a vector $y' \in \mathbb{R}^2 \setminus \{0\}$ that is linearly independent from y.

Then the assignment

$$e_1 \longmapsto y, \qquad e_2 \longmapsto y'$$

defines an isomorphism of $\mathbb{R}^2$. The corresponding linear map has, with respect to the standard basis, a matrix $A \in \mathrm{GL}_2(\mathbb{R})$. In particular,

$$Ae_1 = y.$$

Hence for every $y \in \mathbb{R}^2 \setminus \{0\}$ there exists a matrix $A \in \mathrm{GL}_2(\mathbb{R})$ such that $Ae_1 = y$.

□

3) Let G be a finite group. Then for $|G| \geq 3$, the action of G on $G \setminus \{1\}$ by conjugation is not transitive.

Suggested solution. Assume that the action of G on $G \setminus \{1\}$ by conjugation is transitive. Then for any $x \in G \setminus \{1\}$ we have

$$|G| - 1 = |G \cdot x| = |G : I_x| \mid |G|.$$

Hence

$$|G| - 1 \mid |G|.$$

But for $|G| \geq 3$, the number $|G| - 1$ cannot divide $|G|$. This yields a contradiction.

□

Exercise 3. 1) Show that every group of order 36 has a non-trivial normal subgroup.

Hint: Consider the action of G on the set of Sylow 3-subgroups and use the abstract definition.

Suggested solution. Let $|G| = 36 = 2^2 \cdot 3^2$ and let n_3 be the number of Sylow 3-subgroups of G. By Sylow III we have

$$n_3 \mid 4 \quad \text{and} \quad n_3 \equiv 1 \pmod 3,$$

hence $n_3 \in \{1, 4\}$.

If $n_3 = 1$, then the unique Sylow 3-subgroup is normal in G, and we are done. So assume $n_3 = 4$ and let X be the set of Sylow 3-subgroups of G, so that $|X| = 4$.

Consider the conjugation action of G on X (abstract definition):

$$\varphi : G \longrightarrow S_X \cong S_4.$$

Since there are four distinct Sylow 3-subgroups, the action is non-trivial, so $\operatorname{im}(\varphi) \neq 1$. Now $|\operatorname{im}(\varphi)|$ divides both $|S_4| = 24$ and $|G| = 36$, hence

$$|\operatorname{im}(\varphi)| \mid \gcd(24, 36) = 12,$$

and, as $\operatorname{im}(\varphi) \neq 1$, we obtain

$$|\operatorname{im}(\varphi)| \in \{2, 3, 4, 6, 12\}.$$

Therefore

$$|\ker(\varphi)| = \frac{|G|}{|\operatorname{im}(\varphi)|} \in \{18, 12, 9, 6, 3\}.$$

In particular, $\ker(\varphi)$ is a proper, non-trivial normal subgroup of G. Thus G always has a non-trivial normal subgroup.

□

2) Let G be a group of order 48. Show that G has a normal subgroup of order 8 or 16.

Suggested solution. Let G be a group of order $48 = 2^4 \cdot 3$. Let n_2 denote the number of Sylow 2-subgroups of G.

By Sylow III we have

$$n_2 \mid 3 \qquad \text{and} \qquad n_2 \equiv 1 \pmod 2,$$

hence

$$n_2 \in \{1, 3\}.$$

If $n_2 = 1$, then the unique Sylow 2-subgroup P is normal in G and has order $|P| = 2^4 = 16$, so we are done.

Thus assume $n_2 = 3$. Let X be the set of Sylow 2-subgroups of G. Then $|X| = 3$, and G acts on X by conjugation:

$$\varphi : G \longrightarrow S(X) \cong S_3.$$

By Sylow II the Sylow 2-subgroups are conjugate, so the action of G on X is transitive. Hence $\mathrm{im}(\varphi)$ acts transitively on this 3-element set. For any x in the set we have, by the orbit–stabiliser theorem,

$$|\mathrm{im}(\varphi) \cdot x| = \frac{|\mathrm{im}(\varphi)|}{|I_x|}.$$

Transitivity implies $|\mathrm{im}(\varphi) \cdot x| = 3$, hence $3 \leq |\mathrm{im}(\varphi)|$. Since $\mathrm{im}(\varphi) \leq S_3$, we must have

$$|\mathrm{im}(\varphi)| \in \{3, 6\}.$$

Let $N := \ker(\varphi)$. Then $N \trianglelefteq G$, and by the fundamental theorem of homomorphism

$$|G : N| = |\mathrm{im}(\varphi)| \in \{3, 6\}.$$

Consequently,

$$|N| = \frac{|G|}{|G : N|} \in \left\{ \frac{48}{3}, \frac{48}{6} \right\} = \{16, 8\}.$$

