LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Tutorial sheet 4
Algebra

Winter term 25/26
November 10, 2025

Exercise 1. Let X be a G-set and $Y \subseteq X$ be a subset. Then

$$Y \text{ is } G\text{-invariant} \iff Y \text{ is a disjoint union of orbits.}$$

Exercise 2. Let G be a finite group acting on itself by conjugation, i.e.

$$G \times G \longrightarrow G, \quad (g, x) \longmapsto gxg^{-1}.$$

(a) The corresponding orbit decomposition yields the *class equation*:

$$|G| = |Z(G)| + \sum_{|G \cdot x| > 1} |G \cdot x|,$$

where $Z(G)$ denotes the center of G, and the sum runs over the disjoint, non-trivial orbits $G \cdot x$.

(b) Let G be a group such that $|G| = p^r$ for some prime number p and $r \geq 1$. Show that $Z(G)$ is non-trivial.

(c) Let G be a group such that $|G| = p^2$. Show that G is abelian.

Note: You already proved this result in Exercise Sheet 3.

Exercise 3 (Burnside's Lemma)**.** (a) Let G be a finite group acting on a finite set X. For each $g \in G$, let

$$X^g := \{\, x \in X \mid g \cdot x = x \,\}$$

denote the set of elements fixed by g. Show that the number of orbits of G on X is given by

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

(b) In this exercise, we determine the number of distinct colorings of the vertices of a square using two colors (e.g. blue and green), up to rotation.

Let r denote the rotation of the square by $90°$ counterclockwise. Label the vertices in counterclockwise order by $1, 2, 3, 4$. This rotation can be represented by the 4-cycle

$$r = (1\ 2\ 3\ 4) \in S_4.$$

Hence, the cyclic group

$$G = \langle r \rangle = \{e, r, r^2, r^3\}$$

acts on the set

$$X = \{0, 1\}^4,$$

which represents all possible colorings of the four vertices of the square with two colors (encoded by 0 and 1). The action is defined by

$$r \cdot (x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3),$$

and extended to all of $G = \{e, r, r^2, r^3\}$ by iteration. Use Burnside's Lemma to compute the number of distinct colorings up to rotation.

Bonus Exercise (Not relevant for the final exam)**.**

Definition (Functor). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

(i) an assignment $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$, $x \mapsto F(x)$;

(ii) for each pair of objects $x, y \in \mathcal{C}$ a map

$$\operatorname{Hom}_{\mathcal{C}}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F(x), F(y)), \quad f \mapsto F(f).$$

These must satisfy the following axioms:

(i) for every object $x \in \mathcal{C}$ we have $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$;

(ii) if $f : x \to y$ and $g : y \to z$ are morphisms in $\mathcal{C}$, then

$$F(g \circ f) = F(g) \circ F(f).$$

(a) Show that functors preserve isomorphisms, i.e., let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Let x, y be two objects in $\mathcal{C}$ and let $f \in \operatorname{Hom}_{\mathcal{C}}(x, y)$ be an isomorphism. Then $F(f) \in \operatorname{Hom}_{\mathcal{D}}(F(x), F(y))$ is an isomorphism as well.

(b) Recall from Tutorial Sheet 2 the abelianization. Show that

$$\begin{aligned} (-)^{\mathrm{ab}}\colon \mathbf{Grp} &\longrightarrow \mathbf{Ab}, \\ G &\longmapsto G^{\mathrm{ab}}, \\ f &\longmapsto f^{\mathrm{ab}}, \end{aligned}$$

is a functor, where **Grp** denotes the category of groups and **Ab** the category of abelian groups. Conclude that for $G, H \in \mathbf{Grp}$ such that $G \cong H$, it follows that $G^{\mathrm{ab}} \cong H^{\mathrm{ab}}$.

Note: This was already shown in Tutorial Sheet 2, Exercise 4(d), by using the universal property.