Algebra 2025/2026: Exercise sheet 9

Exercise 1.

Let R be a commutative ring, M and N be R-modules. We denote by $Hom_{R-\mathcal{M}od}(M,N)$ the set of R-linear maps from M to N. For R-linear maps f and $g : M \to N$, we denote by $f+g$ the map $M \to N$, $x \mapsto f(x)+g(x)$ and for $\lambda \in R$ we denote by $\lambda.f$ the map $M \to N$, $x \mapsto \lambda.f(x)$. Show that $f+g$ and $\lambda.f$ are R-linear maps and that in this way $\big(Hom_{R-\mathcal{M}od}(M,N),+,.\big)$ is an R-module. Moreover, show that for R-modules L, M, N the composition map

$$Hom_{R-\mathcal{M}od}(L,M) \times Hom_{R-\mathcal{M}od}(M,N) \to Hom_{R-\mathcal{M}od}(L,N)$$

is an R-bilinear map.

Exercise 2.

Let S be a countable set, that is to say in bijection with a subset of $\mathbb{N}$. Let R be a P.I.D. Show that any sub-R-module of the free R-module on S, $R[S] = \oplus_{s\in S} R$ is a free R-module on a countable set. [Hint: assume $S = \mathbb{N}$, and consider $R[\mathbb{N}]$ as the union of the sub-R-modules $F_n = R[\{0,\dots,n\}]$, free on the subset $\{0,\dots,n\} \subset \mathbb{N}$.]

Exercise 3.

Let R be an integral domain, K its field of fractions.

1) Show that if R is not a field, K is never a finite type R-module. [Hint: consider a finite set of fractions $\frac{a_i}{q_i}$ which generate K as an R-module. Show that $q = \Pi_i q_i$ is in R and conclude that $R = K$]

2) Show that K contains no free sub-R-module of rank ≥ 2.

3) If R is a P.I.D, show that any non zero finite type sub-R-module of K is free of rank 1.

Exercise 4.

Let R be a P.I.D. and K be its field of fractions.

1) Let $I \subset R$ be a non zero ideal. Show that the R-module $Hom_{R-\mathcal{M}od}(R/I, K/R)$ is isomorphic to R/I. [Hint: let $a \in R \setminus \{0\}$, show that the kernel of $a. : K/R \to K/R$ is the sub-R-module generated by $\frac{1}{a}$ and is isomorphic to $R/(a)$.]

2) Let M be a finite type torsion R-module. Show that the R-module $Hom_{R-\mathcal{M}od}(M, K/R)$ is isomorphic to M.