Thus N is a normal subgroup of G of order 8 or 16, as required. □

Exercise 4. 1) Let G be a group of order 30. Show that G has a normal subgroup N of order 15 and that $N \cong \mathbb{Z}/15\mathbb{Z}$.

Suggested solution. Let $|G| = 30 = 2 \cdot 3 \cdot 5$. By Tutorial Sheet 5, Exercise 2(a), we know that G has either a normal subgroup N_5 of order 5, or a normal subgroup N_3 of order 3. Thus it suffices to assume that either N_5 or N_3 is normal. Then the product $N_3 N_5 \leq G$ is a subgroup, and

$$|N_3 N_5| = \frac{|N_3| \cdot |N_5|}{|N_3 \cap N_5|} = \frac{3 \cdot 5}{1} = 15.$$

By Exercise Sheet 4, Exercise 3, we have

$$N_3 N_5 \cong \mathbb{Z}/15\mathbb{Z}.$$

Since $[G : N_3 N_5] = 2$, Exercise Sheet 1, Exercise 4 implies that $N_3 N_5$ is a normal subgroup of G. □

2) Show that every group G of order 45 is abelian.

Suggested solution. Let $|G| = 45 = 3^2 \cdot 5$. By Sylow's theorems, the number of Sylow 5-subgroups satisfies

$$n_5 \mid 9 \qquad \text{and} \qquad n_5 \equiv 1 \pmod 5,$$

hence $n_5 = 1$. Thus the Sylow 5-subgroup N_5 is normal in G.

Similarly, the number of Sylow 3-subgroups satisfies

$$n_3 \mid 5 \qquad \text{and} \qquad n_3 \equiv 1 \pmod 3,$$

so $n_3 = 1$. Hence the Sylow 3-subgroup N_3 is also normal in G.

Both N_3 and N_5 are abelian: N_5 is cyclic of order 5, and every group of order 3^2 is abelian.

Since $|N_3| = 9$ and $|N_5| = 5$ are coprime, we have

$$|N_3 \cap N_5| = 1,$$

and therefore

$$|N_3N_5| = \frac{|N_3|\,|N_5|}{|N_3 \cap N_5|} = 45.$$

Thus

$$G = N_3N_5.$$

We now show that $N_3N_5 \cong N_3 \times N_5$, which is abelian since it is a product of abelian groups. Consider the map

$$\varphi : N_3 \times N_5 \longrightarrow N_3N_5, \qquad (x, y) \longmapsto xy.$$

We first prove that every element of N_3 commutes with every element of N_5. Let $x \in N_3$ and $y \in N_5$. Since both N_3 and N_5 are normal in G, we have

$$yx^{-1}y^{-1} \in N_3, \qquad xyx^{-1} \in N_5.$$

Hence

$$xyx^{-1}y^{-1} \in N_3 \quad \text{and} \quad xyx^{-1}y^{-1} \in N_5.$$

Because $N_3 \cap N_5 = \{1\}$, it follows that

$$xyx^{-1}y^{-1} = 1, \quad \text{i.e.} \quad xy = yx.$$

Thus φ is a group homomorphism. By construction, φ is surjective, since every element of N_3N_5 can be written as a product of an element in N_3 and an element in N_5. If $\varphi(x, y) = 1$, then $xy = 1$, hence $x = y^{-1} \in N_3 \cap N_5 = \{1\}$, so $(x, y) = (1, 1)$. Thus the kernel is trivial. Therefore φ is an isomorphism, and so $N_3N_5 \cong N_3 \times N_5$. Consequently, G is abelian.

□

Alternative argument. It is more convenient to apply Exercise 2 from Exercise Sheet 5: this immediately yields

$$G \cong N_3 \times N_5.$$

One then only has to argue, as above, that N_3 and N_5 are abelian. This is the solution we discussed in the tutorials; nevertheless, I have given a proof here that does not use the notion of a semidirect product.