Prof. Dr. Fabien Morel
Laurenz Wiesenberger

REPETITION WEEK 4
ALGEBRA

Winter term 25/26

Group Action on a Set

We begin by presenting two equivalent definitions of a group action on a set.

Definition (Abstract Definition)**.** Let G be a group and X a set. A *(left) group action* of G on X is a group homomorphism

$$G \longrightarrow S(X),$$

where $S(X)$ denotes the symmetric group of all bijections $X \to X$.

Definition (Concrete Definition)**.** Let G be a group and X a set. A *(left) group action* of G on X is a map

$$G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x,$$

such that the following axioms hold:

(a) $e_G \cdot x = x$ for all $x \in X$;

(b) $g \cdot (h \cdot x) = (g \cdot_G h) \cdot x$ for all $g, h \in G$ and $x \in X$.

In this case, the set X is called a *G-set*.

Remark. (i) Note that the abstract and the concrete definitions are equivalent.

(ii) **Category of G-sets:** Let G be a fixed group. The objects of the category **G-Set** are precisely all G-sets X. For two G-sets X and Y, a *morphism*

$$f\colon X \longrightarrow Y$$

is a map satisfying

$$f(g \cdot x) = g \cdot f(x) \quad \text{for all } g \in G \text{ and } x \in X.$$

We denote by $\mathrm{Hom}_{\mathbf{G\text{-}Set}}(X, Y)$ the set of all morphisms between X and Y. Together with the usual composition of maps, this defines a category. The elements of $\mathrm{Hom}_{\mathbf{G\text{-}Set}}(X, Y)$ are called *G-equivariant maps*.

Example (Examples of Group Actions)**.**

1) The n-th symmetric group S_n operates on the left on $\{1, \dots, n\}$, via:
$$S_n \times \{1, \dots, n\} \to \{1, \dots, n\}, \quad (\sigma, n) \longmapsto \sigma(n).$$

2) Let $H \subseteq G$ be a subgroup. Then the map
$$G \times G/H \longrightarrow G/H, \quad (g, \overline{\alpha}) \longmapsto \overline{g\alpha}$$
defines a left action of G on the set G/H. Note that this map is well defined. Indeed, let $\overline{\alpha_1} = \overline{\alpha_2}$, i.e. $\alpha_1^{-1}\alpha_2 \in H$. But then we also have
$$\alpha_1^{-1} e_G \alpha_2 = \alpha_1^{-1} g^{-1} g \alpha_2 = (g\alpha_1)^{-1}(g\alpha_2) \in H,$$
and hence $\overline{g\alpha_1} = \overline{g\alpha_2}$.

3) For any field k, the general linear group $GL_n(k)$ acts (on the left) on k^n via
$$GL_n(k) \times k^n \longrightarrow k^n, \quad (M, x) \longmapsto Mx.$$

4) The map
$$G \times S \longrightarrow S, \quad (g, x) \longmapsto x$$
is the *trivial* operation of G.

5) The group G operates in two ways on the underlying set of G:
$$\begin{cases} (g, h) \longmapsto gh & \text{(translation)}, \\ (g, h) \longmapsto ghg^{-1} & \text{(conjugation)}. \end{cases}$$

6) Let G be a group and $\mathcal{P}(G)$, the set of all subsets of G. Define the action
$$\theta \colon G \times \mathcal{P}(G) \longrightarrow \mathcal{P}(G), \quad (g, S) \longmapsto g \cdot S := \{\, gs \mid s \in S \,\}.$$
Let $\mathcal{U}(G)$ denote the set of all subgroups of G. Then G acts on $\mathcal{U}(G)$ by
$$G \times \mathcal{U}(G) \longrightarrow \mathcal{U}(G), \quad (g, H) \longmapsto gHg^{-1}.$$

Definition. Let X be a G-set.

a) Let $x \in X$ be an element. Then
$$G_x := I_x := \{\, g \in G \mid g \cdot x = x \,\} \subseteq G$$
is called the *isotropy subgroup* of x (with respect to the given action). Sometimes we write $I_x := \mathrm{Stab}_G(x)$ and call it the *stabilizer* of x.

b) For $x \in X$, the set
$$G{\cdot}x := \{\, g \cdot x \mid g \in G \,\} \subseteq X$$
is called the *orbit* of x.

Remark. Note that the isotropy subgroup G_x of x is a subgroup of G.

Theorem. Let $\emptyset \neq X$ be a G-set. Then the relation
$$x \sim_G y \iff \exists\, g \in G : y = g \cdot x$$
on X is an equivalence relation. The equivalence classes are precisely the orbits of X. We denote by
$$X/G$$
the quotient set of X modulo this equivalence relation.

In particular, we have the disjoint union
$$X = \bigsqcup G{\cdot}x,$$
where the union runs over distinct orbits.

Lemma (Orbit–stabilizer theorem)**.** Let X be a G-set and $x \in X$. Then the surjective map
$$G \twoheadrightarrow G{\cdot}x, \quad g \longmapsto g \cdot x,$$
induces a bijection
$$G/I_x \xrightarrow{\sim} G{\cdot}x.$$

In particular, if G is finite, the orbit–stabilizer theorem states that
$$|G{\cdot}x| = |G/I_x| = \frac{|G|}{|I_x|},$$
where the last equality follows from Lagrange's theorem.

In the tutorials, we will treat and see some applications of the so-called *class equation*. We consider the following situation:

Let G be a finite group acting on itself by conjugation, i.e.
$$G \times G \longrightarrow G, \quad (g, x) \longmapsto gxg^{-1}.$$
(See also Example 5 from the beginning.) Then the corresponding orbit decomposition yields the *class equation*:
$$|G| = |Z(G)| + \sum_{|G\cdot x|>1} |G{\cdot}x|,$$
where $Z(G)$ denotes the center of G, and the sum runs over the disjoint, non-trivial orbits $G{\cdot}x$.

Definition. Let X be a G-set.

(a) Let $g \in G$. Then X^g is the set of points fixed by g, i.e.

$$X^g := \{x \in X \mid g \cdot x = x\}$$

(b) We say that an element $x \in X$ is a *fixed point* if $g \cdot x = x$ for all $g \in G$, i.e. $I_x = G$. We denote by

$$X^G \subseteq X$$

the subset of all fixed points of the action.

In the tutorials, we will get to know Burnside's lemma:
Let G be a finite group acting on a set X. Then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

We will also see an application of this result in the tutorials.

Definition. Let X be a G-set. We say that the action is *transitive* if $X \neq \varnothing$ and for all $(x, y) \in X^2$ there exists a $g \in G$ such that $y = g \cdot x$. In other words, there is only one orbit, i.e. $|G/X| = 1$.

Example. S_n acts transitive on $\{1, \ldots, n\}$, see Example 1 from the beginning.

Definition. Let X be a G-set and $Y \subseteq X$ be a subset. We say that Y is *G-invariant* (or *G-stable*) if

$$g \cdot Y = Y \quad \text{for all } g \in G.$$

Example. Let $Y \subseteq X$ be a G-invariant subset. Then Y is automatically a G-set via the restricted action

$$G \times Y \longrightarrow Y, \quad (g, y) \longmapsto g \cdot y,$$

and the inclusion map $Y \hookrightarrow X$ is G-equivariant.

For instance, for any $x \in X$, the orbit $G \cdot x \subseteq X$ is G-invariant.

We now consider some further examples illustrating the notions and properties of group actions introduced so far.

Example. 1) For every $x \in X$, the bijection

$$G/I_x \xrightarrow{\sim} G \cdot x$$

is G-equivariant. Note that, by Example 2 from the beginning, G/I_x is itself a G-set.

2) Let $Y \subseteq X$ be a subset. Then

$$Y \text{ is } G\text{-invariant} \iff Y \text{ is a disjoint union of orbits.}$$

We will prove this statement in the tutorials.

3) Let $f\colon X \to Y$ be a G-equivariant map. Then f induces a map on the fixed points,

$$f^G\colon X^G \longrightarrow Y^G, \quad x \longmapsto f(x),$$

and a map on the orbit spaces,

$$f/G\colon X/G \longrightarrow Y/G, \quad G\cdot x \longmapsto G\cdot f(x).$$

As a short homework, and to get used to the new definitions, prove that the above maps are well-defined.

4) Let S_n operate on $\{1,\dots,n\}$ as we have already seen earlier. Then this action is transitive (see also Example 3 from the beginning). Thus, for all $i \in \{1,\dots,n\}$ there exists a canonical S_n-equivariant bijection (see again the orbit–stabilizer theorem)

$$S_n/I_i \xrightarrow{\sim} \{1,\dots,n\} = S_n \cdot i.$$

Furthermore, $I_i \cong S_{n-1}$, which implies

$$S_n/S_{n-1} \xrightarrow{\sim} \{1,\dots,n\}.$$

5) Let k be a field and $n \geq 1$. Then $GL_n(k)$ operates on k^n (the same operation as in Example 3 from the beginning). It follows from linear algebra that

$$k^n/GL_n(k) = \{\overline{0}, \overline{e}\}.$$

Hence,

$$k^n = \{0\} \sqcup (k^n \setminus \{0\}),$$

and in particular, $k^n \setminus \{0\}$ is a transitive $GL_n(k)$-set.

6) This example is more of an outlook. Let $G \subseteq \mathrm{Aut}_{\text{field}}(L)$ a finite subgroup. One can show that

$$K := L^G = \{\, a \in L \mid \forall g \in G,\ g(a) = a \,\}$$

is a subfield of L, and that $[L : K] := \dim_K(L) = |G|$. (We will see this later, when we discuss Galois theory.)

Note that the group action is given by

$$G \times L \longrightarrow L, \quad (\sigma, x) \longmapsto \sigma(x),$$

and hence

$$L^G = \{\, a \in L \mid g(a) = a \text{ for all } g \in G \,\}.$$

Let $P \in K[X]$. Then the set $R_P(L)$ of roots of P in L is finite and G-invariant, that is, $R_P(L)$ is a G-invariant subset of L.

Furthermore,

$$R_P(L) = R_P(K) \sqcup \bigsqcup_{\substack{\text{orbits } \alpha \\ |\alpha| \geq 2}} \alpha.$$

Assume now that $P \in L[X]$ splits into a product of linear factors and that P is monic, i.e.

$$P = \prod_{i=1}^{n} (X - \alpha_i), \quad \alpha_i \in L.$$

Assume also that $L = K[\alpha_1, \ldots, \alpha_n]$. Then there is a group homomorphism

$$G \longrightarrow S_{R_P(L)} \cong S_n.$$