LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Tutorial sheet 1
Algebra
Suggested Solutions

Winter term 25/26
October 27, 2025

Exercise 1. Let G be a group with subgroups $H_1, H_2 \subseteq G$. Show that $H_1 \cup H_2$ is a subgroup of G if and only if $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Suggested solution. $\Rightarrow$ Let $H_1 \cup H_2 \leq G$ be a subgroup of G. We aim to show that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. Suppose, for a contradiction, that $H_1 \nsubseteq H_2$ and $H_2 \nsubseteq H_1$. In particular, there exist $h_1 \in H_1 \setminus H_2$ and $h_2 \in H_2 \setminus H_1$. By assumption, $H_1 \cup H_2$ is a subgroup, and so $h_1 h_2 \in H_1 \cup H_2$. Without loss of generality, assume $h_1 h_2 \in H_1$ (the other case is analogous). Then $h_2 = h_1^{-1}(h_1 h_2) \in H_1$, which contradicts $h_2 \in H_2 \setminus H_1$. We therefore conclude $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

$\Leftarrow$ Suppose $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$. We need to show that $H_1 \cup H_2$ is a subgroup of G. Without loss of generality (the argument is analogous otherwise), assume $H_1 \subseteq H_2$. Then $H_1 \cup H_2 = H_2$, which is, by assumption, a subgroup of G.

□

Exercise 2. (a) Prove that the additive group $(\mathbb{R}, +)$ is isomorphic to the multiplicative group $(\mathbb{R}^+, \cdot)$. Does the same hold for $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \cdot)$?

Suggested solution. We aim to show that $(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot)$. A natural approach is to think first about a bijective function from $\mathbb{R}$ to $\mathbb{R}^+$. The obvious candidate is the exponential function $\exp(\cdot)$. Consider

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}^+, \quad x \longmapsto e^x.$$

By the functional equation, $\exp(x + y) = \exp(x)\exp(y)$, the exponential map is a group homomorphism. From basic analysis, it is well known that exp is bijective. Consequently, exp is a group isomorphism, and therefore

$$(\mathbb{R}, +) \cong (\mathbb{R}^+, \cdot).$$

Next, let us assume for contradiction that there exists an isomorphism

$$f : (\mathbb{Q}, +) \longrightarrow (\mathbb{Q}^+, \cdot).$$

In particular, there exists some $x \in \mathbb{Q}$ with $f(x) = 2$. Now consider the element $y := \frac{1}{2}x \in \mathbb{Q}$. Since f is a homomorphism, we obtain

$$2 = f(x) = f(y + y) = f(y) \cdot f(y) = f(y)^2.$$

This implies $f(y) = \sqrt{2}$, which would mean $\sqrt{2} \in \mathbb{Q}$, a contradiction. Hence, no such isomorphism exists, and therefore

$$(\mathbb{Q}, +) \ncong (\mathbb{Q}^+, \cdot).$$

□

(b) Prove that $(\mathbb{R}^\times, \cdot)$ and $(\mathbb{R}, +)$ are not isomorphic as groups.

Suggested solution. Suppose, for a contradiction, that $(\mathbb{R}^\times, \cdot) \cong (\mathbb{R}, +)$. Then there is a group isomorphism

$$f : (\mathbb{R}^\times, \cdot) \longrightarrow (\mathbb{R}, +).$$

Since f is a homomorphism, $f(1) = 0$. Furthermore,

$$0 = f(1) = f((-1) \cdot (-1)) = f(-1) + f(-1) = 2f(-1).$$

Hence $f(-1) = 0$, which contradicts the injectivity of f.

Alternative argument. The group $(\mathbb{R}^\times, \cdot)$ contains an element of order 2, namely -1. However, the group $(\mathbb{R}, +)$ contains no non-trivial element of finite order. Because group isomorphisms preserve the order of elements, no isomorphism between $(\mathbb{R}^\times, \cdot)$ and $(\mathbb{R}, +)$ can exist.

□

(c) Let $m, n \in \mathbb{Z}$ be coprime integers and let $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be a group homomorphism. Prove that f is trivial, i.e. $f \equiv 0$.

Suggested solution. Let $m, n \in \mathbb{Z}$ with $\gcd(m, n) = 1$, and let

$$f : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$$

be a group homomorphism. We will show that f is trivial.

Since $\mathbb{Z}/m\mathbb{Z}$ is a cyclic group generated by $\bar{1}$, the homomorphism f is determined by the image of this generator. Set $\bar{a} := f(\bar{1})$. Then

$$f(\bar{m}) = f(\bar{0}) = \bar{0}$$

and also

$$f(\bar{m}) = f(m \cdot \bar{1}) = m \cdot f(\bar{1}) = m \cdot \bar{a}.$$

Thus

$$m \cdot \bar{a} = \bar{0} \quad \text{in } \mathbb{Z}/n\mathbb{Z},$$

which means

$$n \mid ma.$$

Since $\gcd(m, n) = 1$, it follows by unique prime factorization that $n \mid a$, and hence $\bar{a} = \bar{0}$. Therefore $f(\bar{1}) = \bar{0}$, so f is the trivial homomorphism:

$$f \equiv 0.$$

□

(d) Describe all group homomorphisms $(\mathbb{Z}/4\mathbb{Z}, +) \to (\mathbb{Z}/6\mathbb{Z}, +)$.

Suggested solution. Let $f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ be a group homomorphism. Since f is a homomorphism, we have

$$f(\bar{0}) = \bar{0}.$$

Since $\bar{4} = \bar{0}$ in $\mathbb{Z}/4\mathbb{Z}$, we obtain

$$f(\bar{4}) = 4f(\bar{1}) = \bar{0}.$$

Set $\bar{x} := f(\bar{1})$. In particular,

$$4x \equiv 0 \pmod 6.$$

This implies $\bar{x} = \bar{0}$ or $\bar{x} = \bar{3}$. If $\bar{x} = \bar{0}$, then $f \equiv 0$. If $\bar{x} = \bar{3}$, then

$$f(\bar{2}) = 2\bar{x} = \bar{0} \quad \text{and} \quad f(\bar{3}) = 3\bar{x} = \bar{3}.$$

Hence

$$f(\bar{x}) = 3\bar{x}.$$

Please note, this is indeed a group homomorphism. □

Exercise 3. Let G be a group such that for all $g \in G$, $g^2 = 1$. Show that G is abelian.

Suggested solution. We have to show that for all $a, b \in G$:

$$ab = ba.$$

Let $a, b \in G$ be arbitrary elements. Then

$$ab = 1_G(ab) = (ba)^2(ab) = (ba)(b(aa)b) = ba.$$

Consequently, G is commutative. □

Exercise 4. Let G be a finite abelian group. Show that

$$\prod_{g \in G} g^2 = 1.$$

Suggested solution. Let G be a finite abelian group, say $G = \{g_1, \ldots, g_n\}$ with $n \in \mathbb{N}$ and the identity element is denoted by 1_G. For each $g_i \in G$ there exists a unique $g_j \in G$ such that

$$g_i g_j = 1_G,$$

namely the inverse of g_i. Since G is abelian, we may write

$$\prod_{g \in G} g^2 = g_1^2 g_2^2 \cdots g_n^2$$

in any order. Now pair each element g_i with its inverse $g_j = g_i^{-1}$. If $g_i \neq g_j$, then

$$g_i g_j = g_i g_i^{-1} = 1_G.$$

All such pairs contribute a factor of 1_G to the product.

If $g_i = g_j$, then $g_i^2 = 1_G$, so g_i has order 2 or equals 1_G itself. These elements also contribute 1_G to the product.

Combining all pairs and the remaining self-inverse elements, we obtain

$$\prod_{g \in G} g^2 = 1_G.$$

This completes the proof. □