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Tutorial sheet 5
Algebra
Suggested Solutions

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Exercise 1. (a) Let G be a finite group and let $H \leq G$ be a p-subgroup for a prime number p. Show that if H is a normal subgroup of G, then H is contained in every Sylow p-subgroup of G.

Suggested solution. Let P be any Sylow p-subgroup of G. Since $U \leq G$ is a p-subgroup, Sylow II implies that there exists $g \in G$ such that

$$gUg^{-1} \leq P.$$

Because U is normal in G, we have $gUg^{-1} = U$. Thus

$$U \leq P.$$

Hence every normal p-subgroup of G is contained in every Sylow p-subgroup of G.

□

(b) Let $GL_n(\mathbb{F})$ be the group of invertible $(n \times n)$-matrices over a finite field $\mathbb{F}$ of characteristic $p > 0$, and set $q := |\mathbb{F}|$. Show that the group of upper triangular matrices whose diagonal entries are all equal to 1 forms a Sylow p-subgroup of $GL_n(\mathbb{F})$.

Hint: You may use the formula from Exercise Sheet 2, Exercise 2. You may also use without proof that if $\mathbb{F}$ is a finite field of characteristic p, then $|\mathbb{F}| = q = p^k$ for some $k \geq 1$.

Suggested solution. Let $U_n(\mathbb{F})$ denote the group of upper triangular matrices over $\mathbb{F}$ whose diagonal entries are all equal to 1 (the unitriangular group). There are $\frac{n(n-1)}{2}$ free entries above the diagonal, each of which may be chosen arbitrarily in $\mathbb{F}$. Hence

$$|U_n(\mathbb{F})| = q^{\frac{n(n-1)}{2}}.$$

By Exercise Sheet 2, Exercise 2, we may use the formula

$$|GL_n(\mathbb{F})| = \prod_{i=0}^{n-1} (q^n - q^i).$$

Rewrite each factor as $q^i(q^{n-i} - 1)$ to obtain

$$|GL_n(\mathbb{F})| = \prod_{i=0}^{n-1} q^i \prod_{i=0}^{n-1} (q^{n-i} - 1) = q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1).$$

We now use (without proof) that every finite field $\mathbb{F}$ of characteristic p has $q = p^r$ elements for some $r \geq 1$. Thus,

$$|U_n(\mathbb{F})| = p^{r\frac{n(n-1)}{2}}.$$

Moreover, each term $q^k - 1 = p^{rk} - 1$ is not divisible by p, since $p^{rk} - 1 \equiv -1 \pmod{p}$. Therefore the p-part of $|GL_n(\mathbb{F})|$ is $p^{r\frac{n(n-1)}{2}}$.

We conclude that

$$|U_n(\mathbb{F})| = p^{r\frac{n(n-1)}{2}} \quad \text{and} \quad |GL_n(\mathbb{F})| = p^{r\frac{n(n-1)}{2}} \cdot m, \qquad p \nmid m.$$

Hence $U_n(\mathbb{F})$ is a Sylow p-subgroup of $GL_n(\mathbb{F})$.

□

Exercise 2. (a) Show that every group of order 30 has a non-trivial normal Sylow subgroup.

Suggested solution. Let G be a group of order 30, i.e. $|G| = 2 \cdot 3 \cdot 5$. We denote as usual by n_p the number of Sylow p-subgroups of G. If $n_p = 1$, then by Sylow II (and the remark following it) this unique Sylow p-subgroup is normal. Thus, let us analyse the concrete situation.

By Sylow III we have

$$n_3 \mid 10 \qquad \text{and} \qquad n_3 \equiv 1 \pmod{3}.$$

Since $n_3 \mid 10$, we have $n_3 \in \{1, 2, 5, 10\}$, and the congruence condition forces $n_3 \in \{1, 10\}$.

Analogously,

$$n_5 \mid 6 \qquad \text{and} \qquad n_5 \equiv 1 \pmod{5},$$

hence $n_5 \in \{1, 6\}$.

Suppose for contradiction that $n_5 = 6$ and $n_3 = 10$. Every element of order 5 lies in a Sylow 5-subgroup. Each Sylow 5-subgroup has 4 non-trivial elements, and two distinct Sylow 5-subgroups intersect only in the identity. Therefore, there are

$$6 \cdot 4 = 24$$

elements of order 5.

By the same argument, each Sylow 3-subgroup has 2 non-trivial elements, and two distinct Sylow 3-subgroups intersect only in the identity. Hence the Sylow 3-subgroups contribute

$$10 \cdot 2 = 20$$

elements of order 3.

Thus G would contain at least

$$24 + 20 = 44$$

distinct non-identity elements, which is impossible since $|G| = 30$.

□

(b) Show that every group of order 56 has a non-trivial normal Sylow subgroup.

Suggested solution. Now let G be a group of order $56 = 2^3 \cdot 7$. We want to show that $n_2 = 1$ or $n_7 = 1$.

By Sylow III we have

$$n_7 \mid 8 \qquad \text{and} \qquad n_7 \equiv 1 \pmod 7,$$

and therefore $n_7 \in \{1, 8\}$.

If $n_7 = 1$, we are done. So assume $n_7 = 8$. Then there exist 8 different Sylow 7-subgroups, which intersect pairwise only in the identity. Each Sylow 7-subgroup contains 6 non-trivial elements of order 7, so altogether they contribute

$$8 \cdot 6 = 48$$

distinct elements of order 7.

This already forces $n_2 = 1$, because

$$|G| = 56 = 48 + 8,$$

and a Sylow 2-subgroup has 8 elements, which are obviously different from all elements in the Sylow 7-subgroups.

$\square$