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Tutorial sheet 5
Algebra

Winter term 25/26
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Exercise 1. (a) Let G be a finite group and let $H \leq G$ be a p-subgroup for a prime number p. Show that if H is a normal subgroup of G, then H is contained in every Sylow p-subgroup of G.

(b) Let $GL_n(\mathbb{F})$ be the group of invertible $(n \times n)$-matrices over a finite field $\mathbb{F}$ of characteristic $p > 0$, and set $q := |\mathbb{F}|$. Show that the group of upper triangular matrices whose diagonal entries are all equal to 1 forms a Sylow p-subgroup of $GL_n(\mathbb{F})$.

Hint: You may use the formula from Exercise Sheet 2, Exercise 2. You may also use without proof that if $\mathbb{F}$ is a finite field of characteristic p, then $|\mathbb{F}| = q = p^k$ for some $k \geq 1$.

Exercise 2. (a) Show that every group of order 30 has a non-trivial normal Sylow subgroup.

(b) Show that every group of order 56 has a non-trivial normal Sylow subgroup.

Remark. If we finish the exercises quickly, I will provide some additional problems, either further ones on the Sylow theorems, or we will begin with the classification of all subgroups of $SO(3)$.

Bonus Exercise (Not relevant for the final exam)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then for every object $A \in \mathcal{C}$, the functor F induces a group action on $F(A) \in \mathcal{D}$.

Recall from the lecture that in a category $\mathcal{C}$, an operation of G on an object C in $\mathcal{C}$ is a group homomorphism

$$G \longrightarrow \mathrm{Aut}_{\mathcal{C}}(C).$$

Note that this is indeed a generalisation of the (abstract) definition, since in the category **Set** we have $\mathrm{Aut}_{\mathbf{Set}}(X) = S_X$, the symmetric group of all bijections $X \to X$.

Now let us return to the situation above. Consider the group $G := \mathrm{Aut}_{\mathcal{C}}(A)$. Now show that

$$G \longrightarrow \mathrm{Aut}_{\mathcal{D}}(F(A)), \qquad g \mapsto F(g),$$

is a well-defined group action on $F(A)$.