





Prof. Dr. Fabien Morel Laurenz Wiesenberger

TUTORIAL SHEET 2 ALGEBRA

Winter term 25/26 October 27, 2025

Exercise 1. (a) Let G be an abelian group. Show that any subgroup of G is normal.

- (b) Let $f: G \to H$ be a group homomorphism. Show that $\ker(f) \leq G$ is normal. Note: You have already seen in the lecture that the same does not hold for $\operatorname{im}(f)$.
- (c) Let k be a field. Show that $SL_n(k) \subseteq GL_n(k)$ is normal.
- (d) Let G be a group and [G,G] be the commutator subgroup. Show that $[G,G] \subseteq G$ is normal.

Hint: Note that it suffices to show that $g[a,b]g^{-1} \in [G,G]$.

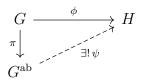
Exercise 2. Let G be a group of prime order p. Show that for every nontrivial element $g \in G$, we have $\langle g \rangle = G$.

Exercise 3. Show that $[S_n, S_n] = A_n$ for all $n \ge 3$.

Exercise 4 (Abelianization). (a) Show that $G^{ab} := G/[G, G]$ is abelian and that [G, G] is the smallest normal subgroup of G satisfying this property.

 G^{ab} is called the *abelianization* of G.

(b) Show that the abelianization G^{ab} , together with the projection $\pi\colon G\to G^{ab}$, satisfies the following universal property: For every abelian group H and every group homomorphism $\phi\colon G\to H$, there exists a unique group homomorphism $\psi\colon G^{ab}\to H$ such that the following diagram commutes:



Note: It also follows directly from the universal property that [G, G] is the smallest normal subgroup of G such that the corresponding quotient is abelian.

- (c) Use the universal property to show that the abelianization G^{ab} , together with π , is unique up to canonical isomorphism.
- (d) Use the universal property to prove the following statement:

$$G \cong H \implies G^{ab} \cong H^{ab}$$
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Algebra Tutorial sheet 2

Bonus Exercise (Not relevant for the final exam).

Definition (Category). A category \mathcal{C} consists of the following data:

- (a) A class of objects Ob(C).
- (b) For any two objects $X, Y \in Ob(\mathcal{C})$, a set $Hom_{\mathcal{C}}(X, Y)$ of morphisms from X to Y.
- (c) For any three objects $X, Y, Z \in Ob(\mathcal{C})$ a composition map

$$\circ \colon \operatorname{Hom}_{\mathcal{C}}(Y,Z) \times \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(X,Z)$$
$$(g,f) \longmapsto g \circ f$$

such that the following axioms are satisfied:

Associativity. For any four objects $X, Y, Z, W \in Ob(\mathcal{C})$, and morphisms $f \in Hom_{\mathcal{C}}(Z, W)$, $g \in Hom_{\mathcal{C}}(Y, Z)$ and $h \in Hom_{\mathcal{C}}(X, Y)$, one has:

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Unity. For each object X there exists an identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ such that for all objects $X,Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ it holds:

$$f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f.$$

- (i) Convince yourself that the class of sets together with functions between them forms a category, denoted by **Set**.
- (ii) Convince yourself that the class of groups together with group homomorphisms forms a category, denoted by **Grp**.

Definition. Let \mathcal{C} be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is called an isomorphism if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, such that

$$g \circ f = \mathrm{id}_X, \quad f \circ g = \mathrm{id}_Y.$$

and g is called the inverse morphism of f.

- (iii) Let \mathcal{C} be a category and let f be an isomorphism in \mathcal{C} . Prove that the inverse of f is unique (hence we can speak of the inverse of f).
- (iv) Every group induces a groupoid, i.e., a category in which all morphisms are isomorphisms. Let G be a group and consider the category \underline{G} , with a single object *. Set $\operatorname{Hom}_{\underline{G}}(*,*) := G$ and define $g \circ h := g \cdot h$, where \cdot denotes the group operation. Convince yourself that \underline{G} is indeed a groupoid.