Algebra 2025/2026: Exercise sheet 8

Exercise 1.

Let R be a commutative ring, $f \in R$ be an element and $n \geq 2$ be an integer. Show that there is a monomorphism of rings $\iota : R \hookrightarrow S$ and an element $x \in S$ such that $x^n = \iota(f)$.

Exercise 2.

Let A be a commutative ring, $n \geq 1$ be an integer. Show that for any commutative ring R the obvious map

$$Hom_{Ring}(A[X_1, \ldots, X_n], R) \to Hom_{Ring}(A, R) \times R^n$$

$$(\phi : A[X_1, \ldots, X_n] \to R) \mapsto \big((\phi|_A : A \to R), (\phi(X_1), \ldots, \phi(X_n) \big)$$

is a bijection.

Exercise 3.

Let R be a commutative ring and $f \in R$ be an element. We let R_f be the ring $R[X]/(f.X - 1)$, quotient of the polynomial ring $R[X]$ by the principal ideal generated by the polynomial $f.X - 1 \in R[X]$ and we consider the induced ring homomorphism $\psi : R \to R_f$ composition of the canonical morphism $R \to R[X]$ and the quotient morphism $R[X] \to R_f$.

1) Show that if f is invertible in R, then $\psi : R \cong R_f$ is an isomorphism of rings.

2) Show that in general $\psi(f)$ is invertible in R_f.

3) Let A be a commutative ring. Show that the map $Hom_{Ring}(R_f, A) \to Hom_{Ring}(R, A)$, $\phi \mapsto \phi \circ \psi$ is injective and its image coincides with the subset of $Hom_{Ring}(R, A)$ consisting of morphisms of rings $\phi : R \to A$ such that $\phi(f) \in A^\times$.

The ring R_f is called "R with f inverted"!

4) Show that the ring $R[X]$ of polynomials with X inverted, $R[X]_X$, is a free R module with basis the family $\{X^n\}_{n \in \mathbb{Z}}$, where X^{-1} is the inverse of X in $R[X]_X$ and $X^{-n} = (X^{-1})^n$ for any natural number n. Conclude it is canonically isomorphic to the group ring $R[\mathbb{Z}]$ of $\mathbb{Z}$ with coefficients in R. This ring is also called the ring of Laurent Polynomials with coefficients in R and denoted by $R[X, X^{-1}]$.

Exercise 4.

Let R be a noetherian ring. Let M be a finite type R-module. Show that any sub R-module N

of M is also of finite type. [Hint: induction on the number of generators of M...]