LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Prof. Dr. Fabien Morel
Laurenz Wiesenberger

Tutorial sheet 2
Algebra

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Exercise 1. (a) Let G be an abelian group. Show that any subgroup of G is normal.

(b) Let $f\colon G \to H$ be a group homomorphism. Show that $\ker(f) \trianglelefteq G$ is normal.

Note: You have already seen in the lecture that the same does not hold for $\mathrm{im}(f)$.

(c) Let k be a field. Show that $\mathrm{SL}_n(k) \trianglelefteq \mathrm{GL}_n(k)$ is normal.

(d) Let G be a group and $[G, G]$ be the commutator subgroup. Show that $[G, G] \trianglelefteq G$ is normal.

Hint: Note that it suffices to show that $g[a, b]g^{-1} \in [G, G]$.

Exercise 2. Let G be a group of prime order p. Show that for every nontrivial element $g \in G$, we have $\langle g \rangle = G$.

Exercise 3. Show that $[S_n, S_n] = A_n$ for all $n \geq 3$.

Exercise 4 (Abelianization)**.** (a) Show that $G^{\mathrm{ab}} := G/[G, G]$ is abelian and that $[G, G]$ is the smallest normal subgroup of G satisfying this property.

G^{ab} is called the *abelianization* of G.

(b) Show that the abelianization G^{ab}, together with the projection $\pi\colon G \to G^{\mathrm{ab}}$, satisfies the following universal property: For every abelian group H and every group homomorphism $\phi\colon G \to H$, there exists a unique group homomorphism $\psi\colon G^{\mathrm{ab}} \to H$ such that the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & H \\ G^{\mathrm{ab}} \arrow[ur, dashed, "\exists!\,\psi"'] & \end{tikzcd}$$

Note: It also follows directly from the universal property that $[G, G]$ is the smallest normal subgroup of G such that the corresponding quotient is abelian.

(c) Use the universal property to show that the abelianization G^{ab}, together with π, is unique up to canonical isomorphism.

(d) Use the universal property to prove the following statement:

$$G \cong H \;\Rightarrow\; G^{\mathrm{ab}} \cong H^{\mathrm{ab}}.$$

Bonus Exercise (Not relevant for the final exam).

Definition (Category). A category $\mathcal{C}$ consists of the following data:

(a) A class of objects $\mathrm{Ob}(\mathcal{C})$.

(b) For any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y.

(c) For any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a composition map

$$\begin{aligned} \circ\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ (g, f) &\longmapsto g \circ f \end{aligned}$$

such that the following axioms are satisfied:

Associativity. For any four objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, one has:

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Unity. For each object X there exists an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that for all objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ it holds:

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f.$$

(i) Convince yourself that the class of sets together with functions between them forms a category, denoted by **Set**.

(ii) Convince yourself that the class of groups together with group homomorphisms forms a category, denoted by **Grp**.

Definition. Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called an isomorphism if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, such that

$$g \circ f = \mathrm{id}_X, \quad f \circ g = \mathrm{id}_Y.$$

and g is called the inverse morphism of f.

(iii) Let $\mathcal{C}$ be a category and let f be an isomorphism in $\mathcal{C}$. Prove that the inverse of f is unique (hence we can speak of *the* inverse of f).

(iv) Every group induces a groupoid, i.e., a category in which all morphisms are isomorphisms. Let G be a group and consider the category $\underline{G}$, with a single object $*$. Set $\mathrm{Hom}_{\underline{G}}(*, *) := G$ and define $g \circ h := g \cdot h$, where $\cdot$ denotes the group operation. Convince yourself that $\underline{G}$ is indeed a groupoid